

EUCLID'S ELEMENTS

BOOK II

THE FUNDAMENTALS OF GEOMETRIC ALGEBRA

Illustrated Edition

Definitions 1–2 and Propositions 1–14

Contents

Introduction

Book II develops Euclid's geometric treatment of relations which, in modern algebraic notation, resemble identities involving products and squares. Euclid does not use symbolic algebra. Instead, rectangles and squares are constructed and compared as geometric magnitudes.

The book begins with two definitions concerning rectangles and gnomons. Propositions 1–10 establish area identities through the dissection and recombination of rectangles, squares, and gnomons. Proposition 11 constructs the division later called the Golden Section. Propositions 12 and 13 give the obtuse- and acute-angle forms underlying the law of cosines. Proposition 14 concludes the book by constructing a square equal in area to an arbitrary rectilinear figure.

The numbered proof actions below correspond to the cinematic action sequences already illustrated for this edition.

Definitions

Definition 1

Any rectangular parallelogram is said to be contained by the two straight lines containing the right angle.

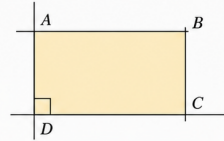
In other words, if two adjacent sides of a rectangle have lengths represented by two straight lines, the rectangle is described as the rectangle contained by those two straight lines.

BOOK II – DEFINITIONS

DEFINITION 1

ΠΑΡΑΛΛΗΛΟΓΡΑΜΜΟΝ ἔστιν ὀρθογωνίῳ ἐμπεριεχόμενον ὑπὸ τῶν τὴν ὀρθὴν γωνίαν ἔχουσῶν εὐθειῶν.

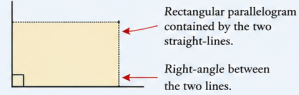
Any rectangular parallelogram is said to be **contained by** the two straight-lines containing the right-angle.



Rectangle $ABCD$ is contained by the two straight-lines AD and DC which contain the right-angle at D .

GEOMETRIC MEANING:

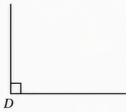
A rectangular parallelogram (a rectangle) is said to be *contained* by two straight-lines that meet at a right-angle if the rectangle lies in the region formed by those two lines and has its sides along them.



CINEMATIC ILLUSTRATION OF DEFINITION 1

Step-by-step geometric actions

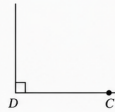
- 1 Draw two straight-lines AD and DC so that they meet at point D at a right-angle.



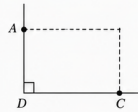
- 2 Take any length along AD . Mark point A on AD .



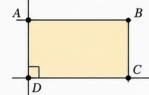
- 3 Take any length along DC . Mark point C on DC .



- 4 Through A draw a line parallel to DC . Through C draw a line parallel to AD .



- 5 Let the two parallels meet at B . Then $ABCD$ is a rectangular parallelogram contained by the two straight-lines AD and DC .



Thus, the rectangle $ABCD$ lies within (is **contained by**) the two straight-lines AD and DC which contain the right-angle at D .

Definition 2

In any parallelogrammic figure, let any one of the parallelograms about its diagonal, together with its two complements, be called a *gnomon*.

For the construction used in this edition, begin with parallelogram $ABCD$ and diagonal BD . Choose an internal point E on BD . Through E , draw $EF \parallel AD$ meeting AB at F , and draw $EG \parallel DC$ meeting BC at G . Then $FEGB$ is an internal parallelogram about the diagonal BD . Subtracting $FEGB$ from the original parallelogram $ABCD$ leaves the L-shaped remainder called the gnomon.

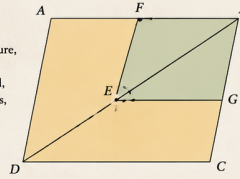


BOOK II – DEFINITIONS

DEFINITION 2

And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a **gnomon**.

A **gnomon** is formed by taking one of the parallelograms that has the diagonal of the given figure as a side and combining it with the two remaining parts (complements) of the original figure.

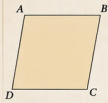


■ Parallelogram *FEGB*
■ Two complements

Parallelogram *FEGB* about diagonal *DB*, taken with its two complements, is a **gnomon**.

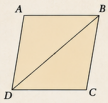
HOW TO FORM A GNOMON IN A PARALLELOGRAMMIC FIGURE

- 1 Start with any parallelogrammic figure *ABCD*.



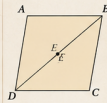
This is our given parallelogrammic figure.

- 2 Draw a diagonal *BD*.



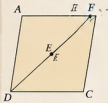
The diagonal *DB* will serve as a side of the inner parallelogram about the diagonal.

- 3 Choose an internal point on diagonal *BD*, call it *E*.



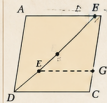
Point *E* lies on the diagonal *DB*.

- 4 From point *E* bring a parallel to *AD* which meets *AB* at *F*.



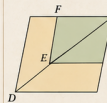
Line through *E* parallel to *AD* meets *AB* at *F*.

- 5 From point *E* bring a parallel to *DC* which meets *BC* at *G*.



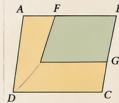
Line through *E* parallel to *DC* meets *BC* at *G*.

- 6 You formed internal parallelogram *FEGB*.



Parallelogram *FEGB* is the parallelogram about diagonal *DB*.

- 7 Subtract *FEGB* from *ABCD*. What remains is the **gnomon**.



The two complements together with *FEGB* form a **gnomon**.



The name "**gnomon**" was first used for an L-shaped figure or a right-angle instrument, because a *gnomon* is like that L-shape made of the inner parallelogram and the two complements. It is the "increment" between two successive figures about the same diagonal.



Proposition 1

Statement.

If there are two straight lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the uncut straight line and every one of the pieces.

For the figure used here,

$$(A, BC) = (A, BD) + (A, DE) + (A, EC).$$

Given / Construction.

The straight line BC is cut at D and E , while the straight line A remains uncut.

Proof Actions.

1. Take the two given straight lines A and BC .
2. Cut BC at D and E , so that $B - D - E - C$.
3. State the goal:

$$(A, BC) = (A, BD) + (A, DE) + (A, EC).$$

4. From B , draw $BF \perp BC$.
5. On BF , choose G such that $BG = A$.
6. Through G , draw $GH \parallel BC$, and through C , draw $CH \parallel BG$, completing the large rectangle.
7. Through D and E , draw lines parallel to BG , partitioning the large rectangle into three smaller rectangles.
8. The whole rectangle is exactly the sum of the three smaller rectangles:

$$BH = BK + DL + EH.$$

9. Since $BG = A$, identify

$$BH = (A, BC).$$

10. Identify

$$BK = (A, BD).$$

11. Identify

$$DL = (A, DE).$$

12. Identify

$$EH = (A, EC).$$

13. Substitute these identities into $BH = BK + DL + EH$, obtaining

$$(A, BC') = (A, BD) + (A, DE) + (A, EC).$$

Conclusion. The rectangle contained by the two original straight lines equals the sum of the rectangles contained by the uncut line and each piece of the cut line.

EUCLID'S ELEMENTS — BOOK II

BOOK II – PROPOSITION 1

IF THERE ARE TWO STRAIGHT LINES, AND ONE OF THEM IS CUT INTO ANY NUMBER OF PIECES, THEN THE RECTANGLE CONTAINED BY THE TWO STRAIGHT LINES IS EQUAL TO THE SUM OF THE RECTANGLES CONTAINED BY THE UNCUT LINE AND EACH PIECE.

To prove:
 $(A, BC) = (A, BD) + (A, DE) + (A, EC)$

1 Take the two straight lines.

Let the straight line A remain uncut, and the straight line BC.

2 Cut BC into any number of pieces.

Choose points D and E on BC. Thus BC is cut into BD, DE, and EC.

3 State what must be proved.

We must prove that $(A, BC) = (A, BD) + (A, DE) + (A, EC)$.

4 Raise a perpendicular at B.

From B draw BF perpendicular to BC (Proposition I.11).

5 Copy the length A onto BF.

On BF take BG = A (Proposition I.3).

6 Complete the large rectangle.

Through G draw GH parallel to BC. Through C draw CH parallel to BG. Rectangle BGCH is formed. Its area is (A, BC) .

7 Raise parallels through D and E.

Through D draw DK $\parallel$ BG. Through E draw EL $\parallel$ BG. The rectangle is divided into three rectangles.

8 Decompose the large rectangle.

The whole rectangle BH is exactly the sum of the three rectangles BK, DL, and EH. So, $BH = BK + DL + EH$.

9 Identify the large rectangle.

BH has sides BG and BC. But BG = A. Therefore $BH = (A, BC)$.

10 Identify the first small rectangle.

BK has dimensions BG $\times$ BD. Since BG = A, therefore $BK = (A, BD)$.

11 Identify the second small rectangle.

DL has dimensions DK $\times$ DE. But DK = BG = A. Therefore $DL = (A, DE)$.

12 Identify the third small rectangle.

EH has dimensions BG $\times$ EC. Since BG = A, therefore $EH = (A, EC)$.

13 Conclude.

From 8, 9, 10, 11, and 12,
 $BH = BK + DL + EH$
 becomes
 $(A, BC) = (A, BD) + (A, DE) + (A, EC)$
 Q.E.D.

The single rectangle formed by the whole line BC and the line A is equal in area to the sum of the rectangles formed by A with each piece of BC.

Thus Proposition 1 is proved.

→

$A \cdot BC = A \cdot BD + A \cdot DE + A \cdot EC$

Proposition 2

Statement.

If a straight line is cut at random, then the sum of the rectangles contained by the whole straight line and each of the pieces is equal to the square on the whole.

For the figure,

$$(AB, BC) + (BA, AC) = AB^2.$$

Given / Construction.

The straight line AB is cut at the arbitrary point C .

Proof Actions.

1. Take the straight line AB .
2. Cut AB at an arbitrary point C , so that $A - C - B$.
3. State the goal:

$$(AB, BC) + (BA, AC) = AB^2.$$

4. Construct the square $ADEB$ on AB .
5. Through C , draw $CF \parallel AD \parallel BE$.
6. The square AE is partitioned into the two rectangles AF and CE :

$$AE = AF + CE.$$

7. Since AE is the square on AB ,

$$AE = AB^2.$$

8. Since $AD = AB = BA$,

$$AF = (BA, AC).$$

9. Since $BE = AB$,

$$CE = (AB, BC).$$

10. Substitute:

$$AB^2 = (BA, AC) + (AB, BC).$$

11. Therefore the rectangles contained by the whole line and its two parts together equal the square on the whole line.

Conclusion.

$$(BA, AC) + (AB, BC) = AB^2.$$

BOOK II — PROPOSITION 2

If a straight line is cut at random, then the sum of the rectangles contained by the whole straight line and each of its two parts is equal to the square on the whole.

To prove:

$(AB, BC) + (BA, AC) = AB^2$

1 Take the straight line AB .

This is the given straight line AB .

2 Cut AB at an arbitrary point C .

Point C is chosen at random on AB . Thus AB is divided into AC and CB .

3 State what must be proved.

We must prove that the sum of the rectangles contained by the whole line AB and each part is equal to the square on AB .

$(AB, BC) + (BA, AC) = AB^2$

4 Construct the square on AB .

With AB as side, construct square $ADEB$ (Proposition I.46). So $AD = AB = BE$.

5 Draw through C a line parallel to AD and BE .

Through C draw CF parallel to AD and BE . It cuts the square into two rectangles.

6 Recognize the partition of the square.

The square AE is composed of two rectangles AF and CE . Therefore, $AE = AF + CE$.

7 Identify the whole figure AE .

Figure AE is the square on AB .

Therefore, $AE = AB^2$.

8 Identify rectangle AF .

Rectangle AF has sides AD and AC . But $AD = AB = BA$. Hence, $AF = (BA, AC)$.

9 Identify rectangle CE .

Rectangle CE has sides BE and BC . But $BE = AB$. Hence, $CE = (AB, BC)$.

10 Substitute the identities.

From 6, $AE = AF + CE$.

From 7, 8, and 9,

$AB^2 = (BA, AC) + (AB, BC)$.

11 Conclude the proposition.

Therefore, if a straight line is cut at random, the sum of the rectangles contained by the whole line and each of its two parts is equal to the square on the whole line.

Q.E.D.

Thus the rectangle formed by the whole line AB with BC , together with the rectangle formed by the whole line AB with AC , is equal in area to the square on AB .

(BA, AC)

+

(AB, BC)

=

AB^2

Proposition 3

Statement.

If a straight line is cut at random, then the rectangle contained by the whole straight line and one of the pieces is equal to the rectangle contained by the two pieces together with the square on that same piece.

For Euclid's figure,

$$(AB, BC) = (AC, CB) + BC^2.$$

Given / Construction.

The straight line AB is cut at C .

Proof Actions.

1. Take the straight line AB .
2. Cut AB at the arbitrary point C , so that $A - C - B$.
3. State the goal:

$$(AB, BC) = (AC, CB) + BC^2.$$

4. Construct the square $CDEB$ on CB .
5. Produce ED beyond D to a point F .
6. Through A , draw $AF \parallel CD \parallel BE$, completing the larger rectangle.
7. Identify the large rectangle:

$$AE = (AB, BC).$$

8. Identify the left rectangle:

$$AD = (AC, CB).$$

9. Identify the right-hand square:

$$DB = BC^2.$$

10. Decompose the large rectangle:

$$AE = AD + DB.$$

11. Substitute the identities:

$$(AB, BC) = (AC, CB) + BC^2.$$

12. Conclude the proposition.

Conclusion.

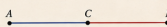
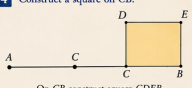
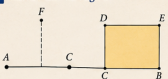
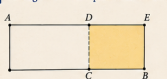
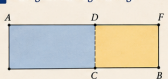
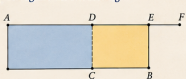
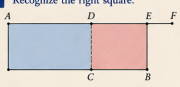
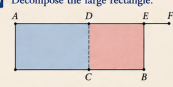
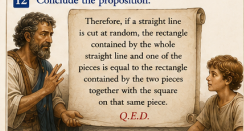
$$(AB, BC) = (AC, CB) + BC^2.$$



BOOK II – PROPOSITION 3

If a straight line is cut at random, then the rectangle contained by the whole straight line and one of the pieces is equal to the rectangle contained by the two pieces together with the square on that same piece.

To prove:
 $(AB, BC) = (AC, CB) + BC^2$

1 Take the straight line AB.  This is the given straight line AB.	2 Cut AB at an arbitrary point C.  Point C is chosen on AB. Thus AB is divided into AC and CB.	3 State what must be proved. We must prove that $(AB, BC) = (AC, CB) + BC^2.$	4 Construct a square on CB.  On CB construct square $CDEB$. Thus $CD = CB = BE$.
5 Extend ED through D.  Produce side ED beyond D to point F. So $E-D-F$ are in a straight line.	6 Through A draw AF parallel to CD and BE.  Draw AF parallel to CD and BE. This completes the figure.	7 Recognize the large rectangle.  Figure AE is a rectangle with sides AB and BE. But $BE = BC$. Therefore, $AE = (AB, BC)$.	8 Recognize the left rectangle.  Rectangle AD has sides AC and CD. But $CD = CB$. Therefore, $AD = (AC, CB)$.
9 Recognize the right square.  Figure DB is the square on CB. Therefore, $DB = BC^2$.	10 Decompose the large rectangle.  The whole rectangle AE is made up of the two figures AD and DB. Therefore, $AE = AD + DB$.	11 Substitute the identities. From 7, 8, and 9 $\begin{aligned} AE &= (AB, BC) \\ AD &= (AC, CB) \\ DB &= BC^2 \end{aligned}$ Hence, $(AB, BC) = (AC, CB) + BC^2.$	12 Conclude the proposition.  Therefore, if a straight line is cut at random, the rectangle contained by the whole straight line and one of the pieces is equal to the rectangle contained by the two pieces together with the square on that same piece. <i>Q.E.D.</i>

Thus the rectangle formed by the whole line AB with BC is equal to the rectangle formed by AC with CB together with the square on CB .

(AB, BC)

$=$

(AC, CB)

$+$

BC^2



Proposition 4

Statement.

If a straight line is cut at random, then the square on the whole straight line is equal to the sum of the squares on the pieces and twice the rectangle contained by the pieces.

$$AB^2 = AC^2 + CB^2 + 2(AC)(CB).$$

Given / Construction.

The straight line AB is cut at C .

Proof Actions.

1. Take the straight line AB .
2. Cut AB at the arbitrary point C .
3. State the goal:

$$AB^2 = AC^2 + CB^2 + 2(AC)(CB).$$

4. Construct square $ADEB$ on AB .
5. Join the diagonal BD .
6. Through C , draw $CF \parallel AD \parallel EB$, meeting BD at G .
7. Through G , draw $HK \parallel AB \parallel DE$.
8. Since $CF \parallel AD$,

$$\angle CGB = \angle ADB.$$

9. Since $AB = AD$, triangle ABD is isosceles, so

$$\angle ADB = \angle ABD.$$

10. Therefore $\angle CGB = \angle GBC$, and hence

$$BC = CG.$$

11. Using opposite sides of the constructed parallelograms, deduce

$$CB = CG = GK = KB.$$

Thus $CGKB$ is equilateral.

12. Show $CGKB$ is right-angled.

13. Therefore $CGKB$ is the square on CB :

$$CGKB = CB^2.$$

14. By the same reasoning, the other internal region HF is the square on AC :

$$HF = AC^2.$$

15. Rectangle AG has sides AC and $CG = CB$, hence

$$AG = (AC, CB).$$

16. By Proposition I.43, the complementary rectangle GE equals AG :

$$AG = GE.$$

17. Therefore

$$AG + GE = 2(AC, CB).$$

18. The four regions HF , CK , AG , and GE reassemble the whole square $ADEB$.

19. Since $ADEB = AB^2$,

$$AB^2 = AC^2 + CB^2 + 2(AC)(CB).$$

20. Conclude the proposition.

Conclusion.

$$AB^2 = AC^2 + CB^2 + 2(AC)(CB).$$



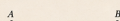
BOOK II – PROPOSITION 4

If a straight line is cut at random,
the square on the whole straight line is equal to the sum of the
squares on the two parts together with twice the rectangle
contained by the two parts.

To prove:

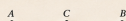
$$AB^2 = AC^2 + CB^2 + 2(AC)(CB)$$

1 Take the straight line AB .



This is the given
straight line AB .

2 Cut AB at an arbitrary point C .



Thus AB is divided into
 AC and CB .

3 State what must be proved.

We must prove that

$$AB^2 = AC^2 + CB^2 + 2(AC)(CB)$$

4 Construct the square on AB .



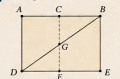
Construct square $ADEB$ on AB .
So $AB = AD = DE = EB$.

5 Draw the diagonal BD .



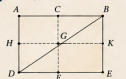
Join B to D .

6 Through C draw $CF \parallel AD$, EB .



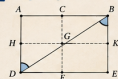
Let it meet BD at G and DE at F .

7 Through G draw $HK \parallel AD$, DE .



A grid of four regions is formed.

8 Compare angles CBG and ADB .



Since $CF \parallel AD$ and BD is a transversal,
 $\angle CGB = \angle ADB$ (Prop. 1.29).

9 In isosceles $\triangle ABD$,
 $\angle ADB = \angle ABD$.



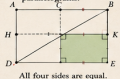
Because $AB = AD$ (in the square),
base angles are equal (Prop. 1.5).

10 Hence $BC = CG$.



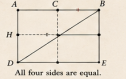
Thus $\angle CGB = \angle GBC$.
So $BC = CG$ (Prop. 1.6).

11 Use opposite sides of
parallelograms.



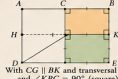
All four sides are equal.

12 $CGKB$ is equilateral.



All four sides are equal.

13 $CGKB$ is right-angled.



With $CG \parallel BK$ and transversal CB ,
and $\angle KBE = 90^\circ$ (square),
so $\angle BCG = 90^\circ$.

14 Thus $CGKB$ is the square on CB .



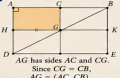
Equilateral and right-angled
"square".
Hence $CGKB = CB^2$.

15 The other inner figure HF
is the square on AC .



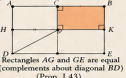
Similarly, HF is square on AC .
Hence $HF = AC^2$.

16 Identify rectangle AG .



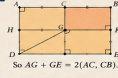
AG has sides AC and CG .
Since $CG = CB$,
 $AG = (AC)(CB)$.

17 Show rectangle $GE = AG$.



Rectangles AG and GE are equal
(complements about diagonal BD)
(Prop. 1.43).

18 Twice the rectangle $AC \times CB$.



So $AG + GE = 2(AC)(CB)$.

19 Reassemble the square.



Square $ADEB$ is the sum of the four
regions: $HF + CG + AG + GE$
 $= AC^2 + CB^2 + 2(AC)(CB)$.

20 Conclude the proposition.



Therefore,
 $AB^2 = AC^2 + CB^2 + 2(AC)(CB)$.
Q.E.D.



Thus the square on the whole straight line AB
is equal to the sum of the squares on the two parts
together with twice the rectangle contained by the two parts.

$$AB^2 = AC^2 + CB^2 + 2 \times (AC)(CB)$$



Proposition 5

Statement.

If a straight line is cut into equal and unequal pieces, then the rectangle contained by the unequal pieces of the whole straight line, together with the square on the difference between the equal and unequal divisions, is equal to the square on half the straight line.

For Euclid's figure,

$$(AD, DB) + CD^2 = CB^2.$$

Given / Construction.

The straight line AB is bisected at C and cut unequally at D .

Proof Actions.

1. Take the straight line AB .
2. Bisect AB at C , so that $AC = CB$.
3. Make a second, unequal cut at D , with $A - C - D - B$.
4. State the goal:

$$(AD, DB) + CD^2 = CB^2.$$

5. Construct the square $CEFB$ on CB .
6. Draw diagonal BE .
7. Through D , draw $DG \parallel CE \parallel BF$, meeting the diagonal.
8. Through the intersection point on the diagonal, draw the corresponding horizontal parallel.
9. Through A , complete the extended rectangular construction.
10. By Proposition I.43, the complements CH and HF are equal:

$$CH = HF.$$

11. Add the same region DM to both, obtaining

$$CM = DF.$$

12. Since $AC = CB$, the corresponding rectangles are equal:

$$CM = AL.$$

13. Hence

$$AL = DF.$$

14. Add the common region CH , obtaining

$$AH = \text{gnomon } NOP.$$

15. Identify

$$AH = (AD, DB),$$

so

$$\text{gnomon } NOP = (AD, DB).$$

16. Add the square $LG = CD^2$.

17. Recognize that the gnomon NOP together with LG restores the entire square $CEFB$.

18. Since $CEFB$ is the square on CB ,

$$(AD, DB) + CD^2 = CB^2.$$

19. Conclude the proposition.

Conclusion.

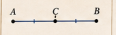
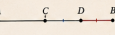
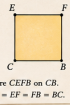
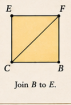
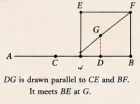
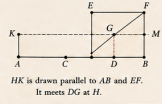
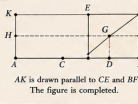
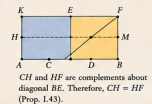
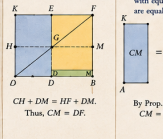
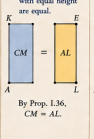
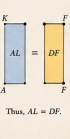
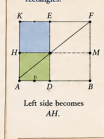
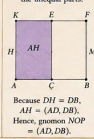
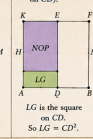
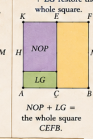
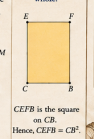
$$(AD, DB) + CD^2 = CB^2.$$



BOOK II — PROPOSITION 5

If a straight line is cut equally and unequally, then the rectangle contained by the two unequal parts, together with the square on the distance between the two cuts, equals the square on half the original line.

To prove:
 $(AD, DB) + CD^2 = CB^2$

1 Take the straight line AB.  This is the given straight line AB.	2 Bisect AB at C.  $AC = CB$. C is the equal division of AB.	3 Make an unequal cut at D.  D is distinct from C. AB is cut equally at C and unequally at D. We have $AC = CB$.	4 State what must be proved. We must prove that $(AD, DB) + CD^2 = CB^2$. Rectangle on unequal parts + square on CD = square on half the line.	5 Construct the square on CB.  Square CEFB on CB. So $CE = EF = FB = BC$.	6 Draw the diagonal BE.  Join B to E.
7 Through D draw DG CE, BE.  DG is drawn parallel to CE and BE. It meets BE at G.	8 Through G draw KM AB, EF.  HK is drawn parallel to AB and EF. It meets DG at H.	9 Through A draw AK CE, BF.  AK is drawn parallel to CE and BF. The figure is completed.	10 The two complements are equal.  CH and HF are complements about diagonal BE. Therefore, $CH = HF$ (Prop. I.43).		
11 Add the same rectangle DM to both complements.  $CH + DM = HF + DM$. Thus, $CM = DF$.	12 Since $AC = CB$, rectangles on them with equal height are equal.  By Prop. I.36, $CM = AL$.	13 Therefore $AL = DF$.  Thus, $AL = DF$.	THE CLIMAX ACTIONS 14-16 14 Add CH to both equal rectangles.  Left side becomes AH.		
15 AH is the rectangle on the unequal parts.  Because $DH = DB$, $AH = (AD, DB)$. Hence, gnomon NOP = (AD, DB).					16 Add LG (the square on CD).  LG is the square on CD. So $LG = CD^2$.
17 Gnomon NOP + LG restore the whole square.  $NOP + LG =$ the whole square CEFB.					18 Identify those whole.  CEFB is the square on CB. Hence, $CEFB = CB^2$.
Therefore, $(AD, DB) + CD^2 = CB^2$.					

Legend (Names of regions):
 AH = rectangle on AD and DB
 LG = square on CD (CD^2)
 NOP = gnomon (rectangle on AD and DB)
 CEFB = square on CB (CB^2)

FINAL GEOMETRIC IDENTITY

(AD, DB)

+

CD^2

=

CB^2

Thus the rectangle on the two unequal parts, together with the square on the distance between the cuts, equals the square on half the original line.

Used: Prop. I.43 (equal complements)
 Prop. I.36 (rectangles on equal bases and equal heights are equal)
 Prop. I.46 (construct square)

Proposition 6

Statement.

If a straight line is cut in half and another straight line is added to it straight-on, then the rectangle contained by the whole extended line and the added part, together with the square on half the original line, equals the square on the sum of half the original line and the added part.

$$(AD, DB) + CB^2 = CD^2.$$

Given / Construction.

The line AB is bisected at C , and BD is added in a straight line beyond B , so $A - C - B - D$.

Proof Actions.

1. Take AB .
2. Bisect AB at C , so that $AC = CB$.
3. Extend the line beyond B by the added segment BD .
4. State the goal:

$$(AD, DB) + CB^2 = CD^2.$$

5. Construct square $CEFD$ on CD .
6. Draw diagonal DE .
7. Through B , draw $BG \parallel CE \parallel DF$, meeting DE at G .
8. Through G , draw $KM \parallel AB \parallel EF$.
9. Through A , complete the extended construction with the required parallels.
10. Since $AC = CB$,

$$AL = CH.$$

11. By Proposition I.43,

$$CH = HF.$$

12. Therefore

$$AL = HF.$$

13. Add rectangle CM to both sides:

$$AM = \text{gnomon } NOP.$$

14. Identify

$$AM = (AD, DB).$$

15. Hence

$$\text{gnomon } NOP = (AD, DB).$$

16. Add the square $LG = CB^2$.

17. The gnomon together with LG restores the whole square $CEFD$.

18. Since $CEFD = CD^2$,

$$(AD, DB) + CB^2 = CD^2.$$

19. Conclude the proposition.

Conclusion.

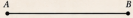
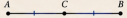
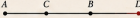
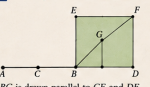
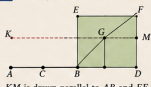
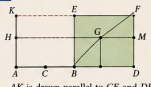
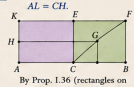
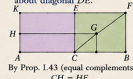
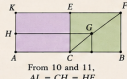
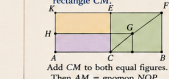
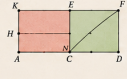
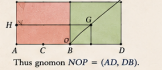
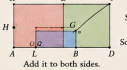
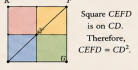
$$(AD, DB) + CB^2 = CD^2.$$



BOOK II – PROPOSITION 6

If a straight line is cut in half and another straight line is added to it in a straight line, then the rectangle contained by the whole extended line and the added part, together with the square on half the original line, equals the square on the line made from half the original line plus the added part.

To prove:
 $(AD, DB) + CB^2 = CD^2$

1 Take the straight line AB.  This is the given straight line AB.	2 Bisect AB at C.  $AC = CB$. C is the equal division of AB.	3 Extend the line beyond B.  Add another straight line BD in the same direction. The whole line is AD.	4 State what must be proved. We must prove that $(AD, DB) + CB^2 = CD^2$. Rectangle on the whole extended line and DB + square on CB = square on CD.	5 Construct the square on CD.  Square $CEFD$ on CD. So $CE = EF = FD = CD$.	6 Draw diagonal DE.  Join D to E.	7 Through B draw $BG \parallel CE, DF$.  BG is drawn parallel to CE and DF. It meets DE at G.	8 Through G draw $KM \parallel AB, EF$.  KM is drawn parallel to AB and EF.	9 Through A draw $AK \parallel CE, DF$.  AK is drawn parallel to CE and DF. The figure is completed.	10 Since $AC = CB$, $AL = CH$.  By Prop. 1.36 (rectangles on equal bases and equal heights are equal).
11 CH and HF are complements about diagonal DE.  By Prop. 1.43 (equal complements), $CH = HF$.	12 Therefore $AL = HF$.  From 10 and 11, $AL = CH = HF$.	13 Add the common rectangle CM.  Add CM to both equal figures. Then $AM =$ gnomon NOP.	14 AM is the rectangle on AD and DB.  AM has sides AD and DM. But $DM = DB$. Hence, $AM = (AD, DB)$.	15 Therefore gnomon $NOP = (AD, DB)$.  Thus gnomon $NOP = (AD, DB)$.	16 Add the square on CB.  Add it to both sides.	17 Gnomon $NOP + LG$ restore the whole square.  Square LG is on CB. So $LG = CB^2$.	18 Identify the whole square.  Hence, gnomon $NOP + LG =$ whole square $CEFD$.	19 Conclude the proposition. Therefore, $(AD, DB) + CB^2 = CD^2$. <i>Q.E.D.</i> 	

Legend (names of regions)

NOP : gnomon (rectangle on AD and DB)

LG : square on CB (CB^2)

$CEFD$: square on CD (CD^2)

FINAL GEOMETRIC IDENTITY

(AD, DB)

+

CB^2

=

CD^2

Thus the rectangle contained by the whole extended line AD and the added part DB , together with the square on half the original line CB , equals the square on CD .

Used:
Prop. 1.43 (equal complements)
Prop. 1.36 (rectangles on equal bases and equal heights)
Prop. 1.46 (construct square)

Proposition 7

Statement.

If a straight line is cut at random, then the sum of the squares on the whole straight line and one of the pieces is equal to twice the rectangle contained by the whole and that piece, together with the square on the remaining piece.

$$AB^2 + BC^2 = 2(AB, BC) + AC^2.$$

Given / Construction.

The line AB is cut at C .

Proof Actions.

1. Take AB .
2. Cut AB at C .
3. State the goal:

$$AB^2 + BC^2 = 2(AB, BC) + AC^2.$$

4. Construct square $ADEB$ on AB .
5. Complete Euclid's internal figure, producing the regions AG , GE , CF , DG , and gnomon KLM .
6. By Proposition I.43,

$$AG = GE.$$

7. Add square CF to both, obtaining

$$AF = CE.$$

8. Therefore

$$AF + CE = 2AF.$$

9. Recognize

$$AF + CE = \text{gnomon } KLM + CF.$$

10. Since $BF = BC$,

$$AF = (AB, BC),$$

and hence

$$2AF = 2(AB, BC).$$

11. Therefore

$$\text{gnomon } KLM + CF = 2(AB, BC).$$

12. Identify

$$DG = AC^2.$$

13. Add DG to both sides.

14. The resulting left-hand figure is exactly the whole square $ADEB$ together with square CF , i.e.

$$AB^2 + BC^2.$$

15. Therefore

$$AB^2 + BC^2 = 2(AB, BC) + AC^2.$$

16. Conclude the proposition.

Conclusion.

$$AB^2 + BC^2 = 2(AB, BC) + AC^2.$$

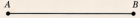


BOOK II – PROPOSITION 7

If a straight line is cut at random,
the sum of the squares on the whole straight line and one of its parts
is equal to twice the rectangle contained by the whole and that part,
together with the square on the remaining part.

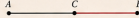
To prove:
 $AB^2 + BC^2 = 2(AB, BC) + AC^2$

1 Take the straight line AB .



This is the given straight line AB .

2 Cut AB at C .



Thus AB is cut at C .
The pieces are AC and CB .

3 State what must be proved.

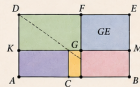
We must prove that
 $AB^2 + BC^2 = 2(AB, BC) + AC^2$.

4 Construct the square on AB .

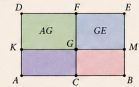


Construct square $ADEB$ on AB .
This square represents AB^2 .

5 Complete the rest of the figure.

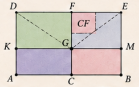


6 AG and GE are complements.



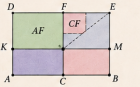
By Prop. I.43 (equal complements),
 $AG = GE$.

7 Add the square CF to both.



Adding the same square CF to both,
 $AG + CF = GE + CF$.
Hence $AF = CE$.

8 Add the two equal rectangles.



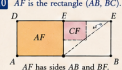
Since $AF = CE$,
 $AF + CE = 2AF$.

9 Recognize a gnomon plus CF .



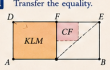
$AF + CE$ is exactly
gnomon KLM together with CF .
Thus gnomon $KLM + CF = 2AF$.

10 AF is the rectangle (AB, BC) .



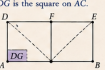
AF has sides AB and BF .
But $BF = BC$.
Hence $AF = (AB, BC)$.
Therefore $2AF = 2(AB, BC)$.

11 Transfer the equality.



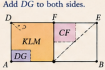
From 9 and 10,
gnomon $KLM + CF = 2(AB, BC)$.

12 DG is the square on AC .



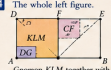
DG is the square on AC .
Thus $DG = AC^2$.

13 Add DG to both sides.



Adding DG to both sides,
gnomon $KLM + CF + DG = 2(AB, BC) + AC^2$.

14 The whole left figure.



Gnomon KLM together with
the squares KG and GD (i.e. CF and DG)
is equivalent to the square $ADEB$
together with the square CF .
Therefore, it equals $AB^2 + BC^2$.

15 Substitute both interpretations.

From 13: gnomon $KLM + CF + DG = 2(AB, BC) + AC^2$.
From 14: same figure = $AB^2 + BC^2$.
Therefore, $AB^2 + BC^2 = 2(AB, BC) + AC^2$.

FINAL GEOMETRIC IDENTITY

AB^2

+

BC^2

=

$2(AB, BC)$

+

AC^2

16 Conclude the proposition.

Thus, if a straight line is cut at random, the sum of the squares on the whole straight line and one of its parts is equal to twice the rectangle contained by the whole and that part, together with the square on the remaining part.
 $Q.E.D.$



Legend (names of regions)

gnomon (rectangle on AB and BC)	square on AC
square on BC (i.e. CF)	the two small squares (CF and DG)

Proposition 8

Statement.

If a straight line is cut at random, then four times the rectangle contained by the whole straight line and one of its pieces, together with the square on the remaining piece, is equal to the square described on the whole and the chosen piece as on one complete straight line.

$$4(AB, BC) + AC^2 = (AB + BC)^2.$$

Given / Construction.

The line AB is cut at C , and it is extended beyond B to D with $BD = BC$.

Proof Actions.

1. Take the straight line AB .
2. Cut AB at C .
3. State the goal:

$$4(AB, BC) + AC^2 = (AB + BC)^2.$$

4. Extend AB beyond B to D .
5. Make

$$BD = BC.$$

6. Construct square $AEFD$ on AD .
7. Draw the doubled internal construction used by Euclid.
8. From $CB = BD$ and the corresponding parts, obtain

$$GK = KN.$$

9. By the same reasoning,

$$QR = RP.$$

10. Deduce

$$CK = KD, \quad GR = RN.$$

11. By Proposition I.43,

$$CK = RN.$$

12. Hence

$$DK = CK = GR = RN,$$

so the four together are four times CK .

13. From the corresponding equal segments, deduce

$$CG = GQ.$$

14. Obtain

$$AG = MQ, \quad QL = RF.$$

15. By Proposition I.43,

$$MQ = QL.$$

16. Therefore

$$AG = MQ = QL = RF,$$

and the four rectangles together are four times AG .

17. Combine the eight regions: together they form the gnomon STU , and

$$\text{gnomon } STU = 4AK.$$

18. Identify

$$AK = (AB, BD).$$

19. Therefore

$$\text{gnomon } STU = 4(AB, BD).$$

20. Add the square $OH = AC^2$.

21. Recognize that

$$\text{gnomon } STU + OH$$

fills the whole square $AEFD$.

22. Since $AEFD = AD^2$,

$$4(AB, BD) + AC^2 = AD^2.$$

23. Replace BD by BC .

24. Since

$$AD = AB + BC,$$

we have

$$AD^2 = (AB + BC)^2.$$

25. Therefore

$$4(AB, BC) + AC^2 = (AB + BC)^2.$$

Conclusion.

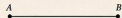
$$4(AB, BC) + AC^2 = (AB + BC)^2.$$

BOOK II – PROPOSITION 8 : If a straight line is cut at random, then four times the rectangle contained by the whole straight line and one of its parts, together with the square on the remaining part, equals the square on the whole line and that part taken together.

To prove : $4(AB, BC) + AC^2 = (AB + BC)^2$

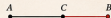
PANEL 1 : ACTIONS 1 – 12

1 Take the straight line AB.



This is the given straight line AB.

2 Cut AB at C.

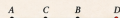


The pieces are AC and CB.

3 State what must be proved.

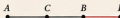
We must prove that $4(AB, BC) + AC^2 = (AB + BC)^2$.

4 Produce AB beyond B to point D.



The points are A – C – B – D.

5 Make BD = BC.



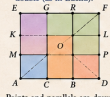
BD is made equal to BC. Hence AD = AB + BC.

6 Construct the square on AD.



Square AEDF on AD. This square is AD².

7 Draw the rest of the figure double (as in Euclid).



Points and parallels are drawn as shown to form the figure.

8 From CB = BD, and the corresponding parts, GK = KN.



GK = KN.

9 By the same reasoning, QR = RP.



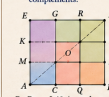
QR = RP.

10 Therefore CK = KD and GR = RN.



CK = KD and GR = RN.

11 But CK and RN are complements.



By Prop. 1.43 (complements), CK = RN.

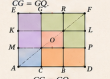
12 Hence all four regions are equal.



DK = CK = GR = RN. Their total area = 4 × (area of CK).

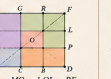
PANEL 2 : ACTIONS 13 – 25

13 From CB = BD, and the corresponding parts, CG = GQ.



CG = GQ.

14 Therefore AG = MQ and QL = RF.



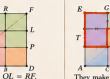
AG = MQ and QL = RF.

15 But MQ and QL are complements.



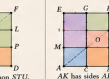
By Prop. 1.43, MQ = QL.

16 Hence AG = MQ = QL = RF.



AG = MQ = QL = RF. Total area = 4 × (area of AG).

17 The eight regions together form the gnomon STU.



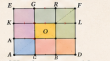
They make the gnomon STU. Hence STU = 4 × (area of AG).

18 AK is the rectangle contained by AB and BD.



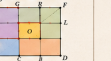
AK has sides AB and BD, but BK = BD. Thus AK = (AB, BD).

20 Let OH (the central square) be added to both sides.



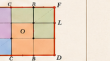
OH is the square on AC. Add it to both sides.

21 Left side: gnomon STU together with OH.



STU + OH.

22 This exactly fills the whole square AEFD.



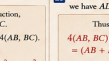
STU + OH = square AEFD.

23 But AEFD is the square on AD.



Therefore STU + OH = AD².

24 Replace BD by BC.



From construction, BD = BC. So 4(AB, BD) = 4(AB, BC). Hence 4(AB, BC) + AC² = AD².

25 Since AD = AB + BC, we have AD² = (AB + BC)².



Thus 4(AB, BC) + AC² = (AB + BC)². Q.E.D.

Legend (names of regions)

STU : gnomon (shaded) DK, CK, GR, RN : four equal regions AG, MQ, QL, RF : second four equal rectangles OH : square on AC

FINAL GEOMETRIC IDENTITY

$$4(AB, BC) + AC^2 = (AB + BC)^2$$

Four times the rectangle contained by the whole line AB and the part BC, together with the square on AC, equals the square on the whole line AB and BC together.



Proposition 9

Statement.

If a straight line is cut into equal and unequal pieces, then the sum of the squares on the unequal pieces is double the sum of the square on half the straight line and the square on the distance between the equal and unequal divisions.

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

Given / Construction.

The line AB is bisected at C and cut unequally at D .

Proof Actions.

1. Take AB .
2. Bisect AB at C , so that $AC = CB$.
3. Make an unequal cut at D .
4. State the goal:

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

5. Draw

$$CE \perp AB.$$

6. Make

$$CE = AC = CB.$$

7. Join EA and EB .
8. Through D , draw $DF \parallel CE$.
9. Through F , draw $FG \parallel AB$.
10. Join AF .
11. Since $AC = CE$,

$$\angle EAC = \angle AEC.$$

12. Since $\angle ACE = 90^\circ$, each of these equal angles is 45° .

13. Similarly,

$$\angle CEB = \angle EBC = 45^\circ,$$

hence

$$\angle AEB = 90^\circ.$$

14. Identify

$$\angle GEF = 45^\circ.$$

15. Since $FG \parallel AB$ and $CE \perp AB$,

$$\angle EGF = 90^\circ.$$

16. Therefore

$$\angle EFG = 45^\circ.$$

17. Hence

$$EG = GF.$$

18. In triangle BDF , the acute angles are each 45° .

19. Therefore

$$DF = DB.$$

20. Since $AC = CE$,

$$AC^2 + CE^2 = 2AC^2.$$

21. By Pythagoras in right triangle ACE ,

$$EA^2 = 2AC^2.$$

22. Since $EG = GF$,

$$EG^2 + GF^2 = 2GF^2.$$

23. By Pythagoras in right triangle EGF , and since $GF = CD$,

$$EF^2 = 2CD^2.$$

24. Add the two results:

$$EA^2 + EF^2 = 2(AC^2 + CD^2).$$

25. Since $\angle AEF = 90^\circ$,

$$AF^2 = EA^2 + EF^2 = 2(AC^2 + CD^2).$$

26. Since $\angle ADF = 90^\circ$,

$$AF^2 = AD^2 + DF^2.$$

27. Replace DF by DB :

$$AF^2 = AD^2 + DB^2.$$

28. Equate the two expressions for AF^2 :

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

29. Conclude the proposition.

Conclusion.

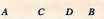
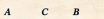
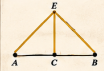
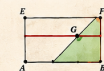
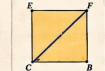
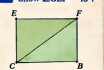
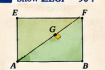
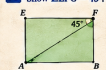
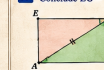
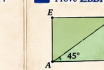
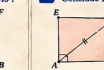
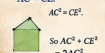
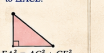
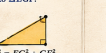
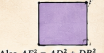
$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$



BOOK II – PROPOSITION 9

If a straight line is cut into equal and unequal parts, then the sum of the squares on the two unequal parts is double the sum of the square on half the straight line and the square on the distance between the equal and unequal divisions.

To prove:
 $AD^2 + DB^2 = 2(AC^2 + CD^2)$

1 Take the straight line AB . 	2 Bisect AB at C .  $AC = CB$.	3 Make an unequal cut at D .  This gives AD and DB .	4 State what must be proved. We must prove that $AD^2 + DB^2 = 2(AC^2 + CD^2)$.	5 Extend AB to E .  Take AB in a straight line to E .	6 Make $BD = BC$.  So $BD = BC$.	7 Construct the square on AD .  Square $AEDF$ on AD .
8 Draw $CE \perp AB$.  CE is perpendicular to AB .	9 Make $CE = AC = CB$.  Thus $AC = CB = CE$.	10 Join EA and EB .  EA and EB are drawn.	11 Through D draw $DF \parallel CE$.  DF is parallel to CE and thus $DF \perp AB$.	12 Through F draw $FG \parallel AB$.  FG is parallel to AB .	13 Join AF .  AF is drawn.	14 Identify 45° angles.  $\angle EAC = \angle AEC = 45^\circ$, and $\angle ECB = \angle EBC = 45^\circ$.
15 Show $\angle AEB = 90^\circ$.  Thus AEB is a right angle.	16 Show $\angle GEF = 45^\circ$.  $\angle GEF = 45^\circ$.	17 Show $\angle EGF = 90^\circ$.  Hence $\angle EGF = 90^\circ$ (since $CE \perp AB$).	18 Show $\angle EFG = 45^\circ$.  Thus $\angle EFG = 45^\circ$.	19 Conclude $EG = GF$.  Hence $EG = GF$.	20 Prove $\angle BDF = 45^\circ$.  Thus $\angle BFD = 45^\circ$.	21 Conclude $DF = DB$.  Hence $DF = DB$.
22 Square the equality $AC = CE$. $AC^2 = CE^2$.  So $AC^2 + CE^2 = 2AC^2$.	23 Apply Pythagoras to $\triangle ACE$.  $EA^2 = AC^2 + CE^2$. Hence $EA^2 = 2AC^2$.	24 Square the equality $EG = GF$. $EG^2 = GF^2 = 2GF^2$. 	25 Apply Pythagoras to $\triangle EGF$.  $EF^2 = EG^2 + GF^2$. Hence $EF^2 = 2CD^2$.	26 Combine the two doublings.  $EA^2 + EF^2 = 2(AC^2 + CD^2)$.	27 Apply Pythagoras to $\triangle AEF$.  $AF^2 = AE^2 + EF^2$. Hence $AF^2 = 2(AC^2 + CD^2)$.	28 Reinterpret AF^2 .  Also $AF^2 = AD^2 + DB^2$. Thus $AD^2 + DB^2 = 2(AC^2 + CD^2)$.
29 Conclude. $AD^2 + DB^2 = 2(AC^2 + CD^2)$. ■ Proposition II.9		KEY: Square on a segment Square on half the line Obvious right triangle Auxiliary construction The beauty of the proof: The same square AF^2 is shown in two ways — as a Pythagorean sum and as the sum of the squares on the unequal parts.				

Proposition 10

Statement.

If a straight line is cut in half, and any straight line is added to it straight-on, then the sum of the square on the whole extended straight line and the square on the added straight line is double the sum of the square on half the original line and the square on the sum of half the original line and the added line.

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

Given / Construction.

The line AB is bisected at C , and BD is added beyond B .

Proof Actions.

1. Take AB .
2. Bisect AB at C , so that $AC = CB$.
3. Extend beyond B by BD , giving $A - C - B - D$.
4. State the goal:

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

5. Draw

$$CE \perp AB.$$

6. Make

$$CE = AC = CB.$$

7. Join EA and EB .
8. Through E , draw $EF \parallel AD$.
9. Through D , draw $FD \parallel CE$.
10. Produce EB and FD until they meet at G .
11. Join AG .
12. Since $AC = CE$, triangle ACE is right isosceles, so

$$\angle EAC = \angle AEC = 45^\circ.$$

13. Likewise,

$$\angle CEB = \angle EBC = 45^\circ,$$

so

$$\angle AEB = 90^\circ.$$

14. Since BG is the produced line BE ,

$$\angle DBG = 45^\circ.$$

15. Because $DG \parallel CE$ and $CE \perp AB$,

$$\angle BDG = 90^\circ.$$

16. Therefore

$$\angle DGB = 45^\circ.$$

17. Hence

$$BD = DG.$$

18. Identify

$$\angle EGF = 45^\circ.$$

19. Since $EF \parallel AD$ and $FG \parallel CE$,

$$\angle EFG = 90^\circ.$$

20. Therefore

$$\angle FEG = 45^\circ.$$

21. Hence

$$GF = EF.$$

22. Since $EC = CA$,

$$EC^2 + CA^2 = 2AC^2.$$

23. By Pythagoras in triangle ACE ,

$$EA^2 = 2AC^2.$$

24. Since $GF = EF$,

$$GF^2 + EF^2 = 2EF^2.$$

25. By Pythagoras in right triangle EFG , and since $EF = CD$,

$$EG^2 = 2CD^2.$$

26. Add the two results:

$$EA^2 + EG^2 = 2(AC^2 + CD^2).$$

27. Since $\angle AEG = 90^\circ$,

$$AG^2 = EA^2 + EG^2 = 2(AC^2 + CD^2).$$

28. In right triangle ADG ,

$$AG^2 = AD^2 + DG^2.$$

29. Since $DG = DB$,

$$AG^2 = AD^2 + DB^2.$$

30. Equate the two expressions for AG^2 :

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

31. Conclude the proposition.

Conclusion.

$$AD^2 + DB^2 = 2(AC^2 + CD^2).$$

BOOK II – PROPOSITION 10

If a straight line is bisected and another straight line is added to it in a straight line, then the sum of the squares on the whole extended line and on the added part is double the sum of the squares on half the original line and the square on half the original line together with the added part.

To prove:
 $AD^2 + DB^2 = 2(AC^2 + CD^2)$

PANEL 1: ACTIONS 1–17

1 Take the straight line AB . 	2 Bisect AB at C . $AC = CB$.	3 Extend the line beyond B . $A = C - B = D$.	4 State what must be proved. We must prove $AD^2 + DB^2$ $= 2(AC^2 + CD^2)$.	5 Raise CE perpendicular to AB . $CE \perp AB$.	6 Make $CE = AC = CB$. $CE = AC = CB$.	7 Join EA and EB . EA and EB are joined.
8 Through E draw $EF \parallel AD$. 	9 Through D draw $FD \parallel CE$. $FD \parallel CE$ (so $FD \perp AD$).	10 Produce EB and FD until they meet at G . EB and FD meet at G.	11 Join AG . AG is joined. The figure is complete.			
12 Since $AC = CE$, in $\triangle ACE$. $\angle EAC = \angle AEC$ (Prop. 1.5). Each equals 45°.	13 In $\triangle CEB$. $\angle CEB = \angle ECB = 45^\circ$. Hence $\angle AEB = 90^\circ$.	14 In $\triangle BDG$. $\angle DBG = 45^\circ$.	15 Since $DG \parallel CE$ and $CE \perp AB$. $\angle BDG = 90^\circ$.	16 Hence in $\triangle BDG$. $\angle DGB = 45^\circ$.	17 Therefore $BD = DG$. (Prop. 1.6) 	

PANEL 2: ACTIONS 18–31

18 In $\triangle EGF$. $\angle EGF = 45^\circ$.	19 Since $EF \parallel AD$ and $FG \parallel CE$ with $CE \perp AD$. $\angle EFG = 90^\circ$. (Prop. 1.29)	20 Hence in $\triangle EGF$. $\angle FEG = 45^\circ$. So $\angle EGF = \angle FEG$.	21 Therefore $GF = EF$. (Prop. 1.6)
22 Square $EC = CA$. $EC = CA$ $\Rightarrow EC^2 = CA^2$ $\Rightarrow EC^2 + CA^2 = 2AC^2$.	23 In right $\triangle ACE$. By Pythagoras, $EA^2 = EC^2 + CA^2 = 2AC^2$.	24 Square $GF = EF$. $GF = EF$ $\Rightarrow GF^2 = EF^2$ $\Rightarrow GF^2 + EF^2 = 2EF^2$.	25 In right $\triangle EGF$. By Pythagoras, $EG^2 = GF^2 + EF^2 = 2EF^2$. But $EF = CD$ (parallel). Hence $EG^2 = 2CD^2$.
26 Add the two results. $EA^2 = 2AC^2$ and $EG^2 = 2CD^2$ $\Rightarrow EA^2 + EG^2 = 2(AC^2 + CD^2)$	27 In right $\triangle AEG$. By Pythagoras, $AG^2 = EA^2 + EG^2$. Hence $AG^2 = 2(AC^2 + CD^2)$.	28 In right $\triangle ADG$. By Pythagoras, $AG^2 = AD^2 + DG^2$.	29 From 17, $DG = DB$. So $DG^2 = DB^2$. Hence $AG^2 = AD^2 + DB^2$.
31 Conclusion: expressions for AG^2 . $AG^2 = 2(AC^2 + CD^2)$ and $AG^2 = AD^2 + DB^2$ $\Rightarrow AD^2 + DB^2 = 2(AC^2 + CD^2)$			

LEGEND
 $\perp$: perpendicular
 $\parallel$: parallel
 : equal segments
 : right angle
 : 45° angle

FINAL GEOMETRIC IDENTITY

$$AD^2 + DB^2 = 2(AC^2 + CD^2)$$

Used:
 Prop. 1.5 (base angles of isosceles)
 1.6 (equal angles $\Rightarrow$ equal sides)
 1.11 (erect a perpendicular)
 1.29 (corresponding angles with parallels)
 1.47 (Pythagoras)
 Q.E.D.

Proposition 11

Statement.

To cut a given straight line so that the rectangle contained by the whole straight line and one of the pieces is equal to the square on the remaining piece.

For the constructed point H ,

$$(AB, BH) = AH^2.$$

Given / Construction.

The given straight line is AB . The construction determines the required point H on AB .

Proof Actions.

1. Take the given straight line AB .
2. State the construction goal:

$$(AB, BH) = AH^2.$$

3. Construct square $ABDC$ on AB .
4. Bisect AC at E :

$$AE = EC.$$

5. Join BE .
6. Produce CA beyond A to F , so $C - E - A - F$ are collinear.
7. Make

$$EF = EB.$$

8. Construct the square on AF .
9. Produce the corresponding side of this square until it meets the original construction at K .
10. The construction determines the required point H on AB .
11. Apply Proposition II.6 to AC , bisected at E , with AF added:

$$(CF, FA) + AE^2 = EF^2.$$

12. Since $EF = EB$,

$$(CF, FA) + AE^2 = EB^2.$$

13. By Pythagoras in right triangle AEB ,

$$EB^2 = AB^2 + AE^2.$$

14. Therefore

$$(CF, FA) + AE^2 = AB^2 + AE^2.$$

15. Subtract the common square AE^2 :

$$(CF, FA) = AB^2.$$

16. Identify rectangle FK with (CF, FA) .

17. Identify the original square AD with AB^2 ; hence

$$FK = AD.$$

18. Subtract the common region AK :

$$FH = HD.$$

19. Since $BD = AB$,

$$HD = (AB, BH).$$

20. Since FH is the square on AH ,

$$FH = AH^2.$$

21. Therefore

$$AH^2 = (AB, BH).$$

22. Thus AB has been cut at H in the required manner.

Conclusion. The constructed division satisfies

$$(AB, BH) = AH^2.$$

This is the division commonly called the Golden Section.

EUCLID'S ELEMENTS — BOOK II



BOOK II – PROPOSITION 11

To cut a given straight line so that the rectangle contained by the whole line and one part is equal to the square on the remaining part.

To construct H on AB such that
 $(AB, BH) = AH^2$

CONSTRUCTION (ACTIONS 1–10)							
1 Take the given straight line AB . We are to find the point H on AB .	2 State what must be constructed. We seek H on AB such that $(AB, BH) = AH^2$.	3 Construct the square on AB . Square $ABDC$ is made.	4 Bisect AC at E . $AE = EC$.	5 Join B to E . BE is joined.	6 Extend CA beyond A to point F . $C-E-A-F$ are collinear.	7 Make $EF = EB$. $EF = EB$.	8 Construct the square on AE . Square FGH is made. Thus $FG = AF$. H on AB is the required point: $(AB, BH) = AH^2$.
PROOF (ACTIONS 11–22)							
11 By Prop. II.6 applied to AC with AF added. $(CF, FA) + AE^2 = EF^2$.	12 But $EF = EB$. Hence $(CF, FA) + AE^2 = EB^2$.	13 In right $\triangle AEB$. By Pythagoras (Prop. I.47), $EB^2 = AB^2 + AE^2$.	14 Equating the two expressions for EB^2 . $(CF, FA) + AE^2 = EB^2 = AB^2 + AE^2$.	15 Subtract AE^2 from both sides. $(CF, FA) = AB^2$.	16 The rectangle FK represents (CF, FA) . Since $AF = FG$ (square on AF), $FK = (CF, FA)$.	17 AD is the square on AB . Thus $AD = AB^2$. Hence $FK = AD$.	18 Subtract the common region AK . Subtract AK from both FK and AD .
19 The remainders are equal. Hence $FH = HD$.	20 HD is the rectangle (AB, BH) . HD has sides AB and BH . Since $AB = BD$ (square), $HD = (AB, BH)$.	21 FH is the square on AH . $FH = AH^2$.	22 Therefore, $FH = HD$ $\Rightarrow AH^2 = (AB, BH)$. Thus the point H on AB is constructed so that $(AB, BH) = AH^2$. Q.E.D.				
FINAL GEOMETRIC RELATION $(AB, BH) = AH^2$				The given straight line AB has been cut at H so that the rectangle contained by the whole line AB and the part BH is equal to the square on the remaining part AH . This is the Golden Section.			

Proposition 12

Statement.

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of those sides and the external segment cut off by the perpendicular.

For Euclid's figure,

$$BC^2 = BA^2 + AC^2 + 2(AC)(AD).$$

Given / Construction.

Triangle ABC has obtuse angle $\angle BAC$. Produce CA beyond A to D , and draw $BD \perp CD$.

Proof Actions.

1. Draw obtuse triangle ABC with $\angle BAC$ obtuse.
2. Extend CA beyond A to D , so $D - A - C$.
3. From B , draw $BD \perp CD$.
4. State the goal:

$$BC^2 = BA^2 + AC^2 + 2(AC)(AD).$$

5. Regard CD as cut at A , so

$$CD = CA + AD.$$

6. By Proposition II.4,

$$CD^2 = CA^2 + AD^2 + 2(CA)(AD).$$

7. Add DB^2 to both sides:

$$CD^2 + DB^2 = CA^2 + AD^2 + DB^2 + 2(CA)(AD).$$

8. In right triangle CDB , Pythagoras gives

$$BC^2 = CD^2 + DB^2.$$

9. Replace the left-hand side by BC^2 .

10. Since $BD \perp AD$, triangle ADB is right-angled.

11. By Pythagoras,

$$AB^2 = AD^2 + DB^2.$$

12. Substitute:

$$BC^2 = CA^2 + AB^2 + 2(CA)(AD).$$

13. Hence

$$BC^2 - (BA^2 + AC^2) = 2(AC)(AD).$$

14. Conclude the proposition.

Conclusion.

$$BC^2 = BA^2 + AC^2 + 2(AC)(AD).$$



BOOK II ~ PROPOSITION 12

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is greater than the sum of the squares on the two sides containing the obtuse angle by twice the rectangle contained by one of those sides and the external segment cut off by the perpendicular.

To prove:
 $BC^2 = BA^2 + AC^2 + 2(AC)(AD)$

1 Draw an obtuse triangle ABC. $\angle BAC$ is obtuse.	2 Extend CA beyond A to a point D. $D-A-C$ are on the same straight line.	3 From B, draw BD perpendicular to CD. $BD \perp CD$. (Prop. I.12)	4 State what must be proved. We must prove $BC^2 = BA^2 + AC^2 + 2(AC)(AD)$. BC is the side subtending the obtuse angle at A.	5 Consider the straight line CD cut at A. $CD = CA + AD$. (For Prop. II.4)
6 By Proposition II.4, expand the square on CD. $CD^2 = CA^2 + AD^2 + 2(CA)(AD)$.	7 Add the square on DB to both sides. $CD^2 = CA^2 + AD^2 + 2(CA)(AD) + DB^2 + DB^2$ $CD^2 + DB^2 = CA^2 + AD^2 + DB^2 + 2(CA)(AD)$.	8 In right triangle CDB, by Pythagoras (Prop. I.47). $BC^2 = CD^2 + DB^2$.	9 Replace $CD^2 + DB^2$ by BC^2. $BC^2 = CA^2 + AD^2 + DB^2 + 2(CA)(AD)$.	10 In right triangle ADB, $BD \perp AD$. By Pythagoras (Prop. I.47), $AB^2 = AD^2 + DB^2$.
11 Replace $AD^2 + DB^2$ by AB^2. $BC^2 = CA^2 + AB^2 + 2(CA)(AD)$.	12 Hence, $BC^2 = BA^2 + AC^2 + 2(AC)(AD)$.	13 Therefore, $BC^2 - (BA^2 + AC^2) = 2(AC)(AD)$. The excess over $BA^2 + AC^2$ is twice the rectangle (AC, AD).	14 Thus, in an obtuse-angled triangle ABC, the square on the side subtending the obtuse angle is greater than the sum of the squares on the two sides containing it by twice the rectangle contained by one of those sides and the external segment cut off by the perpendicular. Q.E.D.	MODERN EQUIVALENT Let $\angle A = \theta$ be obtuse. Then $AD = AC \cos \theta$ ($\cos \theta < 0$). Substituting in the result, $BC^2 = BA^2 + AC^2 + 2(AC)(AC \cos \theta)$ $= BA^2 + AC^2 - 2(AC)^2 \cos \theta$, which is the cosine formula for an obtuse angle.

— ⊥ Perpendicular — Extension / added segment — Square on a segment — Rectangle (AC, AD) — Right triangles

Proposition 13

Statement.

In acute-angled triangles, the square on the side subtending the acute angle is less than the sum of the squares on the sides containing that angle by twice the rectangle contained by one of those sides and the internal segment cut off by the perpendicular.

For Euclid's figure,

$$AC^2 = AB^2 + BC^2 - 2(BC)(BD).$$

Given / Construction.

Triangle ABC has the angle at B acute. Draw $AD \perp BC$, with D lying between B and C .

Proof Actions.

1. Draw acute triangle ABC , with $\angle ABC$ acute.
2. From A , draw $AD \perp BC$, with $B - D - C$.
3. State the goal:

$$BC^2 + BA^2 = AC^2 + 2(BC)(BD).$$

4. Regard BC as cut at D .
5. By Proposition II.7,

$$BC^2 + BD^2 = 2(BC)(BD) + DC^2.$$

6. Add AD^2 to both sides:

$$BC^2 + BD^2 + AD^2 = 2(BC)(BD) + DC^2 + AD^2.$$

7. Triangle ABD is right-angled at D .

8. By Pythagoras,

$$AB^2 = BD^2 + AD^2.$$

9. Replace $BD^2 + AD^2$ by AB^2 .

10. Triangle ADC is right-angled at D .

11. By Pythagoras,

$$AC^2 = AD^2 + DC^2.$$

12. Replace $AD^2 + DC^2$ by AC^2 , obtaining

$$BC^2 + AB^2 = AC^2 + 2(BC)(BD).$$

13. Isolate the square on AC :

$$AC^2 = AB^2 + BC^2 - 2(BC)(BD).$$

14. Conclude the proposition.

Conclusion.

$$AC^2 = AB^2 + BC^2 - 2(BC)(BD).$$



BOOK II – PROPOSITION 13

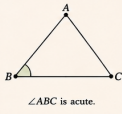
In an acute-angled triangle, the square on the side subtending the acute angle is less than the sum of the squares on the sides containing that acute angle by twice the rectangle contained by one of those sides and the internal segment cut off by the perpendicular.

To prove:

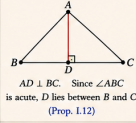
$$AC^2 = AB^2 + BC^2 - 2(BC)(BD)$$

(equivalently, $AB^2 + BC^2 = AC^2 + 2(BC)(BD)$)

1 Draw an acute triangle ABC.



2 From A, draw AD perpendicular to BC.



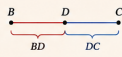
3 State what must be proved.

We must prove

$$BC^2 + BA^2 = AC^2 + 2(BC)(BD).$$

That is, the square on AC is less than the sum of the squares on AB and BC by twice the rectangle (BC, BD).

4 Consider BC cut at D.



5 By Proposition II.7,

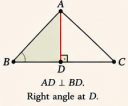
$$BC^2 + BD^2 = 2(BC)(BD) + DC^2.$$

(Whole line and the segment BD.)

6 Add the square on AD to both sides.

$$BC^2 + BD^2 + AD^2 = 2(BC)(BD) + DC^2 + AD^2.$$

7 In right triangle ABD,



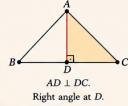
8 By Pythagoras (Prop. 1.47),

$$AB^2 = BD^2 + AD^2.$$

9 Replace $BD^2 + AD^2$ by AB^2 .

$$BC^2 + AB^2 = 2(BC)(BD) + DC^2 + AD^2.$$

10 In right triangle ADC,



11 By Pythagoras (Prop. 1.47),

$$AC^2 = AD^2 + DC^2.$$

12 Replace $DC^2 + AD^2$ by AC^2 .

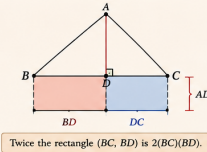
$$BC^2 + AB^2 = 2(BC)(BD) + AC^2.$$

13 Isolate the square on AC.

$$BC^2 + AB^2 = AC^2 + 2(BC)(BD)$$

$$AC^2 = AB^2 + BC^2 - 2(BC)(BD)$$

So the square on AC is less than the sum of the squares on AB and BC by twice the rectangle (BC, BD).



14 Conclusion.

Therefore, in an acute-angled triangle ABC, the square on the side AC subtending the acute angle at B is less than the sum of the squares on the sides AB and BC by twice the rectangle contained by BC and the internal segment BD cut off by the perpendicular.

Q.E.D.

Modern Equivalent

Let $\angle B = \theta$ be acute. Then $BD = AB \cos \theta$ ($\cos \theta > 0$). Substituting in the result,

$$AC^2 = AB^2 + BC^2 - 2(BC)(AB \cos \theta) = AB^2 + BC^2 - 2(AB)(BC) \cos \theta,$$

which is the cosine formula for an acute angle.

Legend: Perpendicular

Segment/part

Rectangle (BC, BD)

Squares

Right triangles

Used: Prop. 1.12 (draw a perpendicular)
Prop. 1.47 (Pythagoras) Prop. II.7 (square on whole and part)

Proposition 14

Statement.

To construct a square equal to a given rectilinear figure.

Given / Construction.

Let A be the given rectilinear figure.

Proof Actions.

1. Take the given rectilinear figure A .
2. State the goal: construct a square equal in area to A .
3. By Proposition I.45, construct a right-angled parallelogram BD equal in area to A :

$$BD = A.$$

4. If the adjacent sides BE and ED are equal, then BD is already the required square.
5. Otherwise, let BE be the longer side.
6. Produce BE in a straight line to F .
7. Make

$$EF = ED.$$

8. Bisect BF at G :

$$BG = GF.$$

9. With center G and radius $GB = GF$, draw the semicircle on diameter BF .
10. Produce DE beyond E until it meets the semicircle at H .
11. Join GH .
12. Regard BF as divided equally at G and unequally at E .
13. By Proposition II.5,

$$(BE, EF) + GE^2 = GF^2.$$

14. Since $GF = GH$,

$$(BE, EF) + GE^2 = GH^2.$$

15. Triangle GEH is right-angled at E .

16. By Pythagoras,

$$GH^2 = GE^2 + EH^2.$$

17. Equate the two expressions for GH^2 :

$$(BE, EF) + GE^2 = GE^2 + EH^2.$$

18. Subtract the common square GE^2 :

$$(BE, EF) = EH^2.$$

19. Since $EF = ED$, the rectangle (BE, EF) equals the original rectangle BD :

$$BD = EH^2.$$

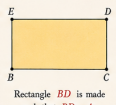
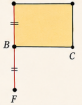
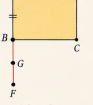
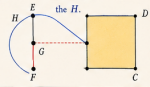
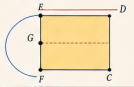
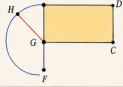
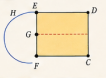
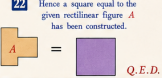
20. Since $BD = A$,

$$A = EH^2.$$

21. Construct the square on EH .

22. Therefore the square on EH is equal in area to the given rectilinear figure A .

Conclusion. A square equal to the given rectilinear figure A has been constructed.

BOOK II – PROPOSITION 14 To construct a square equal to a given rectilinear figure.							
1 Take the given rectilinear figure A .  We must construct a square equal to A.	2 State what must be constructed. Construct a square equal in area to the given figure A .	3 By Prop. I.45, construct a rectangle equal in area to A .  Rectangle BD is made such that $BD = A$.	4 If $BE = ED$, the rectangle BD is already a square. Then we are done.	5 If $BE \neq ED$, take the longer side to be BE .  So $BE > ED$.	6 Produce BE in a straight line to a point F . 	7 On BE take F such that $EF = ED$. 	8 Bisect BF at G .  $BG = GF$.
9 With center G and radius GB , draw the semicircle BHF . 	10 Produce DE until it meets the semicircle at H . 	11 In right triangle CDB , by BC . 	12 Produce DE until it meets the semicircle at H . 	PROOF			
12 Since BF is divided equally at G and unequally at E , by Prop. II.5  $(BE, EF) + GE^2 = GF^2$	13 But $GF = GH$ (G is the center).  $(BE, EF) + GE^2 = GH^2$	14 In right triangle GEH ,  $GH^2 = GE^2 + EH^2$ (Pythagoras, I.47).	15 Hence, $(BE, EF) + GE^2 = GE^2 + EH^2$	16 Subtract GE^2 from both sides. $(BE, EF) = EH^2$	17 Since $EF = ED$ (by construction), 	18 But (BE, ED) is the rectangle BD . $BD = EH^2$	19 From Step 3, we have $BD = A$. $A = EH^2$
21 Construct the square on EH .  The square on EH has area $EH^2 = A$.				22 Hence a square equal to the given rectilinear figure A has been constructed.  $Q.E.D.$			

LEGEND:

- Right angle
- Equal segments
- Diameter / center line
- Produced line
- Constructed square

USED: Prop. I.45 (equal rectilinear figures)
 Prop. I.12 (draw a straight line)
 Prop. I.47 (Pythagoras)

Prop. I.3 (take equal to a given)
 Prop. II.5 (equal & unequal parts)

Conclusion of Book II

Book II has established a geometric theory of products, squares, decompositions, and recombinations of areas. It culminates in the construction of a square equal to any given rectilinear figure. The book also contains the construction later known as the Golden Section and the acute- and obtuse-angle identities corresponding to the modern law of cosines.