

EUCLID'S ELEMENTS

## BOOK IV

### INSCRIBED AND CIRCUMSCRIBED FIGURES

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Circles, Triangles, Squares, Pentagons, Hexagons,  
and the Fifteen-Sided Figure

Illustrated Euclidean constructions presented as cinematic proof sequences

Aristotle teaching the young Alexander





# Contents

# Introduction

Book IV develops the geometry of rectilinear figures and circles through a sequence of exact constructions. It treats figures inscribed in and circumscribed about other figures, circles inscribed in and circumscribed about polygons, chords inserted in circles, the incircle and circumcircle of a triangle, inscribed and circumscribed squares, and the construction of regular pentagons, hexagons, and finally the regular fifteen-sided figure. Its methods build especially on the elementary geometry of Book I and the theory of circles in Book III, combining them into an increasingly sophisticated theory of regular construction.

# Book IV

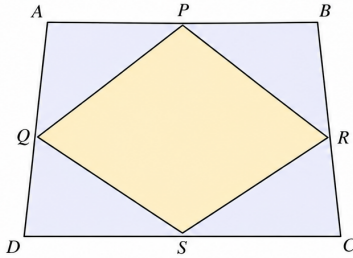
## **Definition 1**

A rectilinear figure is said to be inscribed in another rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the figure in which it is inscribed.

**DEFINITION 1.**

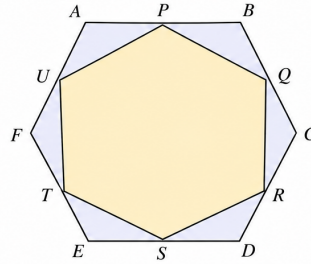
A rectilinear figure is said to be *inscribed in* another rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the figure in which it is inscribed.

Example 1: Quadrilateral inscribed in a quadrilateral



Quadrilateral  $PQRS$  is inscribed in quadrilateral  $ABCD$ , since each of its angles (at  $P, Q, R, S$ ) touches a side of  $ABCD$ .

Example 2: Hexagon inscribed in a hexagon



Hexagon  $PQRSTU$  is inscribed in hexagon  $ABCDEF$ , since each of its angles (at  $P, Q, R, S, T, U$ ) touches a side of  $ABCDEF$ .

## Definition 2

A rectilinear figure is said to be circumscribed about another rectilinear figure when the respective sides of the circumscribed figure touch the respective angles of the figure about which it is circumscribed.

*Editorial note.* The illustration below is intentionally a placeholder so that the incorrectly generated earlier illustration can later be replaced without altering the manuscript structure.

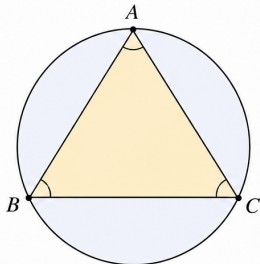
### **Definition 3**

A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed figure touches the circumference of the circle.

**DEFINITION 3.**

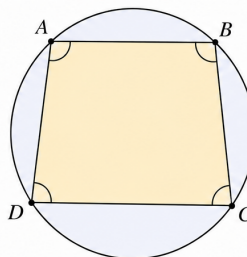
A rectilinear figure is said to be *inscribed in a circle* when each angle of the inscribed figure touches the circumference of the circle.

Example 1: Triangle inscribed in a circle



Triangle  $ABC$  is inscribed in circle ( $O$ ), since each of its angles (at  $A$ ,  $B$ ,  $C$ ) touches the circumference of the circle.

Example 2: Quadrilateral inscribed in a circle



Quadrilateral  $ABCD$  is inscribed in circle ( $O$ ), since each of its angles (at  $A$ ,  $B$ ,  $C$ ,  $D$ ) touches the circumference of the circle.



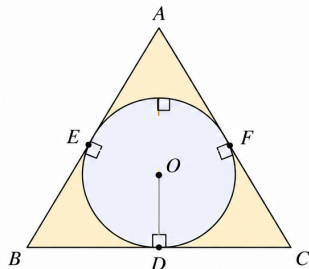
## **Definition 4**

A rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed figure touches the circumference of the circle.

