

EUCLID'S ELEMENTS

BOOK III

THE GEOMETRY OF CIRCLES

Chords, Tangents, Angles, Segments, Secants, and the Power of a
Point

Illustrated Edition

Contents

Book III

Introduction

Book III develops Euclid's classical theory of circles. It studies centers, diameters, chords, the distances of chords from the center, internal and external contact, tangents, angles at the center and circumference, angles in circular segments, cyclic quadrilaterals, equal arcs and equal chords, intersecting chords and secants, and the tangent–secant power relation together with its converse. The book builds upon the constructions and angle theory of Book I and the metric relations of Book II, and establishes the principal circle theorems required in later books.

Definition 1

Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).

EUCLID’S ELEMENTS • BOOK III
DEFINITION I

Equal circles are circles whose diameters are equal,
or whose distances from the centers to the circumferences are equal—
that is, whose radii are equal.

PART 1 – EQUAL CIRCLES

Two circles are called equal circles
when they have the same size.



PART 2 – EQUAL DIAMETERS

This happens when their diameters are equal.
If the diameter of the first circle equals
the diameter of the second circle,
the circles are equal.



PART 3 – EQUAL DISTANCES FROM CENTERS TO CIRCUMFERENCES (EQUAL RADII)

This also happens when the distances from the
centers to the circumferences are equal.
These distances are the radii.



If $r = r'$,
then the circles
are equal.

$$r = r'$$

SUMMARY

Equal circles are circles that are the same size.

They have either equal diameters or equal radii.

In either case:

$$\text{Equal diameters} \Leftrightarrow \text{Equal radii} \Leftrightarrow \text{Equal circles}$$

Definition 2

A straight-line said to touch a circle is any (straightline) which, meeting the circle and being produced, does not cut the circle.

Aristotle:
Some lines
cut a circle,
but others
only touch it.

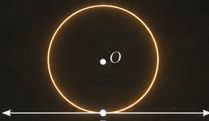
EUCLID'S ELEMENTS • BOOK III

DEFINITION II

A straight-line is said to **TOUCH A CIRCLE**
if, meeting the circle and being produced,
it does not cut the circle.

Alexander (13):
So, master,
a line can
meet a circle
without
cutting it?

PART 1 – THE LINE MEETS THE CIRCLE
A straight-line meets the circle
at exactly one point.



The line and the circle have one
common point, P .

PART 2 – THE LINE IS PRODUCED
The straight-line is extended indefinitely
in both directions.



The line continues beyond the point P
on both sides without stopping.

PART 3 – IT DOES NOT CUT THE CIRCLE
Even when produced, the line
does not enter the circle
at any other point.



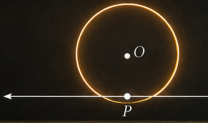
The line shares only one point with the circle.
Therefore, it does not cut the circle.

REMEMBER
If a line cut a circle,
it would meet it in
two points.



This is cutting.

DEFINITION II – IN SUMMARY
A straight-line is said to touch a circle
if, meeting the circle and being produced,
it does not cut the circle.



MEETS in one point.
PRODUCED in both directions.
DOES NOT CUT the circle.
That is, no second point
is common.



The geometer must discern
not by measure, but by reason.
— Aristotle

Definition 3

Circles said to touch one another are any (circles) which, meeting one another, do not cut one another.

Aristotle:
When circles meet, they may cut or touch—but never both.

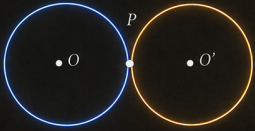
EUCLID'S ELEMENTS • BOOK III

DEFINITION III

Circles are said to **TOUCH ONE ANOTHER** if, meeting one another, they do not cut one another.

Alexander (13):
So, if circles touch but do not cut, they are said to touch one another?
Aristotle:
Exactly, my boy.

PART 1 – THE CIRCLES MEET
Two circles come to a point in common.



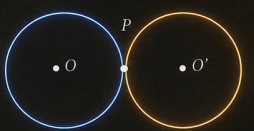
They have one common point, P .

PART 2 – THEY DO NOT CUT
If circles cut one another, they would have two points in common.



This is **cutting**: there are two points in common. ❌

PART 3 – THEY TOUCH
If the circles have only one point in common, they do not cut.



This is **touching**: only one point in common. ✅

IN SUMMARY
Circles are said to **TOUCH ONE ANOTHER** when they meet in one point only and do not cut.

DEFINITION III – IN SYMBOLS
Circles with centers O and O' touch one another if they have exactly one common point.



MATHEMATICAL EXPRESSION
Two circles touch $\Leftrightarrow$ they have exactly one common point.

A GEOMETER'S VIEW



Reason, not sight, reveals the truth of figures.
— Aristotle

Definition 4

In a circle, straight-lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal.

ARISTOTLE:
To understand
the nature of
circles, we must
first understand
what it means for
straight-lines to be
equally distant
from the center.

EUCLID’S ELEMENTS • BOOK III

DEFINITION IV

In a circle, straight-lines are said to be
EQUALLY FAR FROM THE CENTER
when the perpendiculars drawn to
them from the center are equal.

ALEXANDER (13):
So, if the center
reaches out
perpendicularly
and touches two
lines equally,
those lines are
equally far away?

ARISTOTLE:
Exactly, my boy.

PART 1 – THE CIRCLE AND ITS CENTER
Let there be a circle with center O .



O is the center of the circle.

PART 2 – TWO STRAIGHT-LINES IN THE PLANE
Let there be two straight-lines, AB and CD ,
in the same plane as the circle.



AB and CD are straight-lines.

PART 3 – PERPENDICULARS FROM THE CENTER
From the center O , draw perpendiculars
to the straight-lines AB and CD ,
meeting them at E and F .



$OE \perp AB$ at E , and $OF \perp CD$ at F .

PART 4 – THE DEFINITION
The straight-lines AB and CD are said to be
EQUALLY FAR FROM THE CENTER O
when the perpendiculars drawn to them from O are equal.

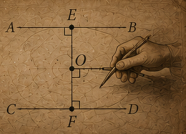
$OE = OF \Rightarrow AB$ and CD are equally far from O .



IN SUMMARY
Two straight-lines in a circle are equally far from the center
when the perpendicular distances from the center to each line are equal.

KEY IDEA
Equal perpendiculars $\Leftrightarrow$ Equal distance
(from the center)





Definition 5

And (that straight-line) is said to be further (from the center) on which the greater perpendicular falls (from the center).

ARISTOTLE:
We measure distance from the center perpendicularly.

EUCLID'S ELEMENTS • BOOK III

DEFINITION V

A straight-line is said to be **FURTHER FROM THE CENTER** when the greater perpendicular falls upon it.

ALEXANDER (13):
So the line with the greater perpendicular is further from the center?
ARISTOTLE:
Exactly.

PART 1 – THE CENTER AND LINES
Let there be a circle with center O , and two straight-lines, AB and CD .



Perpendiculars OE and OF are drawn from O to the lines.

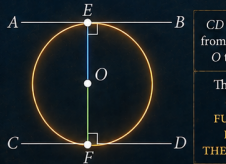
PART 2 – COMPARE PERPENDICULARS
Compare the lengths of the perpendiculars OE and OF .



Suppose OF is greater than OE .
 $OF > OE$

The perpendicular to CD (OF) is greater than that to AB (OE).

PART 3 – WHICH LINE IS FURTHER?
The line on which the greater perpendicular falls is further from the center.



CD is further from the center O than AB .
Therefore, CD is **FURTHER FROM THE CENTER**.

Since $OF > OE$, the line CD is further from the center O than AB .

IN SUMMARY
A straight-line is said to be **FURTHER FROM THE CENTER** when the greater perpendicular falls upon it.

Greater perpendicular
↓
Line is further from the center



THE IDEA IN SYMBOLS



If $OE > OF$, then the line CD is further from the center O than the line AB .
If $OF > OE$, then the line AB is further from the center O than the line CD .

Definition 6

A segment of a circle is the figure contained by a straight-line and a circumference of a circle.

ARISTOTLE:
Look carefully,
Alexander. We
first grasp the
names of things.
Then we grasp
their truths.

EUCLID'S ELEMENTS • BOOK III
DEFINITION VI
A segment of a circle is the figure
contained by a straight-line
and a circumference of a circle.

ALEXANDER (13):
So a segment
is the region
between the
straight-line
and the arc
of the circle?
ARISTOTLE:
Just so.

PART 1 – THE CIRCLE
Let there be a circle with center O .



The boundary of the circle
is its circumference.

PART 2 – A STRAIGHT-LINE (CHORD)
Let a straight-line AB be drawn,
joining two points A and B
on the circumference.



The straight-line AB cuts across
the circle. It is called a chord.

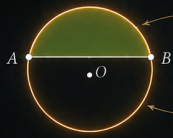
PART 3 – THE SEGMENT
The figure contained by the straight-line AB
(the chord) and the arc AB
(part of the circumference) is shaded.



Segment
of the circle

This shaded region is the
segment of the circle.

IN SUMMARY
A segment of a circle
is the region (area)
that lies between
a straight-line (chord)
and the arc of the circle
endpoints.



Arc AB
(part of the circumference)

Chord AB
(a straight-line)

Segment
of the circle



The segment is not the whole circle.
It is only the part between the chord and the arc.

Definition 7

And the angle of a segment is that contained by a straight-line and a circumference of a circle.

EUCLID'S ELEMENTS
BOOK III

DEFINITION VII

THE ANGLE OF A SEGMENT

ALEXANDER (13):
So the angle here is between
the straight-line and the arc
of the circle?
ARISTOTLE: Just so.

THE ANGLE OF A SEGMENT is that angle contained by a straight-line and the arc of a circle.

1 THE CIRCLE

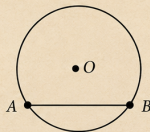
Let there be a circle
with center O .



This is the circle.

2 A STRAIGHT-LINE (CHORD)

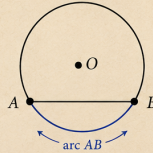
Draw a straight-line AB ,
joining two points on
the circumference.



The straight-line AB is
a chord of the circle.

3 THE ARC

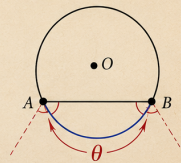
The straight-line AB cuts off
an arc of the circle.



The curved part of the
circumference between A and B
is the arc AB .

4 THE ANGLE OF THE SEGMENT

The angle of the segment is the angle
contained by the straight-line AB
and the arc AB of the circle.



Here θ is the angle of the segment,
the inclination between the arc AB
of the circle and the straight-line AB .

IN SUMMARY

The angle of a segment
is the inclination between
the arc of a circle and the
straight-line (chord) that
subtends it.



EUCLID'S LANGUAGE

Angles are not only between straight-lines.
They can also be between a straight-line
and a curve (such as the circumference
of a circle). This is one kind of angle.



Definition 8

And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.

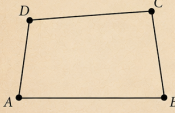
ARISTOTLE:
Let us learn the names of figures, that we may reason truly about them.

EUCLID'S ELEMENTS • BOOK III
DEFINITION IX

A **RECTANGLE** is a figure contained by four straight-lines of which the opposite ones are equal to one another, and the angles contained by the straight-lines are right angles.

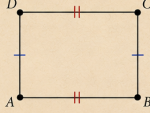
ALEXANDER (13):
What is a rectangle, master?
ARISTOTLE:
I will show you.

1 FOUR STRAIGHT-LINES
Let four straight-lines AB, BC, CD, DA be joined in order, so as to enclose a figure.



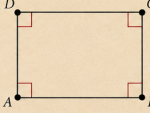
A, B, C, D are the vertices, and the figure $ABCD$ is contained by four straight-lines.

2 OPPOSITE SIDES EQUAL
Let the opposite sides be equal to one another.



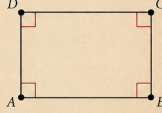
$AB = CD$
and
 $BC = AD$.

3 RIGHT ANGLES
Let the angles contained by the straight-lines be right angles.



$\angle A, \angle B, \angle C, \angle D$ are all right angles.

4 A RECTANGLE
The figure $ABCD$ is therefore a rectangle.



$ABCD$ is a **rectangle**: opposite sides equal, all angles right angles.

IN SUMMARY
A rectangle is a quadrilateral in which opposite sides are equal to one another and all the angles are right angles.

EUCLID'S LANGUAGE
Euclid defines figures by the properties that belong to them. Thus we may know them when we see them, and reason about them with certainty.



Definition 9

And when the straight-lines containing an angle cut off some circumference, the angle is said to stand upon that (circumference).

ARISTOTLE:
Let us learn the names of figures, that we may reason truly about them.

EUCLID’S ELEMENTS • BOOK III
DEFINITION IX

ALEXANDER (13):
What is meant by an angle standing upon a circumference?
ARISTOTLE:
I will show you.

AND WHEN THE STRAIGHT-LINES CONTAINING AN ANGLE CUT OFF SOME CIRCUMFERENCE, THE ANGLE IS SAID TO STAND UPON THAT CIRCUMFERENCE.

EXPLANATION

Let two straight-lines AB and AC contain an angle BAC .

Suppose these straight-lines cut off from a circle with center O the arc BC .

Then the angle BAC is said to **stand upon** that circumference, namely, the arc BC .

DEFINITION ILLUSTRATED

Since the straight-lines AB and AC , which contain the angle BAC , cut off the arc BC of the circle, the angle BAC is said to **stand upon** that circumference, namely, the arc BC .

✦ The angle **stands upon** the circumference (arc) that is cut off by its sides.

IN SUMMARY

An angle is said to **stand upon** a circumference when its two straight-lines cut off some arc of that circumference.

EUCLID’S LANGUAGE

Euclid names figures by the properties that belong to them, so that we may know them when we see them.

Definition 10

And a sector of a circle is the figure contained by the straight-lines surrounding an angle, and the circumference cut off by them, when the angle is constructed at the center of a circle.

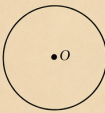
ARISTOTLE:
Let us learn the names of figures, that we may reason truly about them.

ALEXANDER (13):
What figure is contained by an angle at the center of a circle?
ARISTOTLE:
I will show you.

EUCLID'S ELEMENTS • BOOK III
DEFINITION X

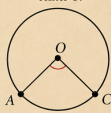
A **SECTOR OF A CIRCLE** is the figure contained by the straight-lines surrounding an angle and the circumference cut off by them, when the angle is constructed at the **center** of a circle.

1 THE CIRCLE
Let there be a circle with center O .



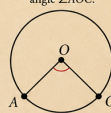
This is the circle.

2 AN ANGLE AT THE CENTER
Let an angle $\angle AOC$ be constructed at the center O .



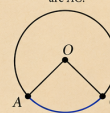
$\angle AOC$ is an angle with vertex at the center O .

3 THE STRAIGHT-LINES SURROUNDING THE ANGLE
Draw the straight-lines OA and OC surrounding the angle $\angle AOC$.



The straight-lines OA and OC surround the angle $\angle AOC$.

4 THE CIRCUMFERENCE CUT OFF BY THEM
The straight-lines OA and OC cut off from the circle the arc AC .



The arc AC of the circle is cut off by OA and OC .

5 THE SECTOR
The figure contained by the straight-lines OA , OC and the arc AC is the sector AOC .



Therefore, the **sector AOC** is the figure contained by the angle at the center and the circumference cut off by it.

IN SUMMARY
A **sector** of a circle is the part of the circle that lies between two radii and the arc they cut off, when the angle is at the center.

EUCLID'S LANGUAGE
A **sector** is determined by three things: the center O , the two straight-lines (radii) surrounding an angle at the center, and the arc of the circle that those lines cut off.



Definition 11

Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.

ARISTOTLE:
Let us learn the names of figures, that we may reason truly about them.

EUCLID’S ELEMENTS • BOOK III
DEFINITION XI

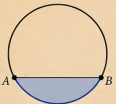
SIMILAR SEGMENTS OF CIRCLES are those accepting equal angles, or in which the angles are equal to one another.

ALEXANDER (13):
What are similar segments of circles?

ARISTOTLE:
I will show you.

WHAT IS A SEGMENT OF A CIRCLE?

A segment of a circle is the figure contained by a chord and the arc it cuts off.

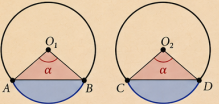


The blue region is a segment of the circle cut off by the chord AB (the arc AB).

DEFINITION XI ILLUSTRATED

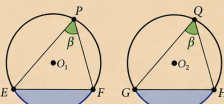
SIMILAR SEGMENTS OF CIRCLES are those accepting equal angles, or in which the angles are equal to one another.

Example 1: Accepting equal angles



The segments cut off by chords AB and CD accept equal angles at the centers $\angle AOB = \angle COD = \alpha$. Therefore, the segments AB and CD are similar.

Example 2: In which the angles are equal



The angles $\angle EPF$ and $\angle GQH$ are equal (both $= \beta$). Therefore, the segments EF and GH are similar.

IN SUMMARY

Two segments of circles are similar if either

- they accept equal angles (at the centers), or
- the angles in the segments are equal to one another.

In both cases, the segments are called **SIMILAR SEGMENTS OF CIRCLES**.

EUCLID’S INSIGHT

Euclid compares parts of circles by the angles they stand upon. Equality of angles ensures similarity of segments.



EUCLID’S LANGUAGE

Angles at the center or within a circle govern the size of the arcs they cut off. Equal angles cut off similar parts, which we call similar segments.



Proposition 1

Statement. To find the center of a given circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be the given circle.
2. So it is required to find the center of circle ABC.
3. Let some straight-line AB have been drawn through (ABC), at random, and let (AB) have been cut in half at point D [Prop. 1.9].
4. And let DC have been drawn from D, at right-angles to AB [Prop. 1.11].
5. And let (CD) have been drawn through to E.
6. And let CE have been cut in half at F [Prop. 1.9].
7. I say that (point) F is the center of the [circle] ABC.
8. For (if) not then, if possible, let G (be the center of the circle), and let GA, GD, and GB have been joined.
9. And since AD is equal to DB, and DG (is) common, the two
10. (straight-lines) AD, DG are equal to the two (straightlines) BD, DG, respectively.
11. And the base GA is equal to the base GB.
12. For (they are both) radii.
13. Thus, angle ADG is equal to angle GDB [Prop. 1.8].
14. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10].
15. Thus, GDB is a right-angle.
16. And FDB is also a right-angle.

17. Thus, FDB (is) equal to GDB, the greater to the lesser. The very thing is impossible.
18. Thus, (point) G is not the center of the circle ABC.
19. So, similarly, we can show that neither is any other (point) except F.
20. Thus, point F is the center of the [circle] ABC.

Conclusion. So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line). – (Which is) the very thing it was required to do.

ARISTOTLE:
Let us learn by reasoning about the figures.

EUCLID'S ELEMENTS • BOOK III

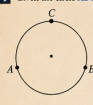
PROPOSITION I

To find the center of a given circle.

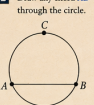
ALEXANDER (13):
What is the center of this circle?

PANEL 1 — CONSTRUCTION AND SETUP (Actions 1–10)

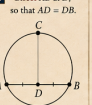
1 Given the circle ABC .



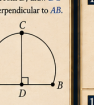
2 Draw any chord AB through the circle.



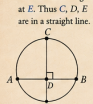
3 Bisect AB at D , so that $AD = DB$.



4 From D , draw DC perpendicular to AB .



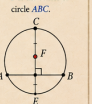
5 Produce DC through the circle to meet it again at E . Thus C, D, E are in a straight line.



6 Bisect CE at F , so that $CF = FE$.



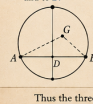
7 We claim that F is the center of circle ABC .



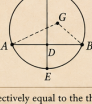
8 Suppose, for the sake of contradiction, that another point G is the center.



9 Join G to A , to D , and to B .



Since D bisects AB , $AD = DB$, and GD is common to triangles ADG and $B DG$.
Also, $GA = GB$ (radii, since G is assumed the center).



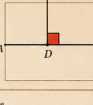
Thus the three sides of triangle ADG are respectively equal to the three sides of triangle $B DG$. Therefore, $\angle ADG = \angle GDB$ (Proposition I.8).

PANEL 2 — PROOF BY CONTRADICTION (Actions 11–20) AND COROLLARY (Action 21)

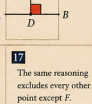
11 Since $\angle ADG = \angle GDB$ and these are adjacent angles on the straight line AB , each is a right angle.



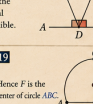
12 Hence $\angle GDB$ is a right angle.



13 But DF was constructed perpendicular to AB , so $\angle FDB$ is also a right angle.



15 But if F and G were distinct points, one of these angles would contain the other; the greater would equal the lesser—impossible.



16 Therefore G cannot be the center.

17 The same reasoning excludes every other point except F .

18 Thus F alone is the center.

19 Hence F is the center of circle ABC .



20 This completes the proof.



COROLLARY

If a straight-line in a circle bisects another straight-line and cuts it at right angles, then the center of the circle lies on that perpendicular bisector.



Geometry is discovered by reason, and its demonstrations leave no room for doubt.

25

Proposition 2

Statement. If two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be a circle, and let two points A and B have been taken at random on its circumference.
2. I say that the straight-line joining A to B will fall inside the circle.
3. For (if) not then, if possible, let it fall outside (the circle), like AEB (in the figure).
4. And let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) D .
5. And let DA and DB have been joined, and let DFE have been drawn through.
6. Therefore, since DA is equal to DB , the angle DAE
7. (is) thus also equal to DBE [Prop. 1.5].
8. And since in triangle DAE the one side, AEB , has been produced, angle DEB (is) thus greater than DAE [Prop. 1.16].
9. And DAE (is) equal to DBE [Prop. 1.5].
10. Thus, DEB (is) greater than DBE .
11. And the greater angle is subtended by the greater side [Prop. 1.19].
12. Thus, DB (is) greater than DE .
13. And DB (is) equal to DF .
14. Thus, DF (is) greater than DE , the lesser than the greater. The very thing is impossible.
15. Thus, the straight-line joining A to B will not fall outside the circle.

16. So, similarly, we can show that neither (will it fall) on the circumference itself.

17. Thus, (it will fall) inside (the circle).

Conclusion. Thus, if two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

ARISTOTLE:
Let us learn by reasoning about the figures.

EUCLID'S ELEMENTS • BOOK III
PROPOSITION 2
If two points are taken at random on the circumference of a circle, then the straight-line joining the points will fall inside the circle.

ALEXANDER (13):
What is the straight-line joining two points on a circle?

CONSTRUCTION AND ASSUMPTION (Actions 1–9)

1 Start with circle ABC .



2 Choose two arbitrary points A and B on the circumference.



3 Join A to B by a straight line.



4 Claim: the straight line AB lies inside the circle.



5 Assume the opposite: suppose the joining line falls outside the circle, as $A'D$.



6 Find the center D of circle ABC . (Prop. 3.1).



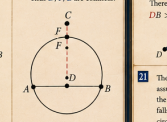
7 Join D to A .



8 Join D to B .



9 Draw the straight line through D and E , meeting the circle again at F . Thus D, F, E are collinear.



PROOF BY CONTRADICTION (Actions 10–25)

10 DA and DB are radii. Therefore $DA = DB$.



11 Hence triangle DAB is isosceles, so $\angle DAB = \angle DBE$.



12 Consider triangle DAE .



13 Produce AE through E to G .



14 By Prop. 1.16, exterior angle $\angle DEB$ is greater than interior opposite angle $\angle DAE$.



15 But $\angle DAE = \angle DBE$. Therefore $\angle DEB > \angle DBE$.



16 In triangle DBE , the greater angle stands opposite the greater side. Therefore $DB > DE$.



17 But DB and DA are radii of the same circle. Therefore $DB = DA$.



18 Therefore $DA > DE$.



19 Yet F lies between D and E . Hence DF is actually less than DE .



20 Thus we obtain the impossible conclusion $DF > DE$ while $DF < DE$.



21 Therefore our assumption that the joining line falls outside the circle is impossible.



22 Similarly, the joining straight line cannot lie on the circumference itself.



23 The only remaining possibility is that the straight line joining A and B lies inside the circle.



24 Hence the straight line joining two points on the circumference lies inside the circle.



25 Therefore, the straight line joining A and B lies inside the circle. **Q.E.D.**



WHAT WE HAVE SHOWN
No matter where two points A and B are taken on a circle, the straight-line joining them must lie inside the circle.

EUCLID'S INSIGHT
The center and the properties of angles lead to a contradiction whenever the joining line is assumed to be outside or on the circle—so it must be inside.

28

Proposition 3

Statement. In a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be a circle, and, within it, let some straightline through the center, CD, cut in half some straight-line not through the center, AB, at the point F.
2. I say that (CD) also cuts (AB) at right-angles.
3. For let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) E, and let EA and EB have been joined.
4. And since AF is equal to FB, and FE (is) common, two (sides of triangle AFE) [are] equal to two (sides of triangle BFE).
5. And the base EA (is) equal to the base EB.
6. Thus, angle AFE is equal to angle BFE [Prop. 1.8].
7. And when a straight-line stood upon (another) straightline makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10].
8. Thus, AFE and BFE are each right-angles.
9. Thus, the
10. (straight-line) CD, which is through the center and cuts in half the (straight-line) AB, which is not through the center, also cuts (AB) at right-angles.
11. And so let CD cut AB at right-angles.
12. I say that it also cuts (AB) in half.
13. That is to say, that AF is equal to FB.

14. For, with the same construction, since EA is equal to EB, angle EAF is also equal to EBF [Prop. 1.5].
15. And the right-angle AFE is also equal to the right-angle BFE.
16. Thus, EAF and EFB are two triangles having two angles equal to two angles, and one side equal to one side—(namely), their common (side) EF, subtending one of the equal angles.
17. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26].
18. Thus, AF (is) equal to FB.

Conclusion. Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half. (Which is) the very thing it was required to show.

ARISTOTLE:
Let us learn by reasoning about the figures.

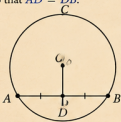
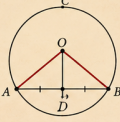
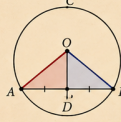
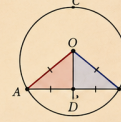
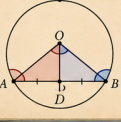
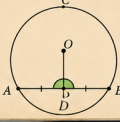
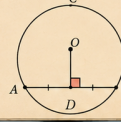
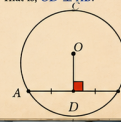
EUCLID'S ELEMENTS • BOOK III

PROPOSITION 3 — PROOF I

I. If any straight-line through the center of a circle cuts in half any straight-line not through the center then it also cuts it at right-angles.

ALEXANDER (13):
Why must it cut at right-angles as well?

Given circle ABC with center O . Let straight-line AB not pass through the center. Let straight-line through O meet AB at D and cut it in half, so that $AD = DB$. I say that the straight-line through O cuts AB at right-angles; that is, OD is perpendicular to AB .

1 Let O be the center of circle ABC. Let AB be a straight-line not through the center. Let the straight-line through O meet AB at D and cut it in half, so that $AD = DB$. 	2 Join OA and OB. Since OA and OB are radii of the same circle, $OA = OB$. 	3 In triangles OAD and OBD, we have $OA = OB$ (proved) $AD = DB$ (given) OD is common. 	4 Therefore, the three sides of triangle OAD are respectively equal to the three sides of triangle OBD. (By Proposition 1.8) 
5 Therefore, the angles of triangle OAD are respectively equal to the angles of triangle OBD. (By Proposition 1.8) 	6 In particular, $\angle ODA = \angle ODB$. These two are adjacent angles on the straight-line AB. 	7 But if two adjacent angles on a straight-line are equal, each of them is a right angle. (By Proposition 1.13) 	8 Therefore $\angle ODA$ is a right angle. Hence the straight-line through the center O cuts AB at right-angles. That is, $OD \perp AB$. 

CONCLUSION (PROOF I)

Hence, if any straight-line through the center of a circle cuts in half any straight-line not through the center, then it also cuts it at right-angles.

WHAT WE HAVE SHOWN

A line through the center that bisects a chord must be perpendicular to that chord.

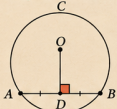
ARISTOTLE:
Let us learn by reasoning about the figures.

EUCLID'S ELEMENTS • BOOK III
PROPOSITION 3 – PROOF I
II. If any straight-line through the center of a circle cuts any straight-line not through the center at right-angles then it also cuts it in half.

ALEXANDER (13):
So if it cuts a chord at right-angles, it must cut it in half?

Given circle ABC with center O . Let straight-line AB not pass through the center. Let straight-line through O meet AB at D and cut AB at right-angles; that is, $OD \perp AB$. I say that the straight-line through O cuts AB in half; that is, $AD = DB$.

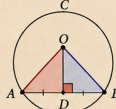
1 Let O be the center of circle ABC . Let AB be a straight-line not through the center. Let the straight-line through O meet AB at D and cut AB at right-angles; that is, $OD \perp AB$.



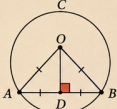
2 Join OA and OB . Since OA and OB are radii of the same circle, $OA = OB$.



3 In triangles OAD and OBD , we have
 $OA = OB$ (proved)
 OD is common
 $\angle ODA = \angle ODB = 90^\circ$ (given)



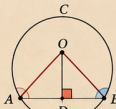
4 Therefore, the three sides of triangle OAD are respectively equal to the three sides of triangle OBD . (By Proposition 1.8)



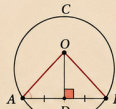
5 Therefore, the angles of triangle OAD are respectively equal to the angles of triangle OBD . (By Proposition 1.8)



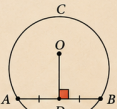
6 In particular, $\angle OAD = \angle OBD$. These are the angles at A and B of triangle OAB .



7 But in triangle OAB , the sides opposite equal angles are equal. (By Proposition 1.5)



8 Therefore, $AD = DB$. Thus the straight-line through O cuts AB in half.



CONCLUSION (PROOF II)
Hence, if any straight-line through the center of a circle cuts any straight-line not through the center at right-angles, then it also cuts it in half.

WHAT WE HAVE SHOWN
A perpendicular from the center to a chord bisects that chord.

Proposition 4

Statement. In a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABCD be a circle, and within it, let two straightlines, AC and BD, which are not through the center, cut one another at (point) E.
2. I say that they do not cut one another in half.
3. For, if possible, let them cut one another in half, such that AE is equal to EC, and BE to ED.
4. And let the center of the circle ABCD have been found [Prop. 3.1], and let it be (at point) F, and let FE have been joined.
5. Therefore, since some straight-line through the center, FE, cuts in half some straight-line not through the center, AC, it also cuts it at right-angles [Prop. 3.3].
6. Thus, FEA is a right-angle.
7. Again, since some straight-line FE
8. cuts in half some straight-line BD, it also cuts it at rightangles [Prop. 3.3].
9. Thus, FEB (is) a right-angle.
10. But FEA was also shown (to be) a right-angle.
11. Thus, FEA (is) equal to FEB, the lesser to the greater. The very thing is impossible.
12. Thus, AC and BD do not cut one another in half.

Conclusion. Thus, in a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half. (Which is) the very thing it was required to show.

ARISTOTLE:
Let us learn
by reasoning
about the
figures.

EUCLID'S ELEMENTS • BOOK III
PROPOSITION 4
In a circle, if two straight-lines, which are not through the center, cut one another, then they do not cut one another in half.

ALEXANDER (13):
Why can't two
lines inside
a circle cut
each other
in half?

1 Let $ABCD$ be a circle.

2 Let the straight-lines AC and BD meet one another at E .

3 Suppose neither AC nor BD passes through the center of the circle.

4 Assume, for the sake of contradiction, that they bisect one another at E .
That is,
 $AE = EC$
and
 $BE = ED$.

5 Thus E is the midpoint of AC and also of BD .

6 Let F be the center of the circle.

7 Join F to E .

8 Since FE passes through the center F and bisects AC , by Proposition 3, FE cuts AC at right angles.

9 Therefore $\angle FEA$ is a right angle.

10 But FE also passes through the center F and (by our assumption) bisects BD . Therefore, by Proposition 3 again, FE cuts BD at right angles.

11 Therefore $\angle FEB$ is a right angle.

12 Hence $\angle FEA = \angle FEB$. But one of these angles contains the other, since the lines AC and BD are distinct and intersect.
Thus the lesser angle would equal the greater, which is impossible.

13 Therefore our assumption is false. AC and BD cannot bisect one another.
Hence, if two straight-lines inside a circle do not pass through the center and intersect one another, they do not cut one another in half.

WHAT WE HAVE SHOWN
Inside a circle, two lines that do not pass through the center can meet, but they can never cut each other in half.

THE INSIGHT
Only a line through the center has the special power to bisect another line at right angles.

Proposition 5

Statement. If two circles cut one another then they will not have the same center.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let the two circles ABC and CDG cut one another at points B and C.
2. I say that they will not have the same center.
3. For, if possible, let E be (the common center), and let EC have been joined, and let EFG have been drawn through (the two circles), at random.
4. And since point E is the center of the circle ABC, EC is equal to EF.
5. Again, since point E is the center of the circle CDG, EC is equal to EG.
6. But EC was also shown (to be) equal to EF.
7. Thus, EF is also equal to EG, the lesser to the greater. The very thing is impossible.
8. Thus, point E is not
9. the (common) center of the circles ABC and CDG.

Conclusion. Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show.

ARISTOTLE:
Let us learn
by reasoning
about the
figures.

EUCLID'S ELEMENTS • BOOK III
PROPOSITION 5
*If two circles cut one another,
then they will not have the same center.*

ALEXANDER (13):
Why can't two
circles that cut
one another
have the same
center?

1 Let the two circles ABC and CDG cut one another at points B and C .

2 Claim: the two circles do not have the same center.

3 Assume for contradiction that they do have the same center.

4 Let that supposed common center be E .

5 Join E to the common point C .

6 Draw a straight-line through E meeting the first circle at F and the second circle at G .

7 Since E is the center of circle ABC , $EC = EF$ (radii of that circle).

8 Since E is also the center of circle CDG , $EC = EG$ (radii of that circle).

9 Therefore, because both EF and EG are equal to EC , we have $EF = EG$.

10 But F lies between E and G . Therefore, $EF < EG$.

11 Thus we have the impossible conclusion that $EF = EG$ and at the same time $EF < EG$.

That is impossible.

12 Therefore the assumption that E is the common center is false.

13 Hence the two intersecting circles cannot have the same center.

CONCLUSION
If two circles cut one another,
then they will not have
the same center.

Proposition 6

Statement. If two circles touch one another then they will not have the same center.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let the two circles ABC and CDE touch one another at point C.
2. I say that they will not have the same center.
3. For, if possible, let F be (the common center), and let FC have been joined, and let FEB have been drawn through (the two circles), at random.
4. Therefore, since point F is the center of the circle ABC, FC is equal to FB.
5. Again, since point F is the center of the circle CDE, FC is equal to FE.
6. But FC was shown (to be) equal to FB.
7. Thus, FE is also equal to FB, the lesser to the greater. The very thing is impossible.
8. Thus, point F is not the (common) center of the circles ABC and CDE.

Conclusion. Thus, if two circles touch one another then they will not have the same center. (Which is) the very thing it was required to show.

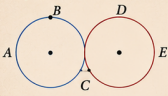
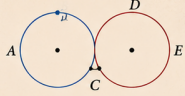
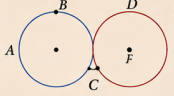
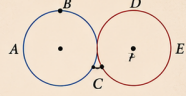
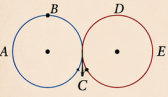
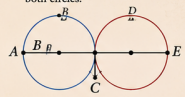
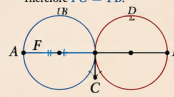
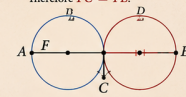
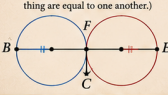
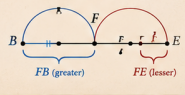
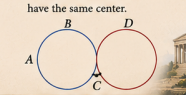
ARISTOTLE:
Let us learn by reasoning about the figures.

EUCLID'S ELEMENTS • BOOK III

PROPOSITION 6

If two circles touch one another, then they will not have the same center.

ALEXANDER (13):
Why can't two circles that touch have the same center?

1 Let the two circles ABC and CDE touch one another at the single point C. 	2 Claim: the two touching circles cannot have the same center. 	3 Assume the contrary: suppose they do have a common center. 	4 Let the supposed common center be F. 
5 Join F to the touching point C. 	6 Through F, draw an arbitrary straight-line EFB across both circles. 	7 Since F is the center of circle ABC, FC and FB are radii of that circle. Therefore $FC = FB$. 	8 Since F is also the center of circle CDE, FC and FE are radii of that circle. Therefore $FC = FE$. 
9 Since $FC = FB$ and $FC = FE$, it follows that $FE = FB$. (Things equal to the same thing are equal to one another.) 	10 But on the straight-line EFB, E lies between F and B. Therefore $FE < FB$. 	11 Thus we have the impossible conclusion that $FE = FB$ and at the same time $FE < FB$. The lesser would equal the greater. This is impossible. 	12 Therefore F cannot be the common center. Hence the two circles which touch one another cannot have the same center. 



CONCLUSION

If two circles touch one another, then they will not have the same center.

WHAT WE HAVE SHOWN

Only coincident circles can have the same center. When circles merely touch or cut, their centers must be different.



Proposition 7

Statement. If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straightlines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straightline) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABCD be a circle, and let AD be its diameter, and let some point F, which is not the center of the circle, have been taken on AD.
2. Let E be the center of the circle.
3. And let some straight-lines, FB, FC, and FG, radiate from F towards (the circumference of) circle ABCD.
4. I say that FA is the greatest (straight-line), FD the least, and of the others, FB (is) greater than FC, and FC than FG.
5. For let BE, CE, and GE have been joined.
6. And since for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20], EB and EF is thus greater than BF.
7. And AE (is) equal to BE [thus, BE and EF is equal to AF].
8. Thus, AF (is) greater than BF.
9. Again, since BE is equal to CE, and FE (is) common, the two (straight-lines) BE, EF are equal to the two (straightlines) CE, EF (respectively).
10. But, angle BEF (is) also greater than angle CEF.
11. Thus, the base BF is greater than the base CF.
12. Thus, the base BF is greater than the base CF [Prop. 1.24].

13. So, for the same (reasons), CF is also greater than FG.
14. Again, since GF
15. and FE are greater than EG [Prop. 1.20], and EG (is) equal to ED, GF and FE are thus greater than ED.
16. Let EF have been taken from
17. Thus, the remainder GF is greater than the rebth. mainder FD.
18. Thus, FA (is) the greatest (straight-line), FD the least, and FB (is) greater than FC, and FC than FG.
19. I also say that from point F only two equal (straightlines) will radiate towards (the circumference of) circle ABCD, (one) on each (side) of the least (straight-line) FD.
20. For let the (angle) FEH, equal to angle GEF, have been constructed on the straight-line EF, at the point E on it [Prop. 1.23], and let FH have been joined.
21. Therefore, since GE is equal to EH, and EF (is) common,
22. the two (straight-lines) GE, EF are equal to the two (straight-lines) HE, EF (respectively).
23. And angle GEF (is) equal to angle HEF.
24. Thus, the base FG is equal to the base FH [Prop. 1.4].
25. So I say that another (straightline) equal to FG will not radiate towards (the circumference of) the circle from point F.
26. For, if possible, let FK (so) radiate.
27. And since FK is equal to FG, but FH [is equal] to FG, FK is thus also equal to FH, the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible.
28. Thus, another (straight-line) equal to GF will not radiate from the point F towards (the circumference of) the circle.
29. Thus, (there is) only one (such straight-line).

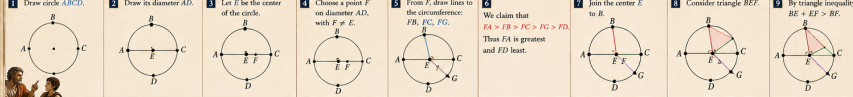
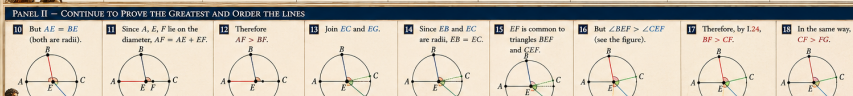
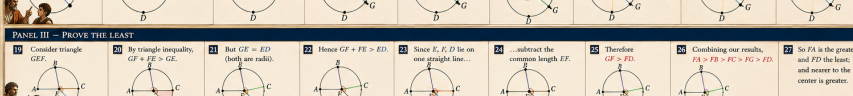
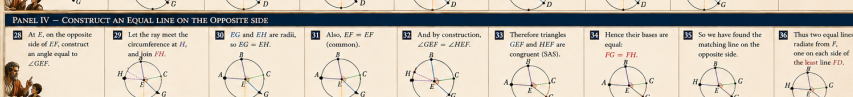
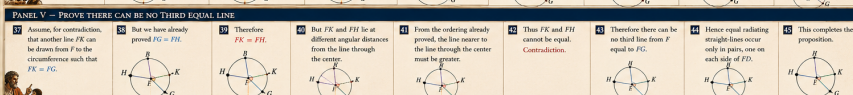
30. Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter).
31. And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away.

Conclusion. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the circumference of the circle, then the greatest straight-line will be that on which the center lies, and the least the remainder of the same diameter. And, of the others, the straight-line nearer to the straight line through the center is always greater than one further away. And only two equal straight-lines will radiate from the point towards the circumference of the circle, one on each side of the least.

PANEL I — SET UP AND PROVE THE GREATEST							
1 Draw circle $ABCD$.	2 Draw its diameter AD .	3 Let E be the center of the circle.	4 Choose a point F on diameter AD , with $F \neq E$.	5 From F , draw lines to the circumference: FB , FC , FG .	6 We claim that $FA > FB > FC > FG > FD$. Thus FA is greatest and FD least.	7 Join the center E to B .	8 Consider triangle BEF .
							
							
							
PANEL II — CONTINUE TO PROVE THE GREATEST AND ORDER THE LINES							
9 But $AE = BE$ (both are radii).	10 Since A, E, F lie on the diameter, $AF = AE + EF$.	11 Therefore $AF > BF$.	12 Join EC and EG .	13 Since EB and EC are radii, $EB = EC$.	14 EF is common to triangles BEF and CEF .	15 But $\angle BEF > \angle CEF$ (see the figure).	16 Therefore, by L24, $BF > CF$.
							
							
PANEL III — PROVE THE LEAST							
17 Consider triangle GEF .	18 By triangle inequality, $GF + FE > GE$.	19 But $GE = ED$ (both are radii).	20 Hence $GF + FE > ED$.	21 Since E, F, D lie on one straight line...	22 subtract the common length EF .	23 Therefore $GF > FD$.	24 Combining our results, $FA > FB > FC > FG > FD$.
							
							
PANEL IV — CONSTRUCT AN EQUAL LINE ON THE OPPOSITE SIDE							
25 At E , on the opposite side of EF , construct an angle equal to $\angle GEF$.	26 Let the ray meet the circumference at H , and join HD .	27 EG and EH are radii, so $EG = EH$.	28 Also, $EF = EF$ (common).	29 And by construction, $\angle GEF = \angle HEF$.	30 Therefore triangles GEF and HEF are congruent (SAS).	31 Hence their bases are equal: $FG = FH$.	32 So we have found the matching line on the opposite side.
							
							
PANEL V — PROVE THERE CAN BE NO THIRD EQUAL LINE							
33 Assume, for contradiction, that another line FK can be drawn from F to the circumference such that $FK = FG$.	34 But we have already proved $FG > FD$.	35 Therefore $FK > FD$.	36 But FK and FH lie at different angular distances from the line through the center: FK is greater.	37 From the ordering already proved, the line nearer to the line through the center must be greater.	38 Thus FK and FH cannot be equal. <i>Contradiction.</i>	39 Therefore there can be no third line from F equal to FG .	40 Hence equal radiating straight-lines occur only in pairs, one on each side of FD .
							
							
							

Proposition 8

Statement. If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For the others, a (straight-line) nearer to the center. (straight-line) through the center is always greater than one further away.
2. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter.
3. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away.
4. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).
5. Let ABC be a circle, and let some point D have been taken outside ABC, and from it let some straight-lines, DA, DE, DF, and DC, have been drawn through (the circle), and let DA be through the center.
6. I say that for the straight-lines radiating towards the concave (part of
7. the) circumference, AEFC, the greatest is the one (passing) through the center, (namely) AD, and (that) DE (is) greater than DF, and DF than DC.
8. For the straight-lines radiating towards the convex (part of the) circumference, HLKG, the least is the one between the point and the diameter AG, (namely) DG, and a (straight-line) nearer to the least (straight-line) DG is always less than one farther away, (so that) DK (is less) than DL, and DL than than DH.
9. L K G

10. For let the center of the circle have been found [Prop. 3.1], and let it be (at point) M [Prop. 3.1].
11. And let ME, MF, MC, MK, ML, and MH have been joined.
12. And since AM is equal to EM, let MD have been added to both.
13. Thus, AD is equal to EM and MD.
14. But, EM and MD is greater than ED [Prop. 1.20].
15. Thus, AD is also greater than ED.
16. Again, since ME is equal to MF, and MD (is) common, the (straight-lines) EM, MD are thus equal to FM, MD.
17. And angle EMD is greater than angle FMD.
18. Thus, the base ED is greater than the base FD [Prop. 1.24].
19. So, similarly, we can show that FD is also greater than CD.
20. Thus, AD (is) the greatest (straight-line), and DE (is) greater than DF, and DF than DC.
21. And since MK and KD is greater than MD [Prop. 1.20], and MG (is) equal to MK, the remainder KD is thus greater than the remainder GD.
22. So GD is less than KD.
23. And since in triangle MLD, the two internal straight-lines MK and KD were constructed on one of the sides, MD, then MK and KD are thus less than ML and LD [Prop. 1.21].
24. And MK (is) equal to ML.
25. Thus, the remainder DK is less than the remainder DL.
26. So, similarly, we can show that DL is also less than DH.
27. Thus, DG (is) the least (straight-line), and DK (is) less than DL, and DL than DH.
28. I also say that only two equal (straight-lines) will radi-

29. ate from point D towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), DG.
30. Let the angle DMB, equal to angle KMD, have been constructed on the straight-line MD, at the point M on it [Prop. 1.23], and let DB have been joined.
31. And since MK is equal to MB, and MD (is) common, the two (straight-lines) KM, MD are equal to the two (straightlines) BM, MD, respectively.
32. And angle KMD (is) equal to angle BMD.
33. Thus, the base DK is equal to the base DB [Prop. 1.4]. [So] I say that another (straightline) equal to DK will not radiate towards the (circumference of the) circle from point D.
34. For, if possible, let (such a straight-line) radiate, and let it be DN.
35. Therefore, since DK is equal to DN, but DK is equal to DB, then DB is thus also equal to DN, (so that) a (straightline) nearer to the least (straight-line) DG [is] equal to one further away. The very thing was shown (to be) impossible.
36. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle ABC from point D, (one) on each side of the least (straightline) DG.
37. Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straightlines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center.
38. For the others, a (straight-line) nearer to the (straightline) through the center is always greater than one further away.
39. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter.

Conclusion. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 8

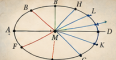
If some point is taken outside a circle, and some straight-lines are drawn from the point to the circumference of the circle, one of which passes through the center, the remainder being random, then:

CONGRUENT LINES (between the part that "bends in").
 Greatest is the line through the center.
 Nearest to it is greater than farther.
 Conversely, using (between the part that "bends out").
 Least is the line to the diameter.
 Nearest to the least is less than farther.
 Only two equal lines can be drawn,
 one on each side of the least.

PANEL I SET-UP AND THE GREATEST LINE (Actions 1–9)	1 Draw circle ABC . 	2 Take a point D outside the circle. 	3 Draw AD through the circle so that it passes through the center. 	4 Draw further rays to the convex circumference: DE, DF, DG . 	5 Draw rays to the concave circumference: DC, DL, DM . 	6 Let M be the center of the circle. 	7 Join ME, MF, MC, MK, ML, MH . 	8 Since $AM = EM$ (radii), $AD = EM + MD$. 	9 In triangle MDH , $EM + MD > ED$. Therefore $AD > ED$. 
PANEL II ORDER THE RAYS TO THE CONVEX CIRCUMFERENCE (Actions 10–18)	10 Since ME and MF are radii, $ME = MF$. 	11 MD is common to triangles EMD and FMD . 	12 Thus two sides are equal in the two triangles. 	13 From the figures, $\angle EMD > \angle FMD$. 	14 By 1.14, the side opposite the greater angle is greater. 	15 Therefore $ED > FD$. 	16 Similarly, using triangles FMD and GMD . 	17 This gives $FD > GD$. 	18 Hence $AD > DE > DF > DG$ on the convex side. 
PANEL III ORDER THE RAYS TO THE CONCAVE CIRCUMFERENCE (Actions 19–27)	19 Consider triangle MDK . 	20 By triangle inequality, $ME + MD > MK$. 	21 Since $ME = MC$ (radii), subtract. 	22 Since D, M, G lie on a diameter, $MD = MG + GD$. 	23 Subtract the common radius MG . 	24 Thus $KD > GD$. 	25 By 1.21 in triangle MDL , $ME + MD < ML + LD$. 	26 But $ME = ML$. Hence $MD < LD$. 	27 Similarly, $DL < DH$. Therefore $DG < DK < DL < DH$ on the concave side. 
PANEL IV CONJECTURE OF EQUAL RAY ON THE OPPOSITE SIDE (Actions 28–36)	28 At M , on the opposite side of MD , construct $\angle EMD = \angle BMD$. 	29 Let the ray meet the circumference at B . 	30 Join DB . 	31 Since ME and MB are radii, $ME = MB$. 	32 MD is common to triangles EMD and BMD . 	33 And by construction, $\angle EMD = \angle BMD$. 	34 Therefore triangles EMD and BMD are congruent (SAS). 	35 Hence their bases are equal: $DM = DB$. 	36 Thus we have two equal rays from D , one on each side of the least DG . 
PANEL V THERE CAN BE NO THIRD EQUAL RAY (Actions 37–45)	37 Assume there is a third ray DN such that $DN = DB$. 	38 But we already have $DB = DM$. 	39 Therefore $DM = DB$. 	40 DM and DB lie at different angular distances from the least DG . 	41 From the ordering on the convex side, a ray nearer DG is shorter than one farther. 	42 Hence DM and DB cannot be equal. 	43 Contradiction. 	44 Therefore no third equal ray can be drawn from D . 	45 Only two equal rays can be drawn from D , one on each side of the least DG . 

WHAT WE HAVE SHOWN

On the convex side:
 $AD > DE > DF > DG$ (greatest is through the center).
 On the concave side:
 $DG < DK < DL < DH$ (least is the line to the diameter).



Between the greatest (AD) and the least (DG), rays are ordered as they approach those lines.

Only two equal rays occur, one on each side of the least line DG .



Proposition 9

Statement. If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be a circle, and D a point inside it, and let more than two equal straight-lines, DA, DB, and DC, radiate from D towards (the circumference of) circle ABC.
2. I say that point D is the center of circle ABC.
3. For let AB and BC have been joined, and (then) have been cut in half at points E and F (respectively) [Prop. 1.10].
4. And ED and FD being joined, let them have been drawn through to points G, K, H, and L.
5. Therefore, since AE is equal to EB, and ED (is) common, the two (straight-lines) AE, ED are equal to the two (straight-lines) BE, ED (respectively).
6. And the base DA (is) equal to the base DB.
7. Thus, angle AED is equal to angle BED [Prop. 1.8].
8. Thus, angles AED and BED (are) each right-angles [Def. 1.10].
9. Thus, GK cuts AB in half, and at right-angles.
10. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on GK.
11. So, for the same (reasons), the center of circle ABC is also on HL.
12. And the straight-lines GK and HL have no common (point) other than point D.

13. Thus, point D is the center of circle ABC.

Conclusion. Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the circumference of the circle, then the point taken is the center of the circle.

ARISTOTLE:

We shall prove that the point is the center.



PANEL I — CONSTRUCTION AND PERPENDICULAR BISECTOR OF AB

1 Draw circle ABC.



2 Take a point D inside the circle.



3 From D, draw more than two equal straight-lines to the circumference: $DA = DB = DC$.



Claim: D is the center of circle ABC.

4 Join A to B.



5 Join B to C.



6 Bisect AB at E.



7 Bisect BC at F.



8 Join E to D, and produce ED in both directions through the circle.



9 Join F to D, and produce FD in both directions through the circle.



PANEL II — PROOF THAT D IS THE CENTER

10 Since $AE = EB$, ED is common, and $DA = DB$, triangles AED and BED have their three sides equal.



Therefore triangles AED and BED are congruent (SSS).

11 Therefore $\angle AED = \angle BED$.



These two angles are equal.

12 These two equal angles are adjacent on straight-line AB. Therefore each is a right angle.



$\angle AED$ and $\angle BED$ are right angles.

13 Hence the straight-line through E and D both bisects AB and cuts it at right angles.



ED is the perpendicular bisector of AB.

14 By the corollary to Proposition 1, the center of the circle lies on this perpendicular bisector.



The center lies on line ED.

15 By the same reasoning, FD is the perpendicular bisector of BC.



FD is the perpendicular bisector of BC.

16 Therefore the center also lies on FD.



The center lies on line FD also.

17 The two perpendicular bisectors ED and FD meet at only one point, namely D.



ED and FD meet at only one point.

18 Therefore the center of the circle must be D. $Q.E.D.$



D is the center of circle ABC.

WHAT WE HAVE USED

- Triangles with three sides equal are congruent (I.4).
- Equal adjacent angles on a straight line are right angles (I.13).
- Perpendicular bisector of a chord passes through the center (Corollary to I.10).

Conclusion

If more than two equal straight-lines radiate from an interior point to the circle, that point is the center.

Proposition 10

Statement. A circle does not cut a(nother) circle at more than two points.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For, if possible, let the circle ABC cut the circle DEF at more than two points, B, G, F, and H.
2. And BH and BG being joined, let them (then) have been cut in half at points K and L (respectively).
3. And KC and LM being drawn at right-angles to BH and BG from K and L (respectively) [Prop. 1.11], let them (then) have been drawn through to points A and E (respectively).
4. Therefore, since in circle ABC some straight-line AC cuts some (other) straight-line BH in half, and at right-angles, the center of circle ABC is thus on AC [Prop. 3.1 corr.].
5. Again, since in the same circle ABC some straight-line NO cuts some (other straight-line) BG in half, and at right-angles, the center of circle ABC is thus on NO [Prop. 3.1 corr.].
6. And it was also shown (to be) on AC.
7. And the straight-lines AC and NO meet at no other (point) than P.
8. Thus, point P is the center of circle ABC.
9. So, similarly, we can show that P is also the center of circle DEF.
10. Thus, two circles cutting one another, ABC and DEF, have the same center P. The very thing is impossible [Prop. 3.5].

Conclusion. Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 10

A circle does not cut another circle at more than two points.

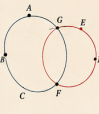



PANEL I — ASSUME MORE THAN TWO COMMON POINTS AND CONSTRUCT

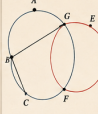
1 Assume, for contradiction, that circle ABC cuts circle DEF at more than two points.



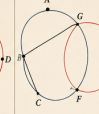
2 Let four common points be B, G, F, H .



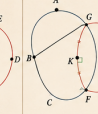
3 Join B to H .



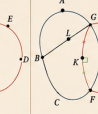
4 Join B to G .



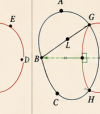
5 Bisect BH at K .



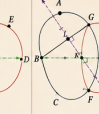
6 Bisect BG at L .



7 From K , draw a line perpendicular to BH , and produce it.



8 From L , draw a line perpendicular to BG , and produce it.



PANEL II — USE PERPENDICULAR BISECTORS AND REACH A CONTRADICTION

9 In circle ABC , the first line both bisects BH and cuts it at right angles.



10 Therefore, by the corollary to Prop. 1, the center of circle ABC lies on this perpendicular bisector.



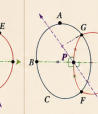
11 The second line similarly bisects BG at right angles.



12 Therefore the center of circle ABC also lies on this second perpendicular bisector.



13 The two perpendicular bisectors intersect at only one point, call it P . Therefore P is the center of circle ABC .



14 But the same chords BH and BG also belong to circle DEF . The same reasoning shows that P is also the center of circle DEF .



15 Thus the two circles, which cut one another, would have the same center P . But by Proposition 5, this is impossible.

Two circles which cut one another cannot have the same center.

16 Therefore the assumption is false: Two circles cannot intersect at more than two points.



WHAT WE HAVE SHOWN

- Through the midpoints of two chords (BH and BG), we constructed perpendicular bisectors.
- The center of a circle lies on the perpendicular bisector of any chord (Corollary to Proposition 1).
- Both circles would be forced to have the same center.

CONCLUSION

A circle does not cut another circle at more than two points.



Proposition 11

Statement. If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let two circles, ABC and ADE, touch one another internally at point A, and let the center F of circle ABC have been found [Prop. 3.1], and (the center) G of (circle) ADE [Prop. 3.1].
2. I say that the straight-line joining G to F, being produced, will fall on A.
3. For (if) not then, if possible, let it fall like FGH (in the figure), and let AF and AG have been joined.
4. Therefore, since AG and GF is greater than FA, that is to say FH [Prop. 1.20], let FG have been taken from
5. Thus, the remainder AG is greater than the rebboth. mainder GH.
6. And AG (is) equal to GD.
7. Thus, GD is also greater than GH, the lesser than the greater. The very thing is impossible.
8. Thus, the straight-line joining F to G will not fall outside (one circle but inside the other).
9. Thus, it will fall upon the point of union (of the circles)
10. at point A.

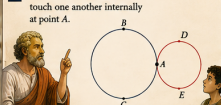
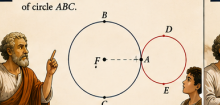
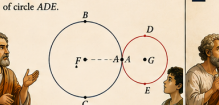
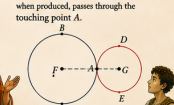
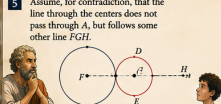
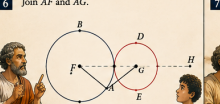
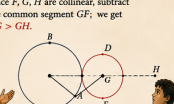
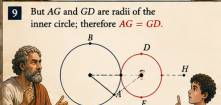
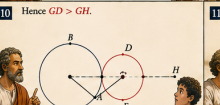
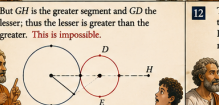
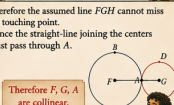
Conclusion. Thus, if two circles touch one another internally, [and their centers are found], then the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 11

If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of contact of the circles.

TO BE SHOWN:
 Let circles ABC and ADE touch internally at A .
 F is the center of the greater circle.
 G the center of the smaller circle.
 Then F, G, A are on one straight line.

1 Let circles ABC and ADE touch one another internally at point A. 	2 Let F be the center of circle ABC. 	3 Let G be the center of circle ADE. 	4 Claim: The straight-line joining F and G, when produced, passes through the touching point A. 
5 Assume, for contradiction, that the line through the centers does not pass through A, but follows some other line FGH. 	6 Join AF and AG. 	7 Consider triangle AFG. By the triangle inequality, $AG + GF > AF$. 	8 Since F, G, H are collinear, subtract the common segment GF; we get $AG > GH$. 
9 But AG and GD are radii of the inner circle; therefore $AG = GD$. 	10 Hence $GD > GH$. 	11 But GH is the greater segment and GD the lesser; thus the lesser is greater than the greater. This is impossible.  A lesser is greater than a greater!	12 Therefore the assumed line FGH cannot miss the touching point. Hence the straight-line joining the centers must pass through A.  Therefore F, G, A are collinear. Q.E.D.

WHAT WE USED

- Triangle inequality.
- Radii of the same circle are equal.
- If two unequal straight lines are compared, the greater is not less than the lesser.

CONCLUSION

When two circles touch internally, the line joining their centers always passes through the point of contact.



Proposition 12

Statement. If two circles touch one another externally then the (straight-line) joining their centers will go through the point of union.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let two circles, ABC and ADE, touch one another externally at point A, and let the center F of ABC have been found [Prop. 3.1], and (the center) G of ADE [Prop. 3.1].
2. I say that the straight-line joining F to G will go through the point of union at A.
3. For (if) not then, if possible, let it go like FCDG (in the figure), and let AF and AG have been joined.
4. Therefore, since point F is the center of circle ABC, FA is equal to FC.
5. Again, since point G is the center of circle ADE, GA is equal to GD.
6. And FA was also shown
7. (to be) equal to FC.
8. Thus, the (straight-lines) FA and AG are equal to the (straight-lines) FC and GD.
9. So the whole of FG is greater than FA and AG.
10. But, (it is) also less [Prop. 1.20]. The very thing is impossible.
11. Thus, the straight-line joining F to G cannot not go through the point of union at A.
12. Thus, (it will go) through it.

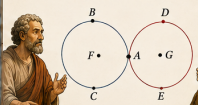
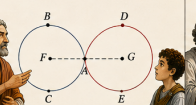
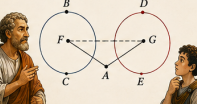
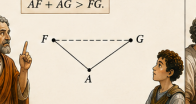
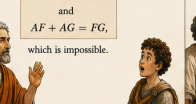
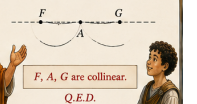
Conclusion. Thus, if two circles touch one another externally then the [straight-line] joining their centers will go through the point of union. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 12

*If two circles touch one another externally,
then the straight-line joining their centers will go through the point of contact.*



1 Let two circles, ABC and ADE, touch one another externally at point A. 	2 Find their centers: let F be the center of the first circle and G the center of the second. 	3 Join the centers F and G with the straight-line FG. 	4 Claim: The straight-line FG passes through the common point of contact A. 
5 Assume the contrary: suppose FG does not pass through A. 	6 Join the touching point A to both centers: AF and AG. 	7 Consider triangle AFG. By the triangle inequality, $AF + AG > FG$. 	8 But AF is a radius of the first circle, so it equals the radius from F along FG to that circle. 
9 Likewise, AG is a radius of the second circle, so it equals the corresponding radius from G along GF to that circle. 	10 Since the circles touch externally, those two radial portions together make the whole distance between the centers. Hence $AF + AG = FG$. 	11 Thus we have both $AF + AG > FG$ and $AF + AG = FG$, which is impossible. 	12 Therefore our assumption was false. The line joining the two centers must pass through A. F, A, G are collinear. <i>Q.E.D.</i> 

Proposition 13

Statement. A circle does not touch a(nother) circle at more than one point, whether they touch internally or externally.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For, if possible, let circle ABDC touch circle EBF $\overline{D}$ — first of all, internally—at more than one point, D and B.
2. And let the center G of circle ABDC have been found [Prop. 3.1], and (the center) H of EBF $\overline{D}$ [Prop. 3.1].
3. Thus, the (straight-line) joining G and H will fall on B and D [Prop. 3.11].
4. Let it fall like BGHD (in the figure).
5. And since point G is the center of circle ABDC, BG is equal to GD.
6. Thus, BG (is) greater than HD.
7. Thus, BH (is) much greater than HD.
8. Again, since point H is the center of circle EBF $\overline{D}$, BH is equal to HD.
9. But it was also shown (to be) much greater than it. The very thing (is) impossible.
10. Thus, a circle does not touch a(nother) circle internally at more than one point.
11. So, I say that neither (does it touch) externally (at more than one point).
12. For, if possible, let circle ACK touch circle ABDC externally at more than one point, A and C.
13. And let AC have been joined.
14. Therefore, since two points, A and C, have been taken at random on the circumference of each of the circles ABDC and ACK, the straight-line joining the points will fall inside each (circle) [Prop. 3.2].

15. But, it fell inside ABDC, and outside ACK [Def. 3.3].
16. The very thing
17. (is) absurd.
18. Thus, a circle does not touch a(nother) circle externally at more than one point.
19. And it was shown that neither (does it) internally.

Conclusion. Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 13

A circle does not touch another circle at more than one point, whether they touch internally or externally.

CASE I – INTERNAL TOUCHING (Two circles cannot touch internally at more than one point)

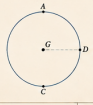
1 Assume, for contradiction, that circle $ABDC$ touches circle $EFDD$ internally at more than one point.



2 Let the two supposed touching points be D and B .



3 Find the center G of circle $ABDC$.



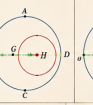
4 Find the center H of circle $EFDD$.



5 By Prop. 11, the line joining centers G and H passes through each point of internal contact, A .



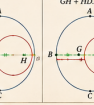
6 Therefore the same line GH must pass through both B and D .



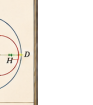
7 Let the order be $B - G - H - D$.



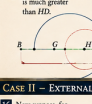
8 Since G is the center of circle $ABDC$, $BG = GD$.



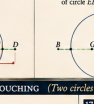
9 Therefore BG is greater than HD , because GD contains $GH = HD$.



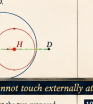
10 Hence $BH = BG + GH$ is much greater than HD .



11 But H is the center of circle $EFDD$.



12 Therefore $BH = HD$, since both are radii of that circle.



13 Thus BH is simultaneously equal to and greater than HD .

$BH = HD$
and
 $BH > HD$



14 This is impossible.

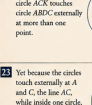


15 Therefore two circles cannot touch internally at more than one point.



CASE II – EXTERNAL TOUCHING (Two circles cannot touch externally at more than one point)

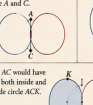
16 Now suppose, for contradiction, that circle ACK touches circle $ABDC$ externally at more than one point.



17 Let the two supposed touching points be A and C .



18 Join AC .



19 Points A and C lie on the circumference of circle $ABDC$.



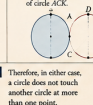
20 Therefore, by Prop. 3, the straight line AC must fall inside circle $ABDC$.



21 But A and C also lie on the circumference of circle ACK .



22 Therefore, again by Prop. 3, AC must also fall inside circle ACK .



23 Yet because the circles touch externally at A and C , the line AC , while inside one circle, falls outside the other.



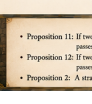
24 Thus AC would have to be both inside and outside circle ACK .



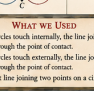
25 This is absurd.



26 Therefore two circles cannot touch externally at more than one point.



27 Therefore, in either case, a circle does not touch another circle at more than one point.



Q.E.D.

A circle does not touch another circle at more than one point, whether internally or externally.



WHAT WE USED

- Proposition 11: If two circles touch internally, the line joining their centers passes through the point of contact.
- Proposition 12: If two circles touch externally, the line joining their centers passes through the point of contact.
- Proposition 2: A straight line joining two points on a circle falls inside the circle.

Circle $ABDC$ (outer in Case I)



Circle ACK



Circle $EFDD$ (inner in Case I)



Circle $ABDC$





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Proposition 14

Statement. In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let $ABDC$ be a circle, and let AB and CD be equal straight-lines within it.
2. I say that AB and CD are equally far from the center.
3. For let the center of circle $ABDC$ have been found [Prop. 3.1], and let it be (at) E .
4. And let EF and EG have been drawn from (point) E , perpendicular to AB and CD (respectively) [Prop. 1.12].
5. And let AE and EC have been joined.
6. Therefore, since some straight-line, EF , through the center (of the circle), cuts some (other) straight-line, AB , not through the center, at right-angles, it also cuts it in half [Prop. 3.3].
7. Thus, AF (is) equal to FB .
8. Thus, AB (is) double AF .
9. So, for the same (reasons), CD is also double CG .
10. And AB is equal to CD .
11. Thus, AF (is) also equal to CG .
12. And since AE is equal to EC , the (square) on AE (is) also equal to the (square) on EC .
13. But, the (sum of the squares) on AF and EF (is) equal to the (square) on AE .

14. For the angle at F (is) a rightangle [Prop. 1.47].
15. And the (sum of the squares) on EG and GC (is) equal to the (square) on EC.
16. For the angle at G (is) a right-angle [Prop. 1.47].
17. Thus, the (sum of the squares) on AF and FE is equal to the (sum of the squares) on CG and GE, of which the (square) on AF is equal to the (square) on CG.
18. For AF is equal to CG.
19. Thus, the remaining (square) on FE is equal to the (remaining square) on EG.
20. Thus, EF (is) equal to EG.
21. And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4].
22. Thus, AB and CD are equally far from the center.
23. So, let the straight-lines AB and CD be equally far from the center.
24. That is to say, let EF be equal to EG.
25. I say that AB is also equal to CD.
26. For, with the same construction, we can, similarly, show that AB is double AF, and CD (double) CG.
27. And since AE is equal to CE, the (square) on AE is equal to the (square) on CE.
28. But, the (sum of the squares) on EF and FA is equal to the (square) on AE [Prop. 1.47].
29. And the (sum of the squares) on EG and GC (is) equal to the (square) on CE [Prop. 1.47].
30. Thus, the (sum of the squares) on EF and FA is equal to the (sum of the squares) on EG and GC, of which the (square) on EF is equal to the (square) on EG.

31. For EF (is) equal to EG.

32. Thus, the remaining (square) on AF is equal to the (remaining square) on CG.

33. Thus, AF (is) equal to CG.

34. And AB is double AF, and CD double CG.

35. Thus, AB (is) equal to CD.

Conclusion. Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 14
*In a circle, equal straight-lines are equally far from the center,
 and straight-lines which are equally far from the center are equal to one another.*

I. EQUAL CHORDS ⇒ EQUAL DISTANCES FROM THE CENTER								
1 Draw circle ABCD.	2 Inside the circle draw two equal chords: $AB = CD$.	3 Let E be the center.	4 From E, draw EF perpendicular to chord AB.	5 From E, draw EG perpendicular to chord CD.	6 Join the center to endpoints A and C.	7 A perpendicular from the center bisects the chord: $AF = FB$.	8 Therefore $AB = 2AF$.	9 Similarly, $CD = 2CG$.
II. EQUAL CHORDS ⇒ EQUAL DISTANCES (CONTINUED)								
10 Given $AB = CD$. $AB = CD$ Since $AB = 2AF$ and $CD = 2CG$, $AF = CG$.	11 EA and EC are radii of the same circle. $EA = EC$	12 Hence $EA^2 = EC^2$. $EA^2 = EC^2$	13 In right triangle EFA, $EA^2 = EF^2 + AF^2$.	14 In right triangle EGC, $EC^2 = EG^2 + CG^2$.	15 Since $EA^2 = EC^2$, we get $EF^2 + AF^2 = EG^2 + CG^2$.	16 But $AF^2 = CG^2$. Subtract equal squares, hence $EF^2 = EG^2$.	17 Therefore $EF = EG$.	18 So AB and CD are equally far from the center E. First direction proved: $AB = CD \Rightarrow EF = EG$
III. EQUAL DISTANCES ⇒ EQUAL CHORDS								
19 Now suppose AB and CD are equally far from the center E. $EF = EG$	20 A perpendicular from the center bisects $AB = 2AF$ and $CD = 2CG$.	21 EA and EC are radii, hence $EA = EC$ $EA^2 = EC^2$.	22 By Pythagoras, $EA^2 = EF^2 + AF^2$, and $EC^2 = EG^2 + CG^2$. $EA^2 = EF^2 + AF^2$ $EC^2 = EG^2 + CG^2$	23 Therefore $EF^2 + AF^2 = EG^2 + CG^2$.	24 But $EF^2 = EG^2$, so $EF^2 = EG^2$. Subtract equal squares, hence $AF^2 = CG^2$.	25 Therefore $AF = CG$.	26 Since $AB = 2AF$ and $CD = 2CG$, $AB = CD$.	27 Thus both directions are established. Therefore in a circle: Equal chords are equally far from the center, and equally distant chords are equal. Q.E.D.

WHAT WE USED

- Prop. 1: To draw a perpendicular from a point to a straight line.
- Prop. 3: To describe a circle.
- Prop. 4: All right angles are equal.
- Prop. 11: The line joining the centers of two circles passing through their point of contact.

KEY

Circle ABCD

Chord AB

Chord CD

Radii

Perpendiculars from E

Proposition 15

Statement. In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABCD be a circle, and let AD be its diameter, and E (its) center.
2. And let BC be nearer to the diameter AD, and FG further away.
3. I say that AD is the greatest (straight-line), and BC (is) greater than FG.
4. For let EH and EK have been drawn from the center E, at right-angles to BC and FG (respectively)
5. And since BC is nearer to the center, [Prop. 1.12]. and FG further away, EK (is) thus greater than EH [Def. 3.5].
6. Let EL be made equal to EH [Prop. 1.3].
7. And LM being drawn through L, at right-angles to EK [Prop. 1.11], let it have been drawn through to N.
8. And let ME, EN, FE, and EG have been joined.
9. And since EH is equal to EL, BC is also equal to MN [Prop. 3.14].
10. Again, since AE is equal to EM, and ED to EN, AD is thus equal to ME and EN.
11. But, ME and EN is greater than MN [Prop. 1.20] [also AD is
12. greater than MN], and MN (is) equal to BC.
13. Thus, AD is greater than BC.
14. And since the two (straight-lines) ME, EN are equal to the two (straight-lines) FE, EG (respectively), and angle MEN [is] greater than angle FEG, the base MN is thus greater than the base FG [Prop. 1.24].

15. But, MN was shown (to be) equal to BC [(so) BC is also greater than FG].
16. Thus, the diameter AD (is) the greatest (straight-line), and BC (is) greater than FG.

Conclusion. Thus, in a circle, a diameter (is) the greatest (straightline), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 15

*In a circle, the diameter is the greatest straight-line,
and among the others the chord nearer to the center is always greater than the one farther away.*

PANEL I — ACTIONS 1–9: SET-UP AND CONSTRUCT AN EQUAL-DISTANCE CHORD

1 Draw circle $ABCD$.

2 Let AD be a diameter.

3 Let E be the center.

4 Let BC be a chord nearer to the center.

5 Let FG be a chord farther from the center.

6 From E , draw $EH \perp BC$.

7 From E , draw $EK \perp FG$.

8 Since BC is nearer to the center than FG , $EH < EK$.

9 On EK , take point L so that $EL = EH$. Through L , draw $MN \perp EK$, meeting the circle at M, N .

PANEL II — ACTIONS 10–18: COMPARE THE LENGTHS

10 Since $EH = EL$, by Proposition 14, $BC = MN$.

11 Join EM and EN .

12 $AE = EM$ and $ED = EN$ (radii). Therefore $AD = AE + ED = EM + EN$.

13 In triangle MEN , by the triangle inequality, $EM + EN > MN$.

14 Therefore $AD = EM + EN > MN$.

15 Since $MN = BC$, we get $AD > BC$.

Thus the diameter is greater than the nearer chord.

16 In triangles MEN and FEG , $EM = EF$ and $EN = EG$ (radii).

17 $\angle MEN > \angle FEG$. By Prop. I.24, the side opposite the greater angle is greater: $MN > FG$.

18 But $MN = BC$. Hence $BC > FG$.

Therefore $AD > BC > FG$.

SUMMARY
A diameter is the greatest chord. Among all other chords, the one nearer to the center is greater than the one farther away.

KEY TO DIAGRAMS

- Circle $ABCD$
- Chord BC (nearer)
- Chord FG (farther)
- Diameter AD
- Chord MN (equal distance)
- + Center E
- Perpendiculars from E

Proposition 16

Statement. A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

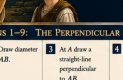
1. Let ABC be a circle around the center D and the diameter AB.
2. I say that the (straight-line) drawn from A, at right-angles to AB [Prop 1.11], from its end, will fall outside the circle.
3. For (if) not then, if possible, let it fall inside, like CA (in the figure), and let DC have been joined.
4. Since DA is equal to DC, angle DAC is also equal to angle ACD [Prop. 1.5].
5. And DAC (is) a right-angle.
6. Thus, ACD (is) also a right-angle.
7. So, in triangle ACD, the two angles DAC and ACD are equal to two rightangles. The very thing is impossible [Prop. 1.17].
8. Thus, the (straight-line) drawn from point A, at right-angles
9. to BA, will not fall inside the circle.
10. So, similarly, we can show that neither (will it fall) on the circumference.
11. Thus, (it will fall) outside (the circle).
12. Let it fall like AE (in the figure).
13. So, I say that another straight-line cannot be inserted into the space between the straight-line AE and the circumference CHA.

14. For, if possible, let it be inserted like FA (in the figure), and let DG have been drawn from point D, perpendicular to FA [Prop. 1.12].
15. And since AGD is a right-angle, and DAG (is) less than a right-angle, AD (is) thus greater than DG [Prop. 1.19].
16. And DA (is) equal to DH.
17. Thus, DH (is) greater than DG, the lesser than the greater. The very thing is impossible.
18. Thus, another straight-line cannot be inserted into the space between the straight-line (AE) and the circumference.
19. And I also say that the semi-circular angle contained by the straight-line BA and the circumference CHA is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference CHA and the straight-line AE is less than any acute rectilinear angle whatsoever.
20. For if any rectilinear angle is greater than the (angle) contained by the straight-line BA and the circumference CHA, or less than the (angle) contained by the circumference CHA and the straight-line AE, then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE—anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line BA and the circumference CHA, or less than the (angle) contained by the circumference CHA and the straightline AE.
21. But (such a straight-line) cannot be inserted.
22. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line BA and the circumference CHA, neither (can it be) less than the (angle) contained by the circumference CHA and the straight-line AE.

Conclusion. So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

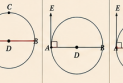
EUCLID'S ELEMENTS • BOOK III • PROPOSITION 16

A straight-line drawn at right angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the straight-line and the circumference. And the angle of the semicircle is greater than any acute rectilinear angle whatsoever, and the remaining angle is less than any acute rectilinear angle.




PANEL I — ACTIONS 1-9: THE PERPENDICULAR CANNOT ENTER THE CIRCLE

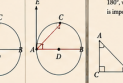
1 Draw circle ABC with center D .



2 Draw diameter AB .



3 At A draw a straight-line perpendicular to AB .



4 Assume the perpendicular falls outside the circle.



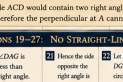
5 Assume the contrary: it enters the circle.



6 Join DC .



7 AD and DC are radii, so $DA = DC$.



8 The right angles of a triangle are together 180° , which is impossible.



9 Therefore the perpendicular cannot enter the circle.



Thus triangle ACD would contain two right angles, which is impossible (Prop. I.17). Therefore the perpendicular at A cannot fall inside the circle.

PANEL II — ACTIONS 10-15: OUTSIDE THE CIRCLE AND NO LINE BETWEEN

10 The right angles of a triangle are together 180° , which is impossible.



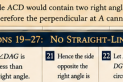
11 We'll drop 117° from the two angles of a triangle are together less than two right angles.



12 Therefore a perpendicular cannot enter the circle.



13 We draw it fall on the circumference.



14 Therefore it must fall outside the circle: let $AE \perp AB$.



15 We claim no other straight line can be inserted between AE and the circumference.



16 Assume contrary: let AF be such a line.

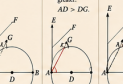


Contradiction.

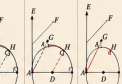
Thus triangle ACD would contain two right angles, which is impossible (Prop. I.17). Therefore the perpendicular at A cannot fall inside the circle.

PANEL III — ACTIONS 19-27: NO STRAIGHT-LINE CAN FIT BETWEEN

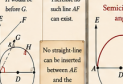
19 In triangle ADC , $\angle ADC = 90^\circ$ right angle.



20 $\angle CDE$ is less than right angle: $AD > DE$.



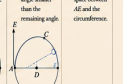
21 Let EF then between AD and the circumference.



22 DA and DE are radii, so $DA = DE$.



23 Therefore $\angle DEF < \angle C$.



24 Hence the base EF is less than AC and the circle, if it falls in, before G .



25 Both AD and DE are radii, so $AD = DE$. Therefore $\angle DEF < \angle C$.



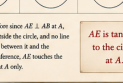
26 Hence the base EF is less than AC and the circle, if it falls in, before G .



27 Therefore $\angle DEF < \angle C$.



28 Hence the base EF is less than AC and the circle, if it falls in, before G .



29 Both AD and DE are radii, so $AD = DE$. Therefore $\angle DEF < \angle C$.



30 Hence the base EF is less than AC and the circle, if it falls in, before G .



No straight line can be inserted between AE and the circumference.

PANEL IV — ACTIONS 28-36: THE SEMICIRCULAR ANGLE AND THE TANGENT

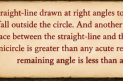
28 Therefore the angle between AE and the circumference is $\angle CEA$ at A .



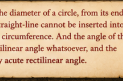
29 Consider the remaining angle at A between AE and AC and the semicircular angle.



30 Suppose there were an acute rectilinear angle greater than the semicircular angle.



31 Or suppose there were an acute rectilinear angle smaller than the remaining angle.



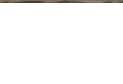
32 The second straight side cannot be drawn between AE and the circumference.



33 The Action 32-37 proved that no straight line can be inserted between AE and the circumference.



34 Therefore the semicircular angle is greater than any acute rectilinear angle.



35 And the remaining angle is less than any acute rectilinear angle.



36 Therefore since $AE \perp AB$ at A , lies outside the circle, and no line can be between it and the circumference, AE touches the circle at <

Proposition 17

Statement. To draw a straight-line touching a given circle from a given point.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let A be the given point, and BCD the given circle.
2. So it is required to draw a straight-line touching circle BCD from point A.
3. For let the center E of the circle have been found [Prop. 3.1], and let AE have been joined.
4. And let (the circle) AFG have been drawn with center E and radius EA.
5. And let DF have been drawn from (point) D, at right-angles to EA [Prop. 1.11].
6. And let EF and AB have been joined.
7. I say that the (straight-line) AB has been drawn from point A touching circle BCD.
8. For since E is the center of circles BCD and AFG, EA is thus equal to EF, and ED to EB.
9. So the two (straight-lines) AE, EB are equal to the two (straightlines) FE, ED (respectively).
10. And they contain a common angle at E.
11. Thus, the base DF is equal to the base AB, and triangle DEF is equal to triangle EBA, and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4].
12. Thus, (angle) EDF (is) equal to EBA.
13. And EDF (is) a right-angle.

14. Thus, EBA (is) also a right-angle.
15. And EB is a radius.
16. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.].
17. Thus, AB touches circle BCD .
18. Thus, the straight-line AB has been drawn touching

Conclusion. the given circle BCD from the given point A . (Which is) the very thing it was required to do.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 17

To draw a straight-line touching a given circle from a given point.



PANEL I — ACTIONS 1–9: CONSTRUCTION OF THE TANGENT

1 Let A be the given point outside the given circle BCD .



2 Find the center of the given circle BCD . Call it E .



3 Join A to the center E .



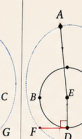
4 With center E and radius EA , draw a circle AFG .



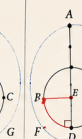
5 Let D be the (given) point on the circle BCD . (D is collinear with A and E .)



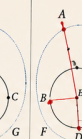
6 At D , draw DF perpendicular to EA .



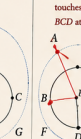
7 Produce DF until it meets the circle AFG at F .



8 Join EF .



9 Draw AB through A meeting the given circle again at B . Claim: AB touches circle BCD at B .



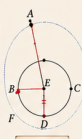
PANEL II — ACTIONS 10–18: PROOF THAT AB IS TANGENT

10 Since F is the center of circle AFG , $EA = EF$.



$EA = EF$

11 Since E is the center of circle BCD , $ED = EB$.



$ED = EB$

12 Thus the two sides AE, EB are respectively equal to FE, ED .



13 Also the angle at E is common to both triangles: $\angle AEB = \angle FED$.



14 Therefore, by Prop. 1.4 (SAS), triangles AEB and FED are equal. Hence $AB = DF$.



$AB = DF$

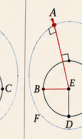
15 Therefore corresponding angles are equal: $\angle ERA = \angle EDF$.



16 But DF was drawn perpendicular to EA ; thus $\angle EDF = 90^\circ$.



17 Therefore $\angle ERA = 90^\circ$. So $EB \perp AB$.



18 EB is a radius of the given circle. A straight-line perpendicular to a radius at its endpoint touches the circle (Prop. 16 cor.). Thus AB is tangent to circle BCD at B .

AB is tangent to circle BCD at B .

KEY TO DIAGRAMS

- Center
- Radius / given line
- Constructed line
- Circle AFG (center E , radius EA)
- Right angle

SUMMARY

From the external point A , the constructed line AB touches the given circle BCD at exactly one point, B .



Proposition 18

Statement. If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let some straight-line DE touch the circle ABC at point C, and let the center F of circle ABC have been found [Prop. 3.1], and let FC have been joined from F to C.
2. I say that FC is perpendicular to DE.
3. For if not, let FG have been drawn from F, perpendicular to DE [Prop. 1.12].
4. Therefore, since angle FGC is a right-angle, (angle) FCG is thus acute [Prop. 1.17].
5. And the greater angle is subtended by the greater side [Prop. 1.19].
6. Thus, FC (is) greater than FG.
7. And FC (is) equal to FB.
8. Thus, FB (is) also greater than FG, the lesser than the greater. The very thing is impossible.
9. Thus, FG is not perpendicular to DE.
10. So, similarly, we can show that neither (is) any other (straight-line) except FC.
11. Thus, FC is perpendicular to DE.

Conclusion. Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 18

If some straight-line touches a circle, and some other straight-line is joined from the center of the circle to the point of contact, then the straight-line so joined will be perpendicular to the tangent.



PANEL I — ACTIONS 1–9: ASSUME FC IS NOT PERPENDICULAR TO THE TANGENT

1 Draw circle ABC .



2 Draw a straight-line DE touching the circle at C .



3 Find the center of the circle and call it F .



4 Join F to the point of contact C .



5 Claim: $FC \perp DE$.



6 Assume the contrary: FC is not perpendicular to DE .



7 From F , draw FG perpendicular to DE , meeting it at G .



8 Join G to C (the portion GC of the tangent).



9 In triangle FGC , $\angle FGC = 90^\circ$.



PANEL II — ACTIONS 10–18: DERIVE A CONTRADICTION

10 Since $\angle FGC = 90^\circ$, the angle at C , $\angle FCG$, is acute ($< 90^\circ$).



11 Hence $\angle FGC > \angle FCG$.



12 In a triangle, the greater angle is opposite the greater side.



13 Side opposite $\angle FGC$ is FC , opposite $\angle FCG$ is FG .



14 Therefore $FC > FG$.



15 Join F to B . Since FC and FB are radii, $FC = FB$.



16 From $FC > FG$ and $FC = FB$, it follows that $FB > FG$.



17 But along the same straight line $F-B-G$, point B is between F and G .



18 We have proved the absurdity $FB > FG$ and $FG > FB$. Therefore our assumption is false.



Hence $FC \perp DE$.

CONCLUSION

The straight-line from the center of the circle to the point of contact is perpendicular to the tangent.

$FC \perp DE$





Thus, if a straight-line touches a circle, the line joining the center to the point of contact is perpendicular to it.

KEY TO DIAGRAMS

- F Center of circle
- Radius
- Tangent
- - - Constructed line
- Right angle

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Proposition 19

Statement. If some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-[angles] to the tangent, then the center (of the circle) will be on the (straight-line) so drawn.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let some straight-line DE touch the circle ABC at point C.
2. And let CA have been drawn from C, at right-
3. angles to DE [Prop. 1.11].
4. I say that the center of the circle is on AC.
5. For (if) not, if possible, let F be (the center of the circle), and let CF have been joined.

Therefore , since some straight-line DE touches the circle ABC, and FC has been joined from the center to the point of contact, FC is thus perpendicular to DE

6. Thus, FCE is a right-angle.

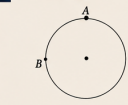
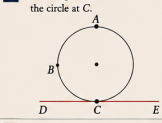
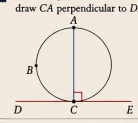
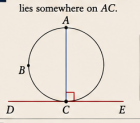
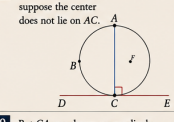
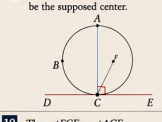
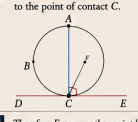
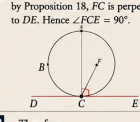
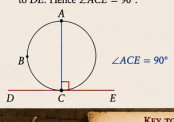
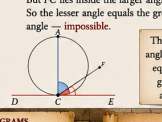
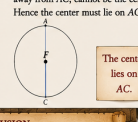
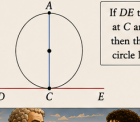
Conclusion. And ACE [Prop. 3.18]. is also a right-angle. Thus, FCE is equal to ACE, the lesser to the greater. The very thing is impossible. Thus, F is not the center of circle ABC. So, similarly, we can show that neither is any (point) other (than one) on AC. Thus, if some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-angles to the tangent, then the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 19

If some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right angles to the tangent, then the center of the circle will be on the straight-line so drawn.



1 Draw circle ABC. 	2 Let straight line DE touch the circle at C. 	3 From the point of contact C, draw CA perpendicular to DE. 	4 Claim: the center of circle ABC lies somewhere on AC. To Prove: Center of the circle lies on AC. 
5 Assume the contrary: suppose the center does not lie on AC. 	6 Let F, somewhere away from AC, be the supposed center. 	7 Join the supposed center F to the point of contact C. 	8 Since DE is tangent to the circle at C, by Proposition 18, FC is perpendicular to DE. Hence $\angle FCE = 90^\circ$. 
9 But CA was drawn perpendicular to DE. Hence $\angle ACE = 90^\circ$. 	10 Thus $\angle FCE = \angle ACE$. But FC lies inside the larger angle ACE. So the lesser angle equals the greater angle — impossible. The lesser angle cannot equal the greater angle. 	11 Therefore F, or any other point lying away from AC, cannot be the center. Hence the center must lie on AC. The center lies on AC. 	12 Therefore: If DE touches the circle at C and $AC \perp DE$, then the center of the circle lies on AC. 



KEY TO DIAGRAMS

- Center (when shown)
- DE Tangent
- AC Constructed perpendicular
- Right angle

CONCLUSION

The center of the circle lies on the straight-line drawn from the point of contact at right angles to the tangent.



Proposition 20

Statement. In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be a circle, and let BEC be an angle at its center, and BAC (one) at (its) circumference.
2. And let them have the same circumference base BC .
3. I say that angle BEC is double (angle) BAC .
4. For being joined, let AE have been drawn through to F .
5. Therefore, since EA is equal to EB , angle EAB (is) also equal to EBA [Prop. 1.5].
6. Thus, angle EAB and EBA is double (angle) EAB .
7. And BEF (is) equal to EAB and EBA [Prop. 1.32].
8. Thus, BEF is also double EAB .
9. So, for the same (reasons), FEC is also double EAC .
10. Thus, the whole (angle) BEC is double the whole (angle) BAC .
11. So let another (straight-line) have been inflected, and let there be another angle, BDC .
12. And DE being joined, let it have been produced to G .
13. So, similarly, we can show that angle GEC is double EDC , of which GEB is double EDB .
14. Thus, the remaining (angle) BEC is double the (remaining angle) BDC .

Conclusion. Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 20

In a circle, the angle at the center is double the angle at the circumference, when the angles have the same circumference as base.

PANEL I — ACTIONS 1–9: FIRST HALF OF THE CENTRAL ANGLE								
1 Draw circle ABC .	2 Let E be the center.	3 Draw the central angle $\angle BEC$.	4 Choose A on the circumference and join AB and AC . Then $\angle BAC$ stands on the same circumference BC .	5 Join AE .	6 Produce AE beyond E to point F .	7 Since $EA = EB$ (radii).	8 Therefore in $\triangle AEB$, $\angle EAB = \angle EBA$ (base angles of an isosceles triangle).	9 Hence $\angle EAB + \angle EBA = 2\angle EAB$.
PANEL II — ACTIONS 10–18: COMPLETE THE DOUBLING								
10 Since AE is produced to F , $\angle BEF$ is the exterior angle of $\triangle AEB$.	11 By Prop. 132, $\angle BEF = \angle EAB + \angle EBA$.	12 Since $\angle EAB = \angle EBA$, $\angle BEF = 2\angle EAB$.	13 Consider $\triangle AEC$.	14 Since $EA = EC$ (radii), $\angle EAC = \angle ECA$.	15 Therefore the exterior angle at E satisfies $\angle FEC = 2\angle EAC$.	16 The whole central angle is the sum of the two parts: $\angle BEC = \angle BEF + \angle FEC$.	17 The whole angle at the circumference is the sum of the two parts: $\angle BAC = \angle BAE + \angle EAC$.	18 Since $\angle BEF = 2\angle EAB$ and $\angle FEC = 2\angle EAC$, therefore $\angle BEC = 2\angle BAC$.

KEY TO DIAGRAMS

- E Center of the circle
- Radii
- Chords
- - - Produced line AE

- ∠ Angle at the circumference ($\angle BAC$)
- ∠ Angle at the center ($\angle BEC$)
- ∠ Exterior angle $\angle BEF$
- ∠ Exterior angle $\angle FEC$

CONCLUSION

In a circle, the angle at the center is double the angle at the circumference, when the angles have the same circumference as base.

$\angle BEC = 2\angle BAC$

Proposition 21

Statement. In a circle, angles in the same segment are equal to one another.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let $ABCD$ be a circle, and let BAD and BED be angles in the same segment $BAED$.
2. I say that angles BAD and BED are equal to one another.
3. For let the center of circle $ABCD$ have been found [Prop. 3.1], and let it be (at point) F .
4. And let BF and FD have been joined.
5. And since angle BFD is at the center, and BAD at the circumference, and they have the same circumference base BCD , angle BFD is thus double BAD [Prop. 3.20].
6. So, for the same (reasons), BFD is also double BED .
7. Thus, BAD (is) equal to BED .

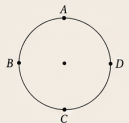
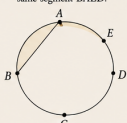
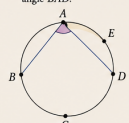
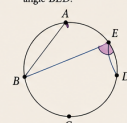
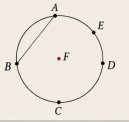
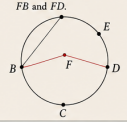
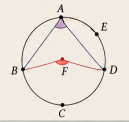
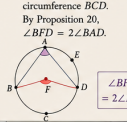
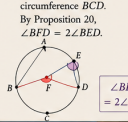
Conclusion. Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 21

In a circle, angles in the same segment are equal to one another.



1 Draw circle $ABCD$. 	2 Choose points A and E in the same segment $BAED$. 	3 Join AB, AD , forming angle BAD . 	4 Join EB, ED , forming angle BED . 	5 Both are angles in the same segment. We claim $\angle BAD = \angle BED$. To Prove: $\angle BAD = \angle BED$
6 Find the center of circle $ABCD$, and call it F . 	7 Join the center to the endpoints of the common arc: FB and FD . 	8 This produces the central angle $\angle BFD$. 	9 Compare $\angle BFD$ with $\angle BAD$. They stand on the same circumference BCD . By Proposition 20, $\angle BFD = 2\angle BAD$.  $\angle BFD = 2\angle BAD$	10 Compare $\angle BFD$ with $\angle BED$. They stand on the same circumference BCD . By Proposition 20, $\angle BFD = 2\angle BED$.  $\angle BFD = 2\angle BED$
11 Thus both $\angle BAD$ and $\angle BED$ are halves of the same angle $\angle BFD$. $2\angle BAD = \angle BFD = 2\angle BED$ Therefore $\angle BAD = \angle BED$.	$\angle BFD = 2\angle BAD$ $\angle BFD = 2\angle BED$ Therefore $\angle BAD = \angle BED$	12 Conclusion: Angles in the same segment of a circle are equal to one another.		

KEY TO DIAGRAMS: • Center F — Radius (FB, FD) — Chords (AB, AD, EB, ED) Angles at the circumference Angle at the center

Proposition 22

Statement. For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral within it.
2. I say that the (sum of the) opposite angles is equal to two right-angles.
3. Let AC and BD have been joined.
4. Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles CAB , ABC , and BCA of triangle ABC are thus equal to two right-angles.
5. And CAB (is) equal to BDC .
6. For they are in the same segment $BADC$ [Prop. 3.21].
7. And ACB (is equal) to ADB .
8. For they are in the same segment $ADCB$ [Prop. 3.21].
9. Thus, the whole of ADC is equal to BAC and ACB .
10. Let ABC have been added to both.
11. Thus, ABC , BAC , and ACB are equal to ABC and ADC .
12. But, ABC , BAC , and ACB are equal to two right-angles.
13. Thus, ABC and ADC are also equal to two right-angles.
14. Similarly, we can show that angles BAD and DCB are also equal to two right-angles.

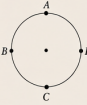
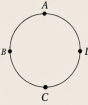
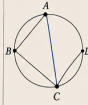
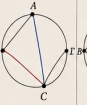
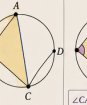
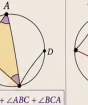
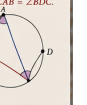
Conclusion. Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 22

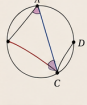
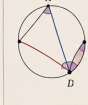
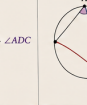
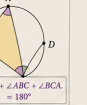
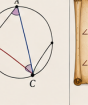
*For quadrilaterals within circles,
the sum of the opposite angles is equal to two right angles.*




PANEL I — ACTIONS 1–9: BUILD THE CYCLIC QUADRILATERAL

1 Draw circle $ABCD$. 	2 Place four points A, B, C, D on its circumference. 	3 Join AB, BC, CD, DA to form the quadrilateral $ABCD$. 	4 Claim: opposite angles together equal two right angles. $\angle ABC + \angle ADC = 180^\circ$ $\angle BAD + \angle DCB = 180^\circ$ To Prove	5 Join the diagonal AC . 	6 Join the other diagonal BD . 	7 Focus on triangle ABC . 	8 By Prop. I.32, the three angles of triangle ABC equal two right angles. $\angle CAB + \angle ABC + \angle BCA = 180^\circ$ 	9 $\angle CAB$ and $\angle BDC$ lie in the same segment $BADC$. By Prop. III.21, $\angle CAB = \angle BDC$. 
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PANEL II — ACTIONS 10–18: REASSEMBLE THE ANGLES

10 $\angle BCA$ and $\angle BDA$ lie in the same segment. By Prop. III.21, $\angle BCA = \angle BDA$. 	11 At D , the two angles $\angle BDA$ and $\angle BDC$ together form the whole angle $\angle ADC$. 	12 Therefore, using Actions 9 and 11, $\angle ADC = \angle CAB + \angle BCA$ 	13 Add $\angle ABC$ to both sides. $\angle ABC + \angle ADC = \angle ABC + \angle CAB + \angle BCA$ 	14 But the right-hand side is exactly the three angles of triangle ABC . $\angle CAB + \angle ABC + \angle BCA = 180^\circ$ 	15 Those three angles equal two right angles. Hence $\angle ABC + \angle ADC = 180^\circ$ 	16 By the same reasoning for the other pair of opposite angles, $\angle BAD + \angle DCB = 180^\circ$ 	17 Therefore both pairs of opposite angles of a cyclic quadrilateral equal two right angles. Conclusion $\angle ABC + \angle ADC = 180^\circ$ and $\angle BAD + \angle DCB = 180^\circ$
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KEY TO DIAGRAMS
 Sides of quadrilateral
 Diagonal AC
 Diagonal BD
 Angles in the same segment (equal)
 Triangle ABC
 Angle $\angle ADC$

CONCLUSION
 In any quadrilateral inscribed in a circle,
 the sum of each pair of opposite angles
 is equal to two right angles (180°).




Proposition 23

Statement. Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For, if possible, let the two similar and unequal segments of circles, ACB and ADB, have been constructed on the same side of the same straight-line AB.
2. And let
3. ACD have been drawn through (the segments), and let CB and DB have been joined.
4. Therefore, since segment ACB is similar to segment ADB, and similar segments of circles are those accepting equal angles [Def. 3.11], angle ACB is thus equal to ADB, the external to the internal. The very thing is impossible [Prop. 1.16].
5. Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

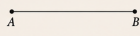


EUCLID'S ELEMENTS • BOOK III • PROPOSITION 23

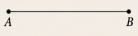
Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.



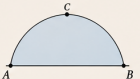
1 Draw the straight line AB .



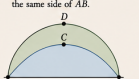
2 Assume, contrary to the proposition, that two unequal circular segments can be constructed on the same side of AB .



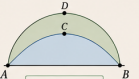
3 Let the first segment be ACB .



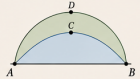
4 Let the second, unequal segment be ADB . Both segments are on the same side of AB .



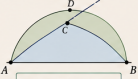
5 Assume that the two segments are similar: $ACB \sim ADB$.



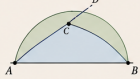
6 Since the segments are unequal, arrange them as shown, one inside the other.



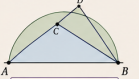
7 Draw the straight line through A and C , and produce it beyond C until it meets the other arc at D .



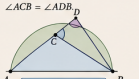
8 Join CB .



9 Join DB .



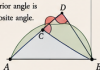
10 Since $ACB \sim ADB$ (Def. III.11), they accept equal angles. Therefore, $\angle ACB = \angle ADB$.



11 But A, C, D are collinear, so $\angle ACB$ is the exterior angle of triangle CDB , and $\angle ADB$ is the interior opposite angle.

By Proposition I.16, the exterior angle is greater than the interior opposite angle.

$\angle ACB > \angle ADB$



12 But similarity gave $\angle ACB = \angle ADB$. This contradicts Proposition I.16.

Contradiction!

KEY TO DIAGRAMS

- Point
- Given straight-line
- Line drawn
- - - Extension of line
- Segment ACB
- Segment ADB
- Exterior angle $\angle ACB$
- Interior opposite angle $\angle ADB$

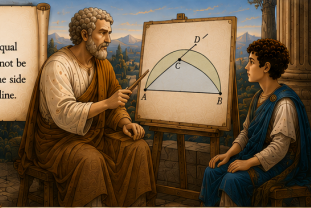
WHAT WE USED

Def. III.11: Similar figures accept equal angles.

Prop. I.16: In any triangle, an exterior angle is greater than the interior opposite angle.

Therefore:

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.



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Proposition 24

Statement. Similar segments of circles on equal straight-lines are equal to one another.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. For let AEB and CFD be similar segments of circles on the equal straight-lines AB and CD (respectively).
2. I say that segment AEB is equal to segment CFD.
3. For if the segment AEB is applied to the segment CFD, and point A is placed on (point) C, and the straight-line AB on CD, then point B will also coincide with point D, on account of AB being equal to CD.
4. And if AB coincides with CD then the segment AEB will also coincide with CFD.
5. For if the straight-line AB coincides with CD, and the segment AEB does not coincide with CFD, then it will surely either fall inside it, outside (it), or it will miss like CGD (in the figure), and a circle (will) cut (another) circle at more than two points. The very
6. thing is impossible [Prop. 3.10].
7. Thus, if the straight-line AB is applied to CD, the segment AEB cannot not also coincide with CFD.
8. Thus, it will coincide, and will be equal to it [C.N. 4].

Conclusion. Thus, similar segments of circles on equal straightlines are equal to one another. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 24

Similar segments of circles on equal straight-lines are equal to one another.



PANEL I — ACTIONS 1–9: SUPERPOSE THE TWO SEGMENTS

1 Draw the straight line AB .



2 On AB , construct the circular segment AEB .



3 Draw another straight line CD such that $AB = CD$.

$AB = CD$

4 On CD , construct the circular segment CFD .



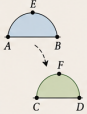
5 The two segments are given to be similar: $AEB \sim CFD$.

$AEB \sim CFD$

6 **Claim:** $AEB = CFD$.

To Prove: $AEB = CFD$

7 Take the segment AEB and apply it to the segment CFD .



8 Place point A exactly upon point C .



9 Place the straight line AB upon the straight line CD .



PANEL II — ACTIONS 10–18: THE ARCS MUST COINCIDE

10 Since $AB = CD$, when A coincides with C , the other end also coincides: B coincides with D .



11 Thus the two equal chords now coincide: $AB = CD$.

$AB = CD$

12 We claim that the segment AEB must coincide with CFD .



13 Suppose, for contradiction, that although the chords coincide, the two segments do not coincide.



14 Then the arc of AEB must lie either inside the arc of CFD ...



15 ... or outside it ...



16 ... or cross it as in this other configuration CGD .



17 In any such case, the two circles would intersect in more than two points.

Impossible: two circles cannot cut one another in more than two points.

18 Therefore the two segments must coincide exactly.

Hence $AEB = CFD$.

CONCLUSION

Therefore, similar segments of circles on equal straight-lines are equal to one another.

KEY TO DIAGRAMS

- Point
- Straight line (chord)
- Segment AEB
- Segment CFD
- Application of AEB onto CFD



Proposition 25

Statement. For a given segment of a circle, to complete the circle, the very one of which it is a segment.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Let ABC be the given segment of a circle.
2. So it is required to complete the circle for segment ABC, the very one of which it is a segment.
3. For let AC have been cut in half at (point) D [Prop. 1.10], and let DB have been drawn from point D, at right-angles to AC [Prop. 1.11].
4. And let AB have been joined.
5. Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD.
6. First of all, let it be greater.
7. And let (angle) BAE, equal to angle ABD, have been constructed on the straight-line BA, at the point A on it [Prop. 1.23].
8. And let DB have been drawn through to E, and let EC have
9. Therefore, since angle ABE is equal to been joined. BAE, the straight-line EB is thus also equal to EA [Prop. 1.6].
10. And since AD is equal to DC, and DE (is) common, the two (straight-lines) AD, DE are equal to the two (straight-lines) CD, DE, respectively.
11. And angle ADE is equal to angle CDE.
12. For each (is) a right-angle.
13. Thus, the base AE is equal to the base CE [Prop. 1.4].
14. But, AE was shown (to be) equal to BE.
15. Thus, BE is also equal to CE.

16. Thus, the three (straight-lines) AE, EB, and EC are equal to one another.
17. Thus, if a circle is drawn with center E, and radius one of AE, EB, or EC, it will also go through the remaining points (of the segment), and the (associated circle) will have been completed [Prop. 3.9].
18. Thus, a circle has been completed from the given segment of a circle.
19. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it.

And , similarly, even if angle ABD is equal to BAD, (since) AD becomes equal to each of BD [Prop. 1.6] and DC, the three (straight-lines) DA, DB, and DC will be

20. And point D will be the center equal to one another. of the completed circle.
21. And ABC will manifestly be a semi-circle.
22. And if ABD is less than BAD, and we construct (angle BAE), equal to angle ABD, on the straight-line BA, at the point A on it [Prop. 1.23], then the center will fall on DB, inside the segment ABC.
23. And segment ABC will manifestly be greater than a semi-circle.

Conclusion. Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 25

For a given segment of a circle, to complete the circle of which it is a segment.

PANEL I — ACTIONS 1–9: SET UP THE CONSTRUCTION

1 Given the circular segment ABC .

2 Our task is to complete the circle of which this is a segment.

3 Bisect the chord AC at D .
 $AD = DC$.

4 Draw DB perpendicular to AC .

5 Thus D is the midpoint of AC and DB is perpendicular to AC .

6 Join AB .

7 Compare the angles $\angle ABD$ and $\angle BAD$.

8 Exactly one of these three possibilities must occur.

 $\angle ABD > \angle BAD$
 $\angle ABD = \angle BAD$
 $\angle ABD < \angle BAD$

9 We shall consider these three cases separately.

 To complete the circle of segment ABC .

PANEL II — ACTIONS 10–18: CASE I — $\angle ABD > \angle BAD$ (center outside the segment)

10 First suppose $\angle ABD > \angle BAD$.

11 At A , construct $\angle BAE$ so that $\angle BAE = \angle ABD$.

12 Produce DB to meet AE at E .
 D, B, E are collinear.

13 Join EC .

14 Since $\angle ABE = \angle ABD$ and $\angle BAE = \angle ABD$,
 $\angle ABE = \angle BAE$.
Therefore $EA = EB$.

15 In right triangles ADE and CDE ,
 $AD = DC$, $DE = DE$,
 $\angle ADE = \angle CDE = 90^\circ$.
Therefore $EA = EC$.

16 Hence $EA = EB = EC$.
 A, B, C are all equally distant from E .

17 With center E and radius EA , draw the circle through A, B , and C .

18 The center E lies outside the segment. Hence ABC is less than a semicircle.

 Case I done.

PANEL III — ACTIONS 19–27: CASES II AND III

19 Second possibility: $\angle ABD = \angle BAD$.

20 Then triangle ABD is isosceles, so $AD = DB$.

21 But D bisects AC , so $AD = DC$.
Thus $DA = DB = DC$.

22 Hence D is the center. With center D and radius DA , draw the circle through A, B, C .

23 Third possibility: $\angle ABD < \angle BAD$.

24 At A , construct $\angle BAE$ so that $\angle BAE = \angle ABD$.

25 DB produced meets AE at E .
(The center E now lies inside the segment.)

26 With center E and radius EA , draw the circle through A, B , and C .

27 The center E lies inside the segment. Hence ABC is greater than a semicircle.

 Case III done.

SUMMARY OF THE THREE CASES

Case I: $\angle ABD > \angle BAD$
Center E is outside the segment.
 ABC is less than a semicircle.

Case II: $\angle ABD = \angle BAD$
Center D is at the midpoint of AC .
 ABC is exactly a semicircle.

Case III: $\angle ABD < \angle BAD$
Center E is inside the segment.
 ABC is greater than a semicircle.

CONCLUSION

In every possible case, we can determine the center and complete the circle of which the given segment ABC is a segment.

Proposition 26

Statement. In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw two equal circles, ABC and DEF.
2. Mark their centers G and H.
3. In the first circle construct the central angle BGC.
4. In the second circle construct the equal central angle EHF, so that angle BGC equals angle EHF.
5. Also mark equal angles at the circumferences: angle BAC equals angle EDF.
6. Mark points K and L on the corresponding circumferences; the goal is to prove arc BKC equals arc ELF.
7. Join the corresponding chords BC and EF.
8. Because the two circles are equal, their radii are equal: BG equals EH and GC equals HF. Together with angle BGC equals angle EHF, there are two sides and the included angle equal.
9. Therefore, by Proposition I.4, triangles BGC and EHF are congruent, and consequently BC equals EF.
10. Return to the given equality of the angles at the circumference: angle BAC equals angle EDF.
11. These equal angles are accepted by the circular segments BAC and EDF.
12. Therefore, by Definition III.11, the two circular segments BAC and EDF are similar.
13. But their bases have already been proved equal: BC equals EF.

14. So there are similar circular segments standing on equal straight-lines.
15. By Proposition III.24, such segments are equal: segment BAC equals segment EDF.
16. But the whole circles ABC and DEF are themselves equal.
17. Subtract the equal segments BAC and EDF from the equal whole circles; the remaining circumferences are equal: arc BKC equals arc ELF.
18. Therefore, in equal circles, equal angles stand upon equal circumferences, whether the equal angles are at the centers or at the circumferences.

Conclusion. or at the circumference. (Which is) the very thing which it was required to show.



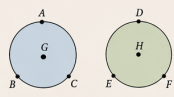
EUCLID'S ELEMENTS • BOOK III • PROPOSITION 26

In equal circles, equal angles stand upon equal circumferences, whether they stand at the centers or at the circumferences.

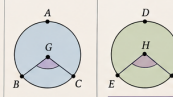


PANEL I — ACTIONS 1–9: FROM EQUAL CENTRAL ANGLES TO EQUAL CHORDS

1 Draw two equal circles ABC and DEF .



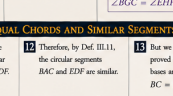
2 Mark their centers G and H .



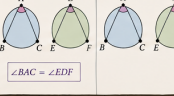
3 In the first circle construct the central angle BGC .



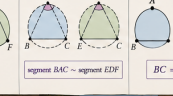
4 In the second circle construct the equal central angle EHF .



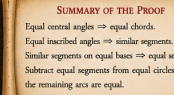
5 Also mark equal angles at the circumferences: $\angle BAC = \angle EDF$.



6 Let K and L be points on the corresponding circumferences. We must prove $\text{arc } BKC = \text{arc } ELF$.



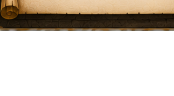
7 Join the corresponding chords BC and EF .



8 Since the circles are equal, $BC = EF$ and $GC = HF$.

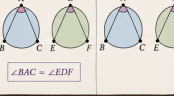


9 With $\angle BGC = \angle EHF$, in triangles BGC and EHF we have two sides and the included angle equal. Therefore, by Prop. I.4, $\triangle BGC \cong \triangle EHF$; and hence $BC = EF$.

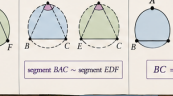


PANEL II — ACTIONS 10–18: FROM EQUAL CHORDS AND SIMILAR SEGMENTS TO EQUAL ARCS

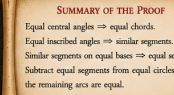
10 We are also given that the angles at the circumference are equal: $\angle BAC = \angle EDF$.



11 These equal angles are accepted by the circular segments BAC and EDF .



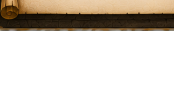
12 Therefore, by Def. III.11, the circular segments BAC and EDF are similar.



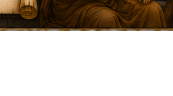
13 But we have already proved that their bases are equal: $BC = EF$.



14 Thus we have similar circular segments standing on equal straight-lines.



15 By Proposition III.24, similar segments on equal lines are equal. Therefore, segment $BAC =$ segment EDF .



16 But the whole circles themselves are equal: circle $ABC =$ circle DEF .



17 Subtract the equal segments BAC and EDF from the equal whole circles. The remaining circumferences are equal.



18 Hence, in equal circles, equal angles stand upon equal circumferences, whether they stand at the centers or at the circumferences.



SUMMARY OF THE PROOF

Equal central angles $\Rightarrow$ equal chords.
 Equal inscribed angles $\Rightarrow$ similar segments.
 Similar segments on equal bases $\Rightarrow$ equal segments.
 Subtract equal segments from equal circles $\Rightarrow$ the remaining arcs are equal.

KEY TO DIAGRAMS

- Point
- Straight line (chord)
- Radius
- Segment BAC
- Segment EDF
- Arc of circumference
- Equal angles

WHAT WE USED

Def. III.11: Similar segments that accept equal angles are similar.
 Prop. I.4: Two sides and the included angle equal $\Rightarrow$ triangles are equal.
 Prop. III.24: Similar segments on equal straight-lines are equal.

Proposition 27

Statement. In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw two equal circles, ABC and DEF.
2. Mark their centers G and H.
3. In the first circle identify the circumference BC.
4. In the second circle identify the circumference EF; they are given equal.
5. Join the radii GB and GC, producing the central angle BGC.
6. Join HE and HF, producing the second central angle EHF. The first goal is to prove angle BGC equals angle EHF.
7. Suppose, for contradiction, that the central angles are unequal.
8. Suppose specifically that angle BGC is greater than angle EHF.
9. Inside the larger angle BGC, construct a smaller angle BGK equal to angle EHF, where K lies on the circumference of the first circle.
10. Now angle BGK equals angle EHF; by Proposition III.26, equal central angles in equal circles stand upon equal circumferences, so arc BK equals arc EF.
11. But from the original hypothesis, arc EF equals arc BC.
12. Hence arc BK equals arc BC.
13. But K lies between B and C on the relevant circumference, so arc BK is only a part of arc BC; a part cannot equal the whole.
14. Therefore the assumption that the central angles are unequal is false, and angle BGC equals angle EHF.

15. Now introduce the corresponding angles at the circumferences, angle BAC and angle EDF.
16. By Proposition III.20, each angle at the circumference is half the corresponding central angle.
17. Since angle BGC equals angle EHF, their halves are equal; therefore angle BAC equals angle EDF.
18. Thus equal circumferences in equal circles support equal angles both at the centers and at the circumferences.

Conclusion. Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

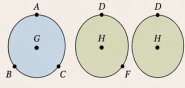


EUCLID'S ELEMENTS • BOOK III • PROPOSITION 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they stand at the center or at the circumference.

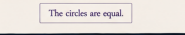
PANEL I — ACTIONS 1–9: ASSUME THE CENTRAL ANGLES ARE UNEQUAL

1 Draw two equal circles ABC and DEF .



The circles are equal.

2 Mark their centers G and H .



3 In the first circle, identify the circumference BC .



4 In the second circle, identify the circumference EF .



5 Join GB and GC to form the central angle BGC .



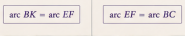
6 Join HE and HF to form the central angle EHF .



7 First goal: Prove that $\angle BGC = \angle EHF$.

Prove $\angle BGC = \angle EHF$

8 Suppose, for contradiction, the central angles are unequal. Suppose $\angle BGC > \angle EHF$.

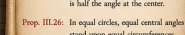


9 In $\angle BGC$, construct $\angle BGC'$ equal to $\angle EHF$. (K lies on the circumference.)



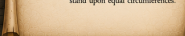
PANEL II — ACTIONS 10–18: CONTRADICTION AND THEN THE ANGLES AT THE CIRCUMFERENCE

10 $\angle BGC' = \angle EHF$. By Prop. III.26, equal central angles in equal circles stand upon equal arcs.



arc BK = arc EF

11 But from the hypothesis, arc EF = arc BC .



arc EF = arc BC

12 Hence, arc BK = arc BC .



arc BK = arc BC

13 But K lies between B and C . So arc BK is a part of arc BC .

A part equal to the whole is impossible.

14 Therefore our assumption is false. Hence, $\angle BGC = \angle EHF$.

$\angle BGC = \angle EHF$

15 Now consider the angles at the circumference: $\angle BAC$ and $\angle EDF$.



$\angle BAC = \frac{1}{2} \angle BGC$
 $\angle EDF = \frac{1}{2} \angle EHF$

16 By Prop. III.20, the angle at the circumference is half the angle at the center.

$\angle BAC = \frac{1}{2} \angle BGC$
 $\angle EDF = \frac{1}{2} \angle EHF$

17 Since $\angle BGC = \angle EHF$, their halves are equal. Therefore, $\angle BAC = \angle EDF$.



$\angle BAC = \angle EDF$

18 Conclusion: Equal circumferences in equal circles stand upon equal angles either at the centers or at the circumferences.

Q. E. D.

WHAT WE HAVE USED

Prop. III.20: The angle at the circumference is half the angle at the center.

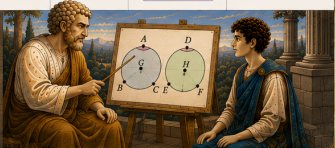
Prop. III.26: In equal circles, equal central angles stand upon equal circumferences.

FINAL CONCLUSION

In equal circles, angles standing upon equal circumferences are equal to one another, whether at the centers or at the circumferences.

KEY TO DIAGRAM

- Center of circle
- Radius
- Arc (circumference)
- Angle



94

Proposition 28

Statement. In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw two equal circles, ABC and DEF.
2. In the first circle draw chord AB.
3. In the second circle draw chord DE.
4. The chords are given equal: AB equals DE.
5. Mark the lesser arcs cut off by the chords, AGB and DHE.
6. Mark the corresponding greater arcs, ACB and DFE; the goal is to prove lesser equals lesser and greater equals greater.
7. Find the centers of the two circles, K and L.
8. Join the radii AK, KB, DL, and LE.
9. Because the circles are equal, their radii are equal: AK equals DL and KB equals LE. Together with AB equals DE, triangles AKB and DLE have all three corresponding sides equal.
10. Therefore, by Proposition I.8, angle AKB equals angle DLE.
11. These are equal central angles in equal circles.
12. By Proposition III.26, equal central angles in equal circles stand upon equal circumferences.
13. Hence the lesser arcs AGB and DHE are equal.
14. But the whole circles are also equal.
15. Remove the equal lesser arcs from the equal whole circles.

16. The remaining greater arcs ACB and DFE are equal.
17. Thus equal chords in equal circles cut off equal arcs in both senses: lesser to lesser and greater to greater.
18. Therefore AB equals DE implies arc AGB equals arc DHE and arc ACB equals arc DFE.

Conclusion. Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.



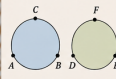
EUCLID'S ELEMENTS • BOOK III • PROPOSITION 28

In equal circles, equal straight-lines cut off equal circumferences: the greater circumference being equal to the greater, and the lesser to the lesser.

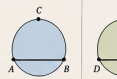


PANEL I — ACTIONS 1–9: SET UP AND SHOW EQUAL CHORDS GIVE EQUAL TRIANGLES

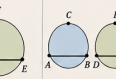
1 Draw two equal circles ABC and DEF .



2 In the first circle draw the chord AB .



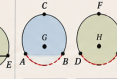
3 In the second circle draw the chord DE .



4 The chords are given equal.

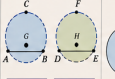
$AB = DE$

5 The chords cut off the lesser circumferences (minor arcs).



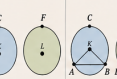
arc AGB
arc DHE

6 And the greater circumferences (major arcs).

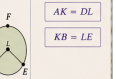


arc ACB
arc DFE

7 Let K and L be the centres of the circles.



8 Join the radii AK, KB, DL, LE .



9 Since the circles are equal, their radii are equal.

$AK = DL$
 $KB = LE$

PANEL II — ACTIONS 10–18: EQUAL CENTRAL ANGLES GIVE EQUAL ARCS

10 From Action 9 and the given $AB = DE$, triangles AKB and DLE have three pairs of equal sides.

$AK = DL$
 $KB = LE$
 $AB = DE$

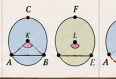
11 Therefore, by Proposition 1.8 (SSS),

$\angle AKB = \angle DLE$

and hence

$\angle AKB = \angle DLE$

12 These are equal central angles in equal circles.



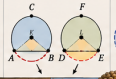
13 By Proposition III.26, equal central angles in equal circles stand upon equal circumferences.

$\text{arc } AGB = \text{arc } DHE$

14 But the whole circles are equal.

$\text{circle } ABC = \text{circle } DEF$

15 Remove the equal lesser circumferences from the equal whole circles.



16 The remaining circumferences (major arcs) must therefore be equal.

$\text{arc } ACB = \text{arc } DFE$

17 Thus, equal straight-lines cut off equal circumferences:

the greater
to the greater,
and the lesser
to the lesser.

18 That is,

$\text{arc } AGB = \text{arc } DHE$

and

$\text{arc } ACB = \text{arc } DFE$

SUMMARY OF THE PROOF

- Equal chords in equal circles give equal central angles (by SSS in the triangles formed by radii).
- Equal central angles in equal circles stand upon equal minor arcs (Prop. III.26).
- The remainders (major arcs) are therefore equal too, since the whole circles are equal.

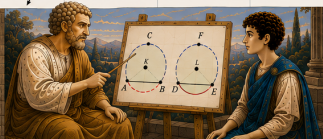
WHAT WE HAVE PROVED

$\text{arc } AGB = \text{arc } DHE$ (lesser arcs)
 $\text{arc } ACB = \text{arc } DFE$ (greater arcs)

Q.E.D.

KEY TO DIAGRAMS

- K, L centers of circles
- radii
- equal chords
- - - lesser (minor) arcs
- - - greater (major) arcs



Proposition 29

Statement. In equal circles, equal straight-lines subtend equal circumferences.

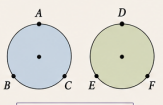
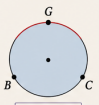
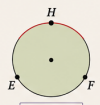
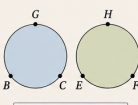
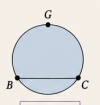
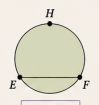
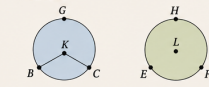
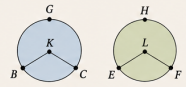
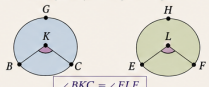
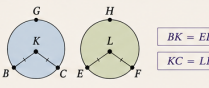
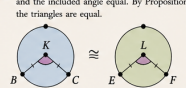
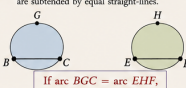
Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw two equal circles, ABC and DEF.
2. In the first circle mark the circumference BGC.
3. In the second circle mark the circumference EHF.
4. These circumferences are given equal: arc BGC equals arc EHF.
5. Join the endpoints of the first arc to form chord BC.
6. Join the endpoints of the second arc to form chord EF; the goal is to prove BC equals EF.
7. Find the centers of the two circles, K and L.
8. Join the radii BK, KC, EL, and LF.
9. Since the equal arcs BGC and EHF stand in equal circles, Proposition III.27 gives angle BKC equals angle ELF.
10. Because the two circles are equal, their radii are equal: BK equals EL and KC equals LF.
11. Thus triangles BKC and ELF have two corresponding sides and the included angle equal; by Proposition I.4 they are congruent, so BC equals EF.
12. Hence in equal circles, equal circumferences are subtended by equal straight-lines.

Conclusion. Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 29
In equal circles, equal circumferences are subtended by equal straight-lines.

1 Draw two equal circles ABC and DEF .  The circles are equal.	2 In the first circle, mark the circumference BGC .  arc BGC	3 In the second circle, mark the circumference EHF .  arc EHF	4 The circumferences are given equal.  arc $BGC = \text{arc } EHF$	5 Join the endpoints of the first arc.  chord BC	6 Join the endpoints of the second arc.  chord EF
To prove: $BC = EF$					
7 Let K and L be the centers of the circles.  $BK = EL$ $KC = LF$	8 Join the radii BK, KC, EL and LF .  $\angle BKC = \angle ELF$	9 Since arc $BGC = \text{arc } EHF$, by Proposition III.27, the central angles they stand upon are equal.  $\angle BKC = \angle ELF$			
10 The circles are equal, therefore their radii are equal.  $BK = EL$ $KC = LF$	11 Thus, triangles BKC and ELF have two sides equal and the included angle equal. By Proposition I.4, the triangles are equal.  Therefore their bases are equal. $BC = EF$		12 Hence, in equal circles, equal circumferences are subtended by equal straight-lines.  If arc $BGC = \text{arc } EHF$, then $BC = EF$.		

SUMMARY OF THE PROOF
 Equal arcs in equal circles $\rightarrow$ Equal central angles (Prop. III.27) $\rightarrow$ Two radii pairs equal (radii of equal circles) $\rightarrow$ Triangles equal (Prop. I.4) $\rightarrow$ Chords are equal $BC = EF$

Proposition 30

Statement. To cut a given circumference in half.

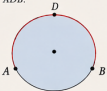
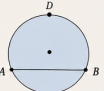
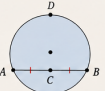
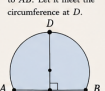
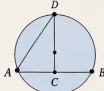
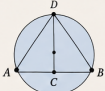
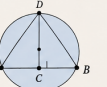
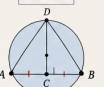
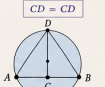
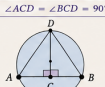
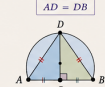
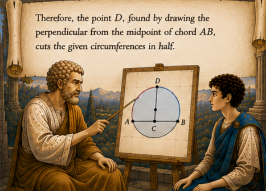
Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Begin with the given circumference ADB; the task is to divide it into two equal circumferences.
2. Join the endpoints A and B by the chord AB.
3. Bisect AB at C, so AC equals CB.
4. From C draw CD perpendicular to AB, meeting the given circumference at D.
5. Join AD.
6. Join DB.
7. Focus on triangles ACD and BCD.
8. Since C bisects AB, AC equals CB.
9. The side CD is common to both triangles.
10. Since CD is perpendicular to AB, angles ACD and BCD are both right angles.
11. Therefore, by Proposition I.4, triangles ACD and BCD are congruent.
12. Hence their corresponding bases are equal: AD equals DB.
13. The two equal chords AD and DB lie in the same circle; by Proposition III.28, equal straight-lines in equal circles cut off equal circumferences.
14. Therefore the corresponding lesser arcs AD and DB are equal.
15. Thus D bisects the given circumference ADB.

Conclusion. Thus, the given circumference has been cut in half at point D. (Which is) the very thing it was required to do.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 30
To cut a given circumference in half.

1 Given the circumference ADB.  To divide it into two equal circumferences.	2 Join the endpoints A and B. 	3 Bisect AB at C.  $AC = CB$	4 From C, draw CD perpendicular to AB. Let it meet the circumference at D. 	5 Join AD. 	6 Join DB.  We must prove that D divides the given circumference into two equal parts.
7 Consider the triangles ACD and BCD. 	8 Since C bisects AB, $AC = CB$.  $AC = CB$	9 CD is common to both triangles.  $CD = CD$	10 Since $CD \perp AB$, both angles at C are right angles.  $\angle ACD = \angle BCD = 90^\circ$	11 Therefore, by Proposition I.4, $\triangle ACD \cong \triangle BCD$.  $\triangle ACD \cong \triangle BCD$	12 Hence their corresponding bases are equal.  $AD = DB$
13 We have two equal chords of the same circle.  $AD = DB$ By Proposition III.28, equal straight-lines in equal circles cut off equal circumferences.	14 Therefore the circumferences cut off by them are equal. These are each less than a semicircle, so the lesser arcs are equal.  $\text{arc } AD = \text{arc } DB$	15 Thus D divides the given circumference ADB in half.  $\text{arc } AD = \text{arc } DB$ $\text{arc } ADB = \text{arc } AD + \text{arc } DB$	Therefore, the point D, found by drawing the perpendicular from the midpoint of chord AB, cuts the given circumference in half. 		

SUMMARY OF THE CONSTRUCTION AND PROOF

Bisect chord AB at C .	→	Draw $CD \perp AB$ to the circle.	→	Show $\triangle ACD \cong \triangle BCD$ ⇒ $AD = DB$.	→	Equal chords in the same circle cut off equal arcs.	→	Hence $\text{arc } AD = \text{arc } DB$. The circumference is cut in half.
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Proposition 31

Statement. In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser segment (is) greater than a right-angle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the angle of a segment less (than a semi-circle) is less than a right-angle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw circle ABCD.
2. Let BC be a diameter.
3. Let E be the center of the circle.
4. Join BA, AC, AD, and DC.
5. Join AE.
6. Produce BA beyond A to F, so B-A-F are collinear.
7. Since BE and EA are radii, BE equals EA; therefore triangle ABE is isosceles and angle ABE equals angle BAE.
8. Likewise CE equals EA, so triangle ACE is isosceles and angle ACE equals angle CAE.
9. Hence angle BAC equals angle ABC plus angle ACB; the exterior angle FAC also equals those same two interior opposite angles. Therefore angle BAC equals angle FAC, and since they are adjacent on the straight line BAF, each is a right angle.
10. Now consider angle ABC.
11. In triangle ABC, angle ABC plus angle BAC is less than two right angles.
12. But angle BAC is one right angle.
13. Therefore angle ABC is less than a right angle.

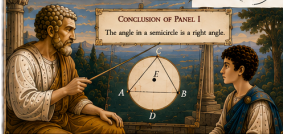
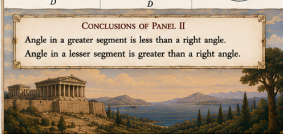
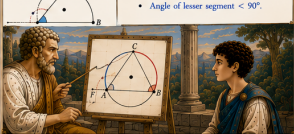
14. Angle ABC lies in the segment ABC, which is greater than a semicircle; hence an angle in a greater segment is less than a right angle.
15. Now consider cyclic quadrilateral ABCD.
16. By Proposition III.22, opposite angles ABC and ADC together equal two right angles.
17. Since angle ABC is less than a right angle, angle ADC is greater than a right angle.
18. Angle ADC lies in segment ADC, which is less than a semicircle; hence an angle in a lesser segment is greater than a right angle.
19. Consider the angle of the greater segment, the inclination between circumference ABC and straight-line AC.
20. The rectilinear angle BAC between BA and AC is a right angle.
21. But the circumference ABC lies beyond BA in forming the angle with AC.
22. Therefore the angle contained by circumference ABC and AC is greater than a right angle.
23. Now consider the angle of the lesser segment, the inclination between straight-line CA and circumference ADC.
24. Since BA has been produced to F, angle FAC is a right angle.
25. The arc ADC lies inside that right angle between CA and AF.
26. Therefore the angle contained by CA and circumference ADC is less than a right angle.
27. Thus the angle in a semicircle is right, the angle in a greater segment is less than right, the angle in a lesser segment is greater than right, the angle of a greater segment is greater than right, and the angle of a lesser segment is less than right.

Conclusion. And, further, the [angle] of a segment greater (than a semicircle) [is] greater than a right-angle, and the [angle] of a segment less (than a semicircle) is less than a right-angle. (Which is) the very thing it was required to show.

EUCLID'S ELEMENTS • BOOK III • PROPOSITION 31

In a circle, the angle in a semicircle is a right angle; the angle in a greater segment is less than a right angle; the angle in a lesser segment is greater than a right angle. And further, the angle of a segment greater than a semicircle is greater than a right angle, while the angle of a segment less than a semicircle is less than a right angle.

PANEL I — ACTIONS 1–9 THE ANGLE IN A SEMICIRCLE IS RIGHT			PANEL II — ACTIONS 10–18 GREATER AND LESSER SEGMENTS			PANEL III — ACTIONS 19–27 THE ANGLES OF THE SEGMENTS		
1 Draw circle ABCD.	2 Let BC be a diameter.	3 Let E be the center.	10 Consider $\angle ABC$.	11 In $\triangle ABC$, $\angle ABC + \angle BAC < 180^\circ$.	12 But $\angle BAC = 90^\circ$.	19 Consider the angle of the greater segment: the inclination between arc ABC and straight line AC.	20 The rectilinear angle between BA and AC is a right angle: $\angle BAC = 90^\circ$.	21 But arc ABC lies beyond the straight line BA in forming the angle with AC.
4 Join BA, AC, AD and DC.	5 Join AE.	6 Produce BA to F.	13 Therefore $\angle ABC < 90^\circ$.	14 $\angle ABC$ lies in segment ABC, which is greater than a semicircle.	15 In cyclic quadrilateral ABCD, opposite angles are supplementary (Prop. III.22).	22 Therefore the angle contained by arc ABC and AC is greater than a right angle: $\angle ACB < 90^\circ$.	23 Consider the angle of the lesser segment: the inclination between straight line CA and arc ADC.	24 Since BA has been produced to F, $\angle FAC = 90^\circ$.
7 $BE = EA$ (radii). Hence $\triangle ABE$ is isosceles, so $\angle ABE = \angle BAE$.	8 $CE = EA$ (radii). Hence $\triangle ACE$ is isosceles, so $\angle ACE = \angle CAE$.	9 Therefore $\angle BAC = \angle ABC + \angle ACB$. But exterior $\angle FAC = \angle ABC + \angle ACB$. Hence $\angle BAC = \angle FAC$. Since they are adjacent on straight line BAF, each is a right angle: $\angle BAC = 90^\circ$.	16 Thus $\angle ABC + \angle ADC = 180^\circ$.	17 Since $\angle ABC < 90^\circ$, therefore $\angle ADC > 90^\circ$.	18 $\angle ADC$ lies in segment ADC, which is less than a semicircle.	25 Therefore the angle contained by CA and arc ADC is less than a right angle: $\angle CAD < 90^\circ$.	26	27
CONCLUSION OF PANEL I The angle in a semicircle is a right angle.			CONCLUSIONS OF PANEL II Angle in a greater segment is less than a right angle. Angle in a lesser segment is greater than a right angle.			ALL FIVE CLAIMS ESTABLISHED - Angle in a semicircle = 90°. - Angle in a greater segment < 90°. - Angle in a lesser segment > 90°. - Angle of greater segment > 90°. - Angle of lesser segment < 90°.		

Proposition 32

Statement. If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw circle ABCD.
2. Let straight-line EF touch the circle at point B.
3. From B draw the chord BD through the circle.
4. The chord BD divides the circle into the two alternate segments BAD and DCB.
5. The goal is to prove angle DBF equals angle BAD and angle DBE equals angle DCB.
6. From B draw BA perpendicular to EF.
7. Choose any point C on the circumference BD in the opposite segment.
8. Join AD, DC, and CB.
9. Since EF is tangent at B and BA is perpendicular to EF, Proposition III.19 says the center lies on BA; hence BA is a diameter, and by Proposition III.31 angle ADB is a right angle.
10. In triangle ADB, since angle ADB is right, angles BAD and ABD together equal one right angle.
11. Because BA is perpendicular to EF, angle ABF is a right angle.
12. Therefore angle ABF equals angle BAD plus angle ABD.
13. Subtract the common angle ABD; hence angle DBF equals angle BAD.
14. Now consider cyclic quadrilateral ABCD.

15. By Proposition III.22, angle BAD plus angle BCD equals two right angles.
16. Since E, B, F are collinear, angle DBF plus angle DBE also equals two right angles.
17. Subtract the equal angles DBF and BAD from these equal sums; hence angle DBE equals angle BCD.
18. Thus each angle made by tangent EF and chord BD equals the angle in the alternate segment.

Conclusion. Thus, if some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle. (Which is) the very thing it was required to show.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 32

If a straight line touches a circle, and another straight line is drawn from the point of contact into the circle, cutting it, then the angles which it makes with the tangent are equal to the angles in the alternate segments of the circle.

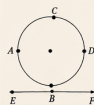


PANEL I — ACTIONS 1–9: PROVE $\angle DBF = \angle BAD$

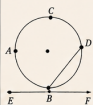
1 Draw circle $ABCD$.



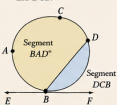
2 Let straight line EF touch the circle at B .



3 From B , draw the chord BD .



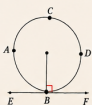
4 Chord BD divides the circle into the segments BAD and DCB .



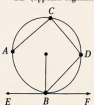
5 To prove:
 $\angle DBF = \angle BAD$
and $\angle DBE = \angle BCD$.

The angle between the tangent and the chord equals the angle in the alternate segment.

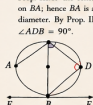
6 From B , draw BA perpendicular to EF .



7 Choose any point C on the circumference BD (opposite segment).



8 Since EF touches the circle at B and $BA \perp EF$, by Prop. III.9 the center lies on BA ; hence BA is a diameter. By Prop. III.31, $\angle ADB = 90^\circ$.



PANEL II — ACTIONS 10–18: COMPLETE THE PROOF

10 In $\triangle ADB$, since $\angle ADB = 90^\circ$, the remaining two angles sum to 90° .



$\angle BAD + \angle ABD = 90^\circ$

11 Since $BA \perp EF$, $\angle ABF = 90^\circ$.



$\angle ABF = 90^\circ$

12 Therefore $\angle ABF = \angle BAD + \angle ABD$.



$\angle ABF = \angle BAD + \angle ABD$

13 Subtract $\angle ABD$ from both sides. Hence $\angle DBF = \angle BAD$.



$\angle DBF = \angle BAD$

14 In cyclic quadrilateral $ABCD$, by Prop. III.22, opposite angles are supplementary.



$\angle BAD + \angle BCD = 180^\circ$

15 Since E, B, F are collinear (a straight line), the adjacent angles with common side BD sum to 180° .



$\angle DBF + \angle DBE = 180^\circ$

16 Therefore $\angle DBF + \angle DBE = \angle BAD + \angle BCD$.



$\angle DBF + \angle DBE = \angle BAD + \angle BCD$

17 But $\angle DBF = \angle BAD$ (from Action 13). Subtract these equal angles from both sides. Hence $\angle DBE = \angle BCD$.



$\angle DBE = \angle BCD$

18 Thus the angles which BD makes with the tangent at B are equal to the angles in the alternate segments of the circle.



$\angle DBE = \angle BAD$
and
 $\angle DBE = \angle BCD$

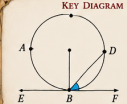
CONCLUSION

The angle between a tangent and a chord is equal to the angle in the alternate segment of the circle.



KEY DIAGRAM (for reference)

EF is tangent at B .
 BD is a chord through B .
 $\angle DBF$ equals the angle in segment BAD (e.g., $\angle BAD$).
 $\angle DBE$ equals the angle in segment DCB (e.g., $\angle BCD$).



Proposition 33

Statement. To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Begin with the given straight-line AB and the given acute rectilinear angle C .
2. At point A on AB , construct angle BAD equal to angle C .
3. Since angle C is acute, angle BAD is acute.
4. Draw AE perpendicular to AD .
5. Bisect AB at F , so AF equals FB .
6. From F draw FG perpendicular to AB .
7. Join GB .
8. In triangles AFG and BFG , AF equals FB , FG is common, and angles AFG and BFG are right; therefore AG equals BG .
9. With center G and radius GA , draw the circle through A and B ; let the circular segment be AEB .
10. Since AE is a diameter and AD is perpendicular to AE at endpoint A , AD is tangent to the circle at A .
11. Chord AB is drawn from the point of contact A .
12. By Proposition III.32, angle DAB equals the angle AEB in the alternate segment.
13. But by construction angle DAB equals angle C .
14. Therefore angle AEB equals angle C ; the required segment has been constructed for the acute case.
15. Now let the given angle C be a right angle.

16. Again construct at A the right angle BAD equal to angle C.
17. Bisect AB at F and, with center F and radius FA, draw the circle through A and B; AB is therefore a diameter.
18. The angle in semicircle AEB is a right angle, so angle AEB equals angle C; the construction works for the right-angle case.
19. Finally, let the given angle C be obtuse.
20. At A on AB construct angle BAD equal to angle C.
21. Draw AE perpendicular to AD.
22. Bisect AB at F, so AF equals FB.
23. From F draw FG perpendicular to AB and join GB.
24. As before, triangles AFG and BFG are congruent, so AG equals BG.
25. Draw the circle with center G and radius GA; it passes through both A and B.
26. Since AD is perpendicular to the diameter AE at endpoint A, AD is tangent at A; by Proposition III.32, angle BAD equals the angle in the alternate segment AHB.
27. But angle BAD equals angle C; therefore the angle in segment AHB equals angle C, completing the obtuse case.

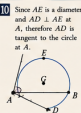
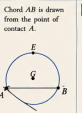
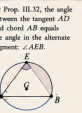
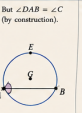
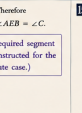
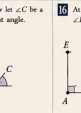
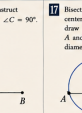
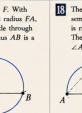
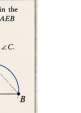
Conclusion. Thus, a segment AHB of a circle, accepting an angle equal to C, has been drawn on the given straight-line AB. (Which is) the very thing it was required to do.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 33

To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.

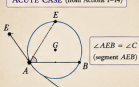


PANEL I — ACTIONS 1–9: ACUTE ANGLE CASE ($\angle C$ is acute)								
1 Given the straight line AB and the given acute angle C . 	2 At A , construct $\angle BAD = \angle C$. 	3 Since $\angle C$ is acute, $\angle BAD$ is acute. 	4 Draw AE perpendicular to AD . 	5 Bisect AB at F . So $AF = FB$. 	6 From F , draw FG perpendicular to AB . 	7 Join GB . 	8 In $\triangle AFG$ and $\triangle BFG$, $AF = FB$, $FG = FG$, and right angles are equal. Therefore $AG = BG$. 	9 With center G and radius GA , draw the circle through A and B . Let the segment be AEB . 
PANEL II — ACTIONS 10–18: COMPLETE ACUTE CASE, THEN RIGHT ANGLE CASE								
10 Since AE is a diameter and $AD \perp AE$ at A , therefore AD is tangent to the circle at A . 	11 Chord AB is drawn from the point of contact A . 	12 By Prop. III.32, the angle between the tangent AD and chord AB equals the angle in the alternate segment: $\angle AEB$. 	13 But $\angle DAB = \angle C$ (by construction). 	14 Therefore $\angle AEB = \angle C$. (Required segment constructed for the acute case.) 	15 Now let $\angle C$ be a right angle. 	16 At A , construct $\angle BAD = \angle C = 90^\circ$. 	17 Bisect AB at F . With center F and radius FA , draw the circle through A and B . Thus AB is a diameter. 	18 The angle in the semicircle AEB is right. Therefore $\angle AEB = \angle C$. 
PANEL III — ACTIONS 19–27: OBTUSE ANGLE CASE ($\angle C$ is obtuse)								
19 Let the given angle C be obtuse. 	20 At A , construct $\angle BAD = \angle C$. 	21 Draw AE perpendicular to AD . 	22 Bisect AB at F . So $AF = FB$. 	23 From F , draw FG perpendicular to AB , and join GB . 	24 In $\triangle AFG$ and $\triangle BFG$, $AF = FB$, $FG = FG$, and right angles are equal. Therefore $AG = BG$. 	25 With center G and radius GA , draw the circle through A and B . 	26 Since $AD \perp AE$ at A and AE is a diameter, (therefore AD) is tangent to the circle at A . 	27 By Prop. III.32, $\angle BAD$ equals the angle in the alternate segment AEB . But $\angle BAD = \angle C$. Therefore the angle in segment $AEB = \angle C$. (Required segment constructed for the obtuse case.) 

CONCLUSION

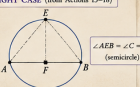
On the given straight line AB , we have constructed a segment of a circle such that the angle in that segment is equal to the given rectilinear angle C , in all three cases: (1) C acute, (2) C right, (3) C obtuse.

ACUTE CASE (from Actions 1–14)



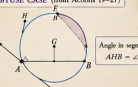
$\angle AEB = \angle C$
(segment AEB)

RIGHT CASE (from Actions 15–18)



$\angle AEB = \angle C = 90^\circ$
(semicircle)

OBTUSE CASE (from Actions 19–27)



Angle in segment $AEB = \angle C$

Thus Euclid accomplishes the construction in every case.

Proposition 34

Statement. To cut off a segment, accepting an angle equal to a given rectilinear angle, from a given circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Begin with the given circle ABC and the given rectilinear angle D.
2. The task is to cut off from circle ABC a segment which accepts an angle equal to angle D.
3. Choose point B on the circle.
4. Draw tangent EF to the circle at B.
5. At point B on tangent ray BF, construct angle FBC equal to the given angle D.
6. Let ray BC meet the circle again at C.
7. The chord BC now cuts off circular segment BAC.
8. Since EF is tangent at B and BC is drawn from the point of contact into the circle, Proposition III.32 applies.
9. Therefore angle FBC between the tangent and chord equals the angle in the alternate segment BAC.
10. But by construction angle FBC equals angle D.
11. Hence the angle in segment BAC equals the given rectilinear angle D.
12. Therefore segment BAC is the required segment.

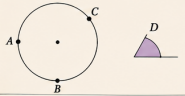
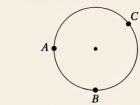
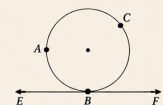
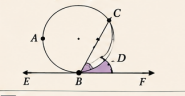
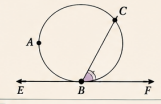
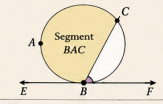
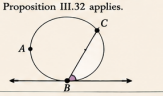
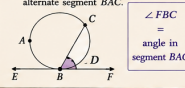
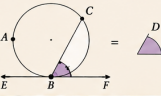
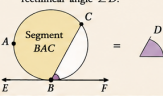
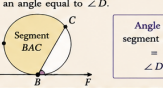
Conclusion. Thus, the segment BAC, accepting an angle equal to the given rectilinear angle D, has been cut off from the given circle ABC. (Which is) the very thing it was required to do.



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 34

To cut off from a given circle a segment accepting an angle equal to a given rectilinear angle.



1 Given the circle ABC and the given rectilinear angle D. 	2 The task is to cut off from circle ABC a segment which accepts an angle equal to $\angle D$. Required: a segment of the circle whose angle = $\angle D$.	3 Choose point B on the circle. 	4 Draw the tangent EF to the circle at B. 
5 At point B, on the tangent ray BF, construct the angle $\angle FBC$ equal to the given angle $\angle D$. 	6 Let the ray BC meet the circle again at C. 	7 The chord BC now cuts off the circular segment BAC. 	8 Since EF is tangent to the circle at B, and BC is drawn from the point of contact into the circle, Proposition III.32 applies. 
9 Therefore the angle between the tangent and chord, $\angle FBC$, is equal to the angle in the alternate segment BAC. $\angle FBC$ = angle in segment BAC 	10 But by construction, $\angle FBC = \angle D$. 	11 Hence the angle in segment BAC is also equal to the given rectilinear angle $\angle D$. 	12 Therefore the required segment has been cut off from the given circle: segment BAC, and it accepts an angle equal to $\angle D$. Angle in segment BAC = $\angle D$ 

Thus Euclid has cut off from the given circle a segment accepting an angle equal to the given rectilinear angle.

Key Idea:
Angle between tangent and chord
= angle in alternate segment (Prop. III.32).

Proposition 35

Statement. If two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw circle ABCD.
2. Draw straight-line AC through the circle.
3. Draw straight-line BD through the circle so that AC and BD intersect at E.
4. The goal is to prove AE multiplied by EC equals DE multiplied by EB.
5. First consider the special case where both AC and BD pass through the center.
6. Then their intersection point E is the center of the circle.
7. Therefore AE equals EC, because both are radii.
8. Likewise DE equals EB.
9. Since all four pieces are equal, AE multiplied by EC equals DE multiplied by EB.
10. Now suppose AC and BD do not pass through the center.
11. Find the center of the circle and call it F.
12. From F draw FG perpendicular to AC.
13. From F draw FH perpendicular to BD.
14. Join FB, FC, and FE.
15. Since FG passes through the center and is perpendicular to chord AC, Proposition III.3 gives AG equals GC.

16. Thus chord AC is cut equally at G and unequally at E; by Book II Proposition 5, AE multiplied by EC plus GE squared equals GC squared.
17. Add FG squared to both sides.
18. By Pythagoras, GE squared plus FG squared equals FE squared, and GC squared plus FG squared equals FC squared; therefore AE multiplied by EC plus FE squared equals FC squared.
19. Since FC and FB are radii, FC equals FB.
20. Hence FC squared equals FB squared, so AE multiplied by EC plus FE squared equals FB squared.
21. Now repeat the same argument for chord BD.
22. Since FH is perpendicular to BD and F is the center, DH equals HB.
23. Thus BD is bisected at H and cut unequally at E; by Book II Proposition 5, DE multiplied by EB plus EH squared equals HB squared.
24. Add FH squared to both sides and use Pythagoras: EH squared plus FH squared equals FE squared, and HB squared plus FH squared equals FB squared; hence DE multiplied by EB plus FE squared equals FB squared.
25. Therefore AE multiplied by EC plus FE squared equals DE multiplied by EB plus FE squared.
26. Subtract the common square FE squared from both sides.
27. Hence AE multiplied by EC equals DE multiplied by EB.

$$AE \cdot EC = DE \cdot EB$$



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 35

If two straight-lines in a circle cut one another, then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.



PANEL I — ACTIONS 1–9: STATEMENT AND THE CASE WHEN THE LINES PASS THROUGH THE CENTER									
1 Draw circle $ABCD$.	2 Draw straight line AC through the circle.	3 Draw straight line BD through the circle. Let the lines intersect at E .	4 To prove: $AE \cdot EC = DE \cdot EB$	5 First, suppose both AC and BD pass through the center. Then E is the center of the circle.	6 Therefore AE and EC are radii. So $AE = EC$.	7 And DE and EB are also radii. So $DE = EB$.	8 Thus all four pieces are equal.	9 Hence $AE \cdot EC = DE \cdot EB$	

PANEL II — ACTIONS 10–18: THE GENERAL CASE — CONVERT ONE PRODUCT INTO A RADIUS SQUARE									
10 Now suppose AC and BD do NOT pass through the center.	11 Let F be the center of the circle.	12 Draw $FG \perp AC$.	13 Draw $FH \perp BD$.	14 Join FB , FC , and FE .	15 Since FG is through the center and $\perp$ to chord AC , Prop. III.3 gives $AG = GC$.	16 So AC is bisected at G and cut unequally at E . By Prop. II.5, $AE \cdot EC + GE^2 = GC^2$.	17 Add FG^2 to both sides: $AE \cdot EC + GE^2 + FG^2 = GC^2 + FG^2$	18 By Pythagoras, $GE^2 + FG^2 = FE^2$ and $GC^2 + FG^2 = FC^2$. Therefore $AE \cdot EC + FE^2 = FC^2$	

PANEL III — ACTIONS 19–27: DO THE SAME FOR THE OTHER CHORD AND CONCLUDE									
19 Since FC and FB are radii of the same circle, $FC = FB$. Hence from Action 18, $AE \cdot EC + FE^2 = FB^2$	20 Therefore $FC^2 = FB^2$.	21 Now consider chord BD . Since $FH \perp BD$ and F is the center, Prop. III.3 gives $DH = HB$.	22 So BD is bisected at H and cut unequally at E . By Prop. II.5, $DE \cdot EB + EH^2 = HB^2$	23 Add FH^2 to both sides: $DE \cdot EB + EH^2 + FH^2 = HB^2 + FH^2$	24 By Pythagoras, $EH^2 + FH^2 = FE^2$ and $HB^2 + FH^2 = FB^2$. Therefore $DE \cdot EB + FE^2 = FB^2$	25 From Actions 20 and 24, $AE \cdot EC + FE^2 = FB^2$ and $DE \cdot EB + FE^2 = FB^2$. Thus $AE \cdot EC + FE^2 = DE \cdot EB + FE^2$	26 Subtract FE^2 from both sides. $AE \cdot EC = DE \cdot EB$	27 Therefore, whenever two straight-lines in a circle cut one another, $AE \cdot EC = DE \cdot EB$ Q.E.D.	

Thus the rectangle contained by the pieces of one line is equal to the rectangle contained by the pieces of the other.

In symbols:

 $AE \cdot EC = DE \cdot EB$

Key Propositions Used

III.3 — The straight line drawn from the center of a circle perpendicular to a chord, bisects the chord.

II.5 — If a straight line is drawn cutting another straight line, it makes the rectangle contained by the pieces of the one equal to the rectangle contained by the pieces of the other.

Key Idea

Drop perpendiculars from the center to the two intersecting chords. Using Props. III.3, II.3, and Pythagoras, both products become equal to the same radius squared after adding the common square FE^2 .

Proposition 36

Statement. If some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and the (other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line).

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw circle ABC.
2. Take a point D outside the circle.
3. From D draw a secant through the circle in the order D-C-A.
4. From D draw tangent DB touching the circle at B.
5. The goal is to prove AD multiplied by DC equals DB squared.
6. First suppose the secant DCA passes through the center of the circle.
7. Let F be the center, lying on A-F-C-D.
8. Join FB; since DB is tangent at B, FB is perpendicular to DB, so angle FBD is a right angle.
9. Since F is the center and AC is a diameter, AF equals FC; thus AC is bisected at F and the external part CD has been added to it.
10. By Book II Proposition 6, AD multiplied by DC plus FC squared equals FD squared.
11. Since FC and FB are radii, FC equals FB; therefore AD multiplied by DC plus FB squared equals FD squared.
12. Triangle FBD is right-angled at B.
13. By Proposition I.47, FD squared equals FB squared plus BD squared.

14. Hence AD multiplied by DC plus FB squared equals FB squared plus BD squared.
15. Subtract the common square FB squared from both sides.
16. Therefore AD multiplied by DC equals BD squared.
17. So Proposition 36 is proved when the secant passes through the center.
18. Now consider the general case where DCA does not pass through the center.
19. Let E be the center of circle ABC.
20. From E draw EF perpendicular to AC.
21. Join EB, EC, and ED; since DB is tangent at B, angle EBD is a right angle.
22. Because the perpendicular from the center to chord AC bisects it, AF equals FC; by Book II Proposition 6, AD multiplied by DC plus FC squared equals FD squared.
23. Add FE squared to both sides.
24. By Pythagoras, FC squared plus FE squared equals EC squared and FD squared plus FE squared equals ED squared; hence AD multiplied by DC plus EC squared equals ED squared.
25. Since EC and EB are radii, EC equals EB; thus AD multiplied by DC plus EB squared equals ED squared.
26. Triangle EBD is right-angled at B, so ED squared equals EB squared plus BD squared; subtract EB squared to obtain AD multiplied by DC equals BD squared.
27. Therefore, in both cases, the whole secant multiplied by its external part equals the square on the tangent.

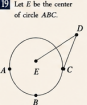
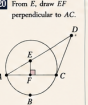
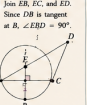
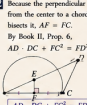
$$AD \cdot DC = DB^2$$



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 36

If some point is taken outside a circle, and two straight-lines are drawn from it toward the circle, one cutting the circle and the other touching it, then the rectangle contained by the whole secant and its external part is equal to the square on the tangent.



PANEL I — ACTIONS 1–9: SET UP THE THEOREM AND THE CENTRAL CASE									
1	Draw circle ABC.	2	Take a point D outside the circle.	3	From D, draw a secant through the circle: D – C – A.	4	From D, draw the tangent DB touching the circle at B.	5	To prove: $AD \cdot DC = DB^2$
   									
PANEL II — ACTIONS 10–18: COMPLETE THE CENTRAL CASE									
10	By Book II, Prop. 6, $AD \cdot DC + FC^2 = FD^2$.	11	Since FC and FB are radii, $FC = FB$. Therefore $AD \cdot DC + FB^2 = FD^2$.	12	Triangle FBD is right-angled at B.	13	By Prop. 147, $FD^2 = FB^2 + BD^2$.	14	Hence $AD \cdot DC + FB^2 = FB^2 + BD^2$.
   									
$AD \cdot DC + FC^2 = FD^2$		$AD \cdot DC + FB^2 = FD^2$		$FD^2 = FB^2 + BD^2$		$AD \cdot DC + FB^2 = FB^2 + BD^2$		$AD \cdot DC = BD^2$	
   									
15. Subtract the common square FB^2 from both sides. $AD \cdot DC = BD^2$ 16. Therefore, $AD \cdot DC = BD^2$. $AD \cdot DC = BD^2$ 17. So Prop. 36 is proved when the secant passes through the center. $AD \cdot DC = BD^2$ 18. Now consider the general case where DCA does NOT pass through the center. $AD \cdot DC = DB^2$									
PANEL III — ACTIONS 19–27: THE GENERAL CASE (THE SECANT DOES NOT PASS THROUGH THE CENTER)									
19	Let E be the center of circle ABC.	20	Join EB, EC, and ED. Since DB is tangent at B, $\angle EBD = 90^\circ$.	21	Join EB, EC, and ED. Since DB is tangent at B, $\angle EBD = 90^\circ$. By Book II, Prop. 6, $AD \cdot DC + FC^2 = FD^2$.	22	Because the perpendicular from the center to a chord bisects it, $AF = FC$. $AD \cdot DC + FC^2 = FD^2$.	23	Add FE^2 to both sides: $AD \cdot DC + FC^2 + FE^2 = FD^2 + FE^2$.
   									
$AD \cdot DC + FC^2 = FD^2$		$AD \cdot DC + FC^2 + FE^2 = FD^2 + FE^2$		$FC^2 + FE^2 = EC^2$ and $FD^2 + FE^2 = ED^2$ Hence $AD \cdot DC + EC^2 = ED^2$ $AD \cdot DC + EC^2 = ED^2$		$AD \cdot DC + EC^2 = ED^2$ $AD \cdot DC = ED^2 - EC^2$		$AD \cdot DC = EB^2$ $AD \cdot DC = ED^2$	
24. By Pythagoras, $FC^2 + FE^2 = EC^2$ and $FD^2 + FE^2 = ED^2$ Hence $AD \cdot DC + EC^2 = ED^2$ $AD \cdot DC + EC^2 = ED^2$ 25. Since EC and EB are radii, $EC = EB$. Thus $AD \cdot DC + EB^2 = ED^2$ $AD \cdot DC + EB^2 = ED^2$ $AD \cdot DC = ED^2 - EB^2$ 26. But $\angle EBD = 90^\circ$, so $ED^2 = EB^2 + BD^2$. Therefore $AD \cdot DC + EB^2 = EB^2 + BD^2$. Subtract EB^2. $AD \cdot DC = BD^2$ 27. Therefore, in other cases, $AD \cdot DC = DB^2$. (Proposition 36)									

THE THEOREM

Whole secant $\times$ external part
= (tangent)²

IN SYMBOLS

$AD \cdot DC = DB^2$

KEY PROPOSITIONS USED

II.6 If a straight line is drawn cutting another straight line (bisecting the chord), rectangle = square on half-chord = square on external part.

I.47 In a right-angled triangle, the square on the side subtending the right angle is equal to the sum of the squares on the other two sides.

Proposition 37

Statement. If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle.

Given / Construction. The complete labelled configuration and construction are preserved in the numbered actions below; every previously constructed object and incidence relation persists unless an action explicitly modifies it.

Proof Actions.

1. Draw circle ABC.
2. Take a point D outside the circle.
3. Draw secant D-C-A through the circle.
4. Draw another straight-line DB from D to the circle, meeting it at B.
5. We are given AD multiplied by DC equals DB squared; the goal is to prove that DB is tangent to the circle.
6. From the same external point D, construct an actual tangent DE to the circle.
7. Find the center of the circle and call it F.
8. Join FE, FB, and FD; since DE is tangent at E, angle FED is a right angle.
9. Apply Proposition III.36 to tangent DE and secant DCA: AD multiplied by DC equals DE squared. But AD multiplied by DC also equals DB squared; therefore DE squared equals DB squared and hence DE equals DB.
10. Since E and B lie on the same circle with center F, FE equals FB.
11. Also FD is common to triangles DEF and DBF.
12. Thus DE equals DB, FE equals FB, and FD equals FD.

13. Therefore, by Proposition I.8, triangles DEF and DBF are congruent.
14. Hence the corresponding angles DEF and DBF are equal.
15. But angle DEF is a right angle because DE is tangent at E; therefore angle DBF is also a right angle.
16. So DB is perpendicular to FB.
17. Since FB is a radius drawn to point B and DB is perpendicular to that radius at its endpoint, Proposition III.16 corollary gives that DB touches the circle at B.
18. Therefore AD multiplied by DC equals DB squared implies that DB is tangent to circle ABC.

$AD \cdot DC = DB^2 \implies DB \text{ is tangent to the circle at } B$



EUCLID'S ELEMENTS • BOOK III • PROPOSITION 37

If a point is taken outside a circle, and from that point two straight-lines are drawn toward the circle, one cutting the circle and the other merely meeting it, and if the rectangle contained by the whole secant and its external part is equal to the square on the line which meets the circle, then that line will touch the circle.



PANEL I — ACTIONS 1–9: CONSTRUCT A TANGENT AND COMPARE LENGTHS								
1 Draw circle ABC. 	2 Take a point D outside the circle. 	3 Draw the secant D - C - A. 	4 Draw the line DB meeting the circle at B. 	5 Given: $AD \cdot DC = DB^2$ To prove: DB is tangent to the circle.	6 From D, construct a tangent DE to the circle. 	7 Let F be the center of the circle. 	8 Join FE, FB, and FD. 	9 By Prop. III.36, $AD \cdot DC = DE^2$. But given, $AD \cdot DC = DB^2$. Therefore $DE^2 = DB^2$ Hence $DE = DB$.
PANEL II — ACTIONS 10–18: FORCE DB TO BE PERPENDICULAR TO THE RADIUS								
10 Since E and B are points on the same circle with center F,  $FE = FB$	11 Also,  $FD = FD$ (common).	12 Thus we have: $DE = DB$ $FE = FB$ $FD = FD$	13 Therefore, by Prop. I.8, $\triangle DEF \cong \triangle DBF$. 	14 Hence corresponding angles are equal: $\angle DEF = \angle DBF$	15 But $\angle DEF = 90^\circ$ (tangent $\perp$ radius). Therefore $\angle DBF = 90^\circ$	16 So $DB \perp FB$. 	17 Since FB is a radius, and DB is perpendicular to the radius at B, by Prop. III.16 (cor.), DB touches the circle at B.	18 Therefore, from $AD \cdot DC = DB^2$, it follows that DB is tangent to the circle at B. Q.E.D.

THE THEOREM

If from a point outside a circle a secant and a line meeting the circle are drawn, and if (whole secant) $\times$ (external part) = square on the other line, then the line is tangent.

IN SYMBOLS

If $AD \cdot DC = DB^2$,
then DB touches circle ABC at B.

KEY PROPOSITIONS USED

I.8 If in two triangles two sides and the included angle of one are equal to two sides and the included angle of the other, the triangles are congruent.

III.16 If a straight line is drawn from a point outside a circle, touching the circle, then it is perpendicular to the radius drawn to the point of contact.

III.36 (Tangent-Secant Theorem) In a circle, the square on the tangent equals the rectangle contained by the whole secant and its external part.

Conclusion

Book III has established the principal elementary relations of the circle, culminating in the power relation for a tangent and secant and its converse criterion for tangency.