**DEFINITION 4.**

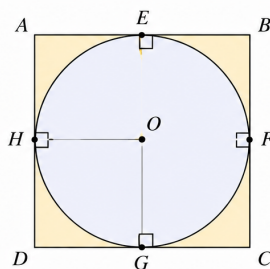
A rectilinear figure is said to be *circumscribed about a circle* when each side of the circumscribed figure touches the circumference of the circle.

Example 1: Triangle circumscribed about a circle



Triangle  $ABC$  is circumscribed about circle  $(O)$ , since each of its sides  $AB$ ,  $BC$ ,  $CA$  touches the circumference of the circle.

Example 2: Square circumscribed about a circle



Square  $ABCD$  is circumscribed about circle  $(O)$ , since each of its sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  touches the circumference of the circle.

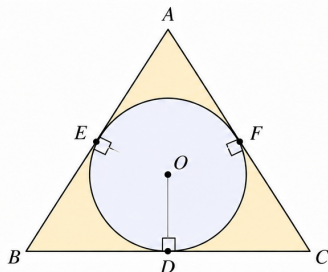
## **Definition 5**

A circle is said to be inscribed in a rectilinear figure when the circumference of the circle touches each side of the figure in which it is inscribed.

**DEFINITION 5.**

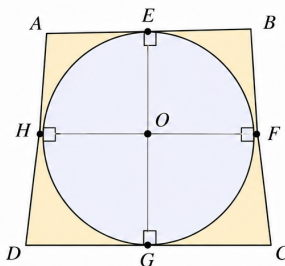
A circle is said to be *inscribed in* a rectilinear figure when the circumference of the circle touches each side of the figure in which it is inscribed.

Example 1: Circle inscribed in a triangle



Circle ( $O$ ) is inscribed in triangle  $ABC$ , since the circumference of the circle touches each side  $AB$ ,  $BC$ ,  $CA$  of the triangle.

Example 2: Circle inscribed in a quadrilateral



Circle ( $O$ ) is inscribed in quadrilateral  $ABCD$ , since the circumference of the circle touches each side  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  of the quadrilateral.

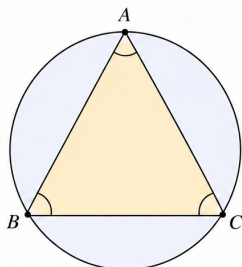
## **Definition 6**

A circle is said to be circumscribed about a rectilinear figure when the circumference of the circle touches each angle of the figure about which it is circumscribed.

**DEFINITION 6.**

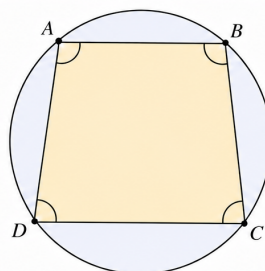
A circle is said to be *circumscribed about a* rectilinear figure when it touches each angle of the figure about which it is circumscribed.

Example 1: Circle circumscribed about a triangle



Circle ( $O$ ) is circumscribed about triangle  $ABC$ , since the circle touches each angle of the triangle, namely,  $\angle A$ ,  $\angle B$ , and  $\angle C$ .

Example 2: Circle circumscribed about a quadrilateral



Circle ( $O$ ) is circumscribed about quadrilateral  $ABCD$ , since the circle touches each angle of the quadrilateral, namely,  $\angle A$ ,  $\angle B$ ,  $\angle C$ , and  $\angle D$ .

## Definition 7

A straight-line is said to be inserted into a circle when its extremities are on the circumference of the circle.

*Editorial note.* The illustration below is intentionally a placeholder so that the incorrectly generated earlier illustration can later be replaced without altering the manuscript structure.

## Proposition 1

*To insert a straight-line equal to a given straight-line into a circle, the given straight-line not being greater than the diameter of the circle.*

Let  $ABC$  be the given circle and let  $D$  be the given straight-line, with  $D$  not greater than the diameter of the circle.

1. Begin with the given objects: the circle  $ABC$  and the given straight-line  $D$ .
2. Draw a diameter  $BC$  of the circle.
3. If  $BC = D$ , the required line has already been inserted into the circle.
4. Otherwise suppose  $BC > D$ .
5. On  $BC$ , construct  $CE = D$  by Proposition I.3.
6. With centre  $C$  and radius  $CE$ , draw the auxiliary circle  $EAF$ .
7. Let the auxiliary circle meet the given circle at  $A$ .
8. Join  $C$  to  $A$ .
9. Since  $C$  is the centre of circle  $EAF$ , its radii are equal, so  $CA = CE$ .
10. Since  $CE = D$ , it follows that  $CA = D$ .
11. Both  $C$  and  $A$  lie on the circumference of the given circle, so  $CA$  is inserted into it.
12. Therefore  $CA$ , equal to the given straight-line  $D$ , has been inserted into the given circle.

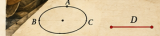
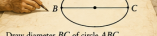
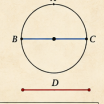
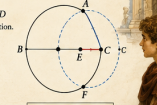
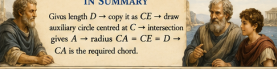
$$\boxed{CA = CE = D}$$

*Which is the very thing it was required to do.*



**BOOK IV – PROPOSITION 1:** To insert a straight-line equal to a given straight-line into a given circle, the given straight-line not being greater than the diameter of the circle.

- Given circle
- Given straight-line
- Constructed line

|                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                     |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given objects.</p> <p>Let <math>ABC</math> be the given circle and let <math>D</math> be the given straight-line, with <math>D</math> not greater than the diameter of the circle.</p>                                                                                             | <p><b>2</b> Draw a diameter of the circle.</p>  <p>Draw diameter <math>BC</math> of circle <math>ABC</math>.</p>                                                                                                                                          | <p><b>3</b> Check the easy case.</p> <p>If <math>BC = D</math>, then <math>BC</math> itself is a straight-line equal to <math>D</math> inserted into the circle.</p>  <p>If <math>BC = D</math>, we are done.</p>                                     | <p><b>4</b> Move to the nontrivial case.</p> <p>Suppose instead that the diameter <math>BC</math> is greater than <math>D</math>.</p>  <p>So assume <math>BC &gt; D</math>.</p>   |
| <p><b>5</b> Copy the given length onto the diameter.</p> <p>On <math>BC</math>, construct <math>CE = D</math> using Proposition 1.3.</p>  <p>On <math>BC</math>, construct <math>CE = D</math>.</p>                                                                                                           | <p><b>6</b> Use <math>C</math> as a new centre.</p> <p>With centre <math>C</math> and radius <math>CE</math>, draw the auxiliary circle <math>EAF</math>.</p>  <p>Draw circle <math>EAF</math> with centre <math>C</math> and radius <math>CE</math>.</p> | <p><b>7</b> Locate the intersection point <math>A</math>.</p> <p>The auxiliary circle meets the original circle at <math>A</math>.</p>                                                                                                                | <p><b>8</b> Join <math>C</math> to <math>A</math>.</p> <p>Draw the straight-line <math>CA</math>.</p>                                                                             |
| <p><b>9</b> Use the radii of the auxiliary circle.</p> <p>Since <math>C</math> is the centre of circle <math>EAF</math>, its radii are equal, so <math>CA = CE</math>.</p>  <p><math>CA = CE</math></p>                                                                                                       | <p><b>10</b> Substitute the copied length.</p> <p>But <math>CE = D</math> by construction. Therefore <math>CA = D</math>.</p>  <p><math>CA = CE = D</math></p>                                                                                            | <p><b>11</b> Recognize <math>CA</math> as the required inserted line.</p> <p>Both endpoints <math>C</math> and <math>A</math> lie on the circumference of the original circle <math>ABC</math>, so <math>CA</math> is inserted into that circle.</p>  | <p><b>12</b> Conclude.</p> <p>Thus <math>CA</math>, equal to the given straight-line <math>D</math>, has been inserted into the given circle <math>ABC</math>.</p>  <p>Q.E.D.</p> |
| <p><b>IN SUMMARY</b></p> <p>Given length <math>D \rightarrow</math> copy it as <math>CE \rightarrow</math> draw auxiliary circle centred at <math>C \rightarrow</math> intersection gives <math>A \rightarrow</math> radius <math>CA = CE = D \rightarrow</math> <math>CA</math> is the required chord.</p>  |                                                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                     |

## Proposition 2

*To inscribe a triangle, equiangular with a given triangle, in a given circle.*

Let  $ABC$  be the given circle and  $DEF$  the given triangle.

1. Begin with the given circle  $ABC$  and the given triangle  $DEF$ .
2. Draw the straight-line  $GH$  tangent to circle  $ABC$  at  $A$ .
3. At  $A$ , on ray  $AH$ , construct  $\angle HAC = \angle DEF$  by Proposition I.23.
4. At  $A$ , on ray  $AG$ , construct  $\angle GAB = \angle DFE$ .
5. Join  $B$  to  $C$ , completing the candidate inscribed triangle  $ABC$ .
6. Since  $AH$  is tangent at  $A$  and  $AC$  is a chord, Proposition III.32 gives  $\angle HAC = \angle ABC$ .
7. Since  $\angle HAC = \angle DEF$ , conclude  $\angle ABC = \angle DEF$ .
8. By the same tangent–chord theorem applied to chord  $AB$ ,  $\angle GAB = \angle ACB$ .
9. Since  $\angle GAB = \angle DFE$ , conclude  $\angle ACB = \angle DFE$ .
10. Thus two angles of triangle  $ABC$  are respectively equal to two angles of triangle  $DEF$ .
11. By Proposition I.32, the remaining angles are equal:  $\angle BAC = \angle EDF$ .
12. Hence triangle  $ABC$  is equiangular with triangle  $DEF$ .
13. Since  $A, B, C$  lie on the circumference, triangle  $ABC$  is the required inscribed triangle.

|                                                                                         |
|-----------------------------------------------------------------------------------------|
| $\angle ABC = \angle DEF, \quad \angle ACB = \angle DFE, \quad \angle BAC = \angle EDF$ |
|-----------------------------------------------------------------------------------------|

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 2: To inscribe a triangle, equiangular with a given triangle, in a given circle.

— Given  
— Construction / Tangent  
— Result / Used

|                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                             |                                                                                                                                                        |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given objects.</p> <p>Given circle <math>ABC</math>.<br/>Given triangle <math>DEF</math>.</p>                                                                                       | <p><b>2</b> Draw a tangent at <math>A</math>.</p> <p>Draw tangent <math>GH</math> to the circle at <math>A</math>.</p>                                                                                                                                              | <p><b>3</b> Copy <math>\angle E</math> of triangle <math>DEF</math> at <math>A</math> on ray <math>AH</math>.</p> <p>Construct <math>\angle HAC = \angle DEF</math> (using Prop. I.23).</p>                                                                                        | <p><b>4</b> Copy <math>\angle F</math> of triangle <math>DEF</math> at <math>A</math> on ray <math>AG</math>.</p> <p>Construct <math>\angle GAB = \angle DFE</math> (using Prop. I.23).</p> | <p><b>5</b> Join <math>B</math> to <math>C</math>.</p> <p>Draw chord <math>BC</math>.</p>                                                              |
| <p><b>6</b> Use tangent-chord theorem on <math>AC</math>.</p> <p>Since <math>AH</math> is tangent at <math>A</math> and <math>AC</math> is a chord, by Prop. III.32, <math>\angle HAC = \angle ABC</math>.</p> | <p><b>7</b> Transfer the first copied angle.</p> <p>But we constructed <math>\angle HAC = \angle DEF</math>.<br/>Therefore <math>\angle ABC = \angle DEF</math>.</p>                                                                                                | <p><b>8</b> Use tangent-chord theorem on <math>AB</math>.</p> <p>By the same reasoning, <math>\angle GAB = \angle ACB</math>.</p>                                                                                                                                                  | <p><b>9</b> Transfer the second copied angle.</p> <p>Since we constructed <math>\angle GAB = \angle DFE</math>,<br/>therefore <math>\angle ACB = \angle DFE</math>.</p>                     | <p><b>10</b> Compare the two triangles.</p> <p>We have <math>\angle ABC = \angle DEF</math><br/>also<br/>and <math>\angle ACB = \angle DFE</math>.</p> |
| <p><b>11</b> Use angle-sum of a triangle.</p> <p>The third angles are therefore equal also, by Prop. I.32.<br/><math>\angle BAC = \angle EDF</math>.</p> <p><math>\angle BAC = \angle EDF</math>.</p>          | <p><b>12</b> Conclude equiangularity.</p> <p>Thus <math>\angle ABC = \angle DEF</math><br/><math>\angle ACB = \angle DFE</math><br/><math>\angle BAC = \angle EDF</math><br/>Therefore triangle <math>ABC</math> is equiangular with triangle <math>DEF</math>.</p> | <p><b>13</b> Conclude the construction.</p> <p>Since <math>A, B, C</math> lie on the circumference of the given circle <math>ABC</math>, triangle <math>ABC</math> is inscribed in the given circle and is equiangular with the given triangle <math>DEF</math>.</p> <p>Q.E.D.</p> |                                                                                                                                                                                             |                                                                                                                                                        |

### Proposition 3

*To circumscribe a triangle, equiangular with a given triangle, about a given circle.*

Let  $ABC$  be the given circle and  $DEF$  the given triangle.

1. Begin with the given circle  $ABC$  and the given triangle  $DEF$ .
2. Produce  $EF$  in both directions to points  $G$  and  $H$ .
3. Find the centre  $K$  of circle  $ABC$  by Proposition III.1.
4. Draw a radius  $KB$  across the circle.
5. At  $K$ , construct  $\angle BKA = \angle DEG$  by Proposition I.23.
6. At  $K$ , construct  $\angle BKC = \angle DFH$ .
7. Through  $A, B, C$ , draw the tangents  $LAM, MBN$ , and  $NCL$ , forming the outer triangle  $LMN$ .
8. Since radii to points of contact are perpendicular to the tangents, the angles at  $A, B, C$  are right angles by Proposition III.18.
9. The quadrilateral  $AMBK$  has its four angles equal to four right angles.
10. Since  $\angle KAM$  and  $\angle KBM$  are right angles,  $\angle AKB + \angle AMB$  equals two right angles.
11. Since  $EG$  is a straight continuation of  $EF$ ,  $\angle DEG + \angle DEF$  also equals two right angles.
12. By construction  $\angle AKB = \angle DEG$ , so the remaining angles are equal:  $\angle AMB = \angle DEF$ .
13. In the same way,  $\angle LNB = \angle DFE$ .
14. By Proposition I.32, the remaining angle of triangle  $LMN$  satisfies  $\angle MLN = \angle EDF$ .
15. Therefore triangle  $LMN$  is equiangular with triangle  $DEF$ .
16. Each side of  $LMN$  is tangent to the given circle.

17. Therefore a triangle equiangular with the given triangle has been circumscribed about the given circle.

$$\angle AMB = \angle DEF, \quad \angle LNB = \angle DFE, \quad \angle MLN = \angle EDF$$

*Which is the very thing it was required to do.*

### BOOK IV – PROPOSITION 3.

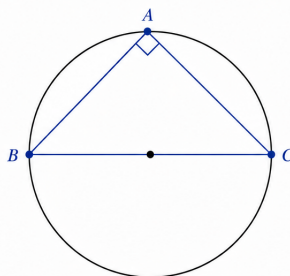
To inscribe a right-angled triangle in a given circle.

#### STATEMENT.

Let  $ABC$  be the given circle. To inscribe in it a triangle having a right angle.

#### VERBAL PROOF BROKEN INTO ACTIONS.

1. **Begin** with the given circle. Let  $ABC$  be the given circle.
2. **Draw** a diameter of the circle. Draw diameter  $BC$ .
3. **Take** any point on the circle. On the circumference of the circle, take any point  $A$ .
4. **Join**  $A$  to the ends of the diameter. Draw the straight-lines  $AB$  and  $AC$ .
5. **Apply** the angle-in-a-semicircle theorem. Since  $BC$  is a diameter and  $A$  is a point on the circle, the angle subtended by the diameter at the circumference is a right angle (Proposition III.31).
6. **Conclude** that the inscribed triangle is right-angled. Therefore  $\angle BAC$  is a right angle.
7. **Hence** triangle  $ABC$  has a right angle at  $A$ .
8. **And** it is inscribed in the given circle  $ABC$ .
9. **Therefore**, a right-angled triangle has been inscribed in the given circle.



#### Given:

○ Circle  $ABC$ .

#### To construct:

A right-angled triangle inscribed in the circle.

#### Construction summary:

Draw diameter  $BC$ .

Take any point  $A$  on the circumference.

Join  $AB$  and  $AC$ .

#### Result:

Triangle  $ABC$  is right-angled at  $A$ .

### BOOK IV – PROPOSITION 3: To inscribe a right-angled triangle in a given circle.

1 Begin with the given circle.

Let  $ABC$  be the given circle.



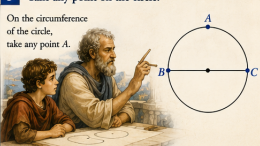
2 Draw a diameter of the circle.

Draw diameter  $BC$ .



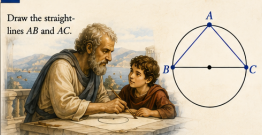
3 Take any point on the circle.

On the circumference of the circle, take any point  $A$ .



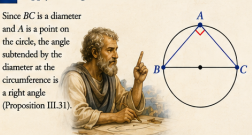
4 Join  $A$  to the ends of the diameter.

Draw the straight-lines  $AB$  and  $AC$ .



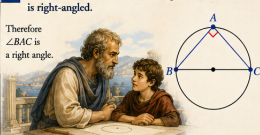
5 Apply the angle-in-a-semicircle theorem.

Since  $BC$  is a diameter and  $A$  is a point on the circle, the angle subtended by the diameter at the circumference is a right angle (Proposition III.31).



6 Conclude that the inscribed triangle is right-angled.

Therefore  $\angle BAC$  is a right angle.



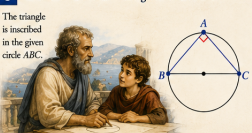
7 Hence triangle  $ABC$  has a right angle at  $A$ .

Triangle  $ABC$  has a right angle at  $A$ .



8 And it is inscribed in the given circle  $ABC$ .

The triangle is inscribed in the given circle  $ABC$ .



9 Therefore, a right-angled triangle has been inscribed in the given circle.

A right-angled triangle has been inscribed in the given circle  $ABC$ .



**RESULT:** Triangle  $ABC$  is right-angled at  $A$ .

## Proposition 4

*To inscribe a circle in a given triangle.*

Let  $ABC$  be the given triangle.

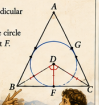
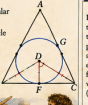
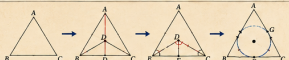
1. Begin with the given triangle  $ABC$ .
2. Bisect  $\angle ABC$  with  $BD$  by Proposition I.9.
3. Bisect  $\angle ACB$  with  $CD$ ; let the two bisectors meet at  $D$ .
4. From  $D$ , draw  $DE \perp AB$ .
5. From  $D$ , draw  $DF \perp BC$ .
6. From  $D$ , draw  $DG \perp CA$ .
7. Compare triangles  $EBD$  and  $FBD$ :  $\angle EBD = \angle FBD$ , the angles at  $E, F$  are right, and  $BD$  is common.
8. By Proposition I.26, conclude  $DE = DF$ .
9. Repeat the same reasoning at  $C$  to obtain  $DG = DF$ .
10. Therefore  $DE = DF = DG$ .
11. With centre  $D$  and radius  $DE$ , draw a circle.
12. Since  $DE = DF = DG$ , the circle also passes through  $F$  and  $G$ .
13. Since  $DE \perp AB$ , the circle touches  $AB$  at  $E$ .
14. Since  $DF \perp BC$ , the circle touches  $BC$  at  $F$ .
15. Since  $DG \perp CA$ , the circle touches  $CA$  at  $G$ .
16. Proposition III.16 excludes the possibility that any of these sides cuts the circle at the perpendicular endpoint.
17. Hence the circle centred at  $D$  and passing through  $E, F, G$  touches all three sides.
18. Therefore circle  $EFG$  has been inscribed in triangle  $ABC$ .

$$\boxed{DE = DF = DG}$$

*Which is the very thing it was required to do.*



## BOOK IV – PROPOSITION 4: To inscribe a circle in a given triangle.

|                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                    |                                                                                    |
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| <p><b>1</b> Begin with the given triangle.</p> <p>Let <math>ABC</math> be the given triangle.</p>                                                                                                                                                                                                                                                            | <p><b>2</b> Bisect angle <math>ABC</math>.</p> <p>Draw <math>BD</math> so that <math>\angle ABD = \angle DBC</math> (Proposition I.9).</p>                                                                            | <p><b>3</b> Bisect angle <math>ACB</math>.</p> <p>Draw <math>CD</math> so that <math>\angle ACD = \angle DCB</math>. The two bisectors meet at <math>D</math>.</p>                                                    | <p><b>4</b> Drop perpendicular from <math>D</math> to <math>AB</math>.</p> <p>Draw <math>DE \perp AB</math>.</p>  <p>(Perpendiculars by Proposition I.12.)</p>                                                                                              | <p><b>5</b> Drop perpendicular from <math>D</math> to <math>BC</math>.</p> <p>Draw <math>DF \perp BC</math>.</p>                                             | <p><b>6</b> Drop perpendicular from <math>D</math> to <math>CA</math>.</p> <p>Draw <math>DG \perp CA</math>.</p>                                                                 |                                                                                    |
| <p><b>7</b> Compare triangles <math>EBD</math> and <math>FBD</math>.</p> <p>We have:<br/> <math>\angle EBD = \angle FBD</math><br/> <math>(BD \text{ bisects } \angle ABC)</math><br/> <math>\angle BED = \angle BFD = 90^\circ</math><br/> <math>BD</math> is common.</p>                                                                                   | <p><b>8</b> Use Proposition I.26.</p> <p>Therefore triangles <math>EBD</math> and <math>FBD</math> are congruent in the required sense. Hence <math>DE = DF</math>.</p>                                               | <p><b>9</b> Repeat the same reasoning at <math>C</math>.</p> <p>Comparing the corresponding right triangles at <math>C</math>, we obtain <math>DG = DF</math>.</p>                                                    | <p><b>10</b> Conclude the three perpendicular distances are equal.</p> <p>Therefore <math>DE = DF = DG</math>.</p>                                                                                                                                          | <p><b>11</b> Use <math>D</math> as the centre.</p> <p>With centre <math>D</math>, draw a circle with radius <math>DE</math>.</p>                             | <p><b>12</b> The circle also passes through <math>F</math> and <math>G</math>.</p> <p>Since <math>DE = DF = DG</math>, the same circle passes through <math>E, F, G</math>.</p>  |                                                                                    |
| <p><b>13</b> The circle touches <math>AB</math> at <math>E</math>.</p> <p><math>DE</math> is perpendicular to <math>AB</math> at <math>E</math>. Therefore the circle cannot cut <math>AB</math> there; it touches <math>AB</math> at <math>E</math>.</p>                                                                                                    | <p><b>14</b> The circle touches <math>BC</math> at <math>F</math>.</p> <p><math>DF</math> is perpendicular to <math>BC</math> at <math>F</math>. Therefore the circle touches <math>BC</math> at <math>F</math>.</p>  | <p><b>15</b> The circle touches <math>CA</math> at <math>G</math>.</p> <p><math>DG</math> is perpendicular to <math>CA</math> at <math>G</math>. Therefore the circle touches <math>CA</math> at <math>G</math>.</p>  | <p><b>16</b> Use Proposition III.16 to justify tangency.</p> <p>If any side were to cut the circle instead of touching it, a line perpendicular to a diameter at its extremity would fall inside the circle, which is impossible (Proposition III.16).</p>  | <p><b>17</b> Conclude.</p> <p>The circle centred at <math>D</math>, passing through <math>E, F, G</math>, touches <math>AB, BC, CA</math> respectively.</p>  | <p><b>18</b> Therefore, circle <math>EPG</math> has been inscribed in triangle <math>ABC</math>.</p> <p>Q.E.D.</p>                                                               |                                                                                    |
| <p><b>CONSTRUCTION SUMMARY:</b></p> <p>Bisect <math>\angle ABC</math> and <math>\angle ACB</math> to meet at <math>D</math>. From <math>D</math> draw perpendiculars to the three sides, meeting them at <math>E, F, G</math>. With centre <math>D</math> and radius <math>DE</math> (<math>= DF = DG</math>) draw the circle touching all three sides.</p>  |                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                    |  |

## Proposition 5

*To circumscribe a circle about a given triangle.*

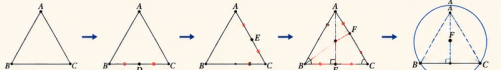
Let  $ABC$  be the given triangle.

1. Begin with the given triangle  $ABC$ .
2. Bisect  $AB$  at  $D$ .
3. Bisect  $AC$  at  $E$ .
4. Through  $D$ , draw  $DF \perp AB$ .
5. Through  $E$ , draw  $EF \perp AC$ , meeting  $DF$  at  $F$ .
6. Join  $FA$ ,  $FB$ , and  $FC$ .
7. Compare triangles  $ADF$  and  $BDF$ :  $AD = DB$ ,  $DF$  is common, and the included angles are right.
8. By Proposition I.4, conclude  $FA = FB$ .
9. Compare triangles  $AEF$  and  $CEF$  in the same way.
10. By Proposition I.4, conclude  $FA = FC$ .
11. Therefore  $FA = FB = FC$ .
12. With centre  $F$  and radius  $FA$ , draw a circle.
13. Since  $FB = FA$ , the circle passes through  $B$ .
14. Since  $FC = FA$ , the circle passes through  $C$ .
15. Thus the circle passes through all three vertices  $A, B, C$ .
16. Therefore a circle has been circumscribed about the given triangle.

$$\boxed{FA = FB = FC}$$

*Which is the very thing it was required to do.*

## BOOK IV – PROPOSITION 5: To circumscribe a circle about a given triangle.

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                       |                                                                                                                                                                                       |
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| <p><b>1</b> Begin with the given triangle.</p> <p>Let <math>ABC</math> be the given triangle.</p>                                                                                                                                                                                                                                                                                                                                             | <p><b>2</b> Bisect side <math>AB</math>.</p> <p>Bisect <math>AB</math> at <math>D</math>. (Prop. I.10)</p>                                  | <p><b>3</b> Bisect side <math>AC</math>.</p> <p>Bisect <math>AC</math> at <math>E</math>. (Prop. I.10)</p>                                                                                                                                                   | <p><b>4</b> Erect a perpendicular at <math>D</math>.</p> <p>From <math>D</math>, draw <math>DF</math> perpendicular to <math>AB</math>.</p>                          |                                                                                                                                                                                       |
| <p><b>5</b> Erect a perpendicular at <math>E</math>.</p> <p>From <math>E</math>, draw <math>EF</math> perpendicular to <math>AC</math>. Let the two perpendiculars meet at <math>F</math>.</p>                                                                                                                                                                                                                                                | <p><b>6</b> Join <math>F</math> to the vertices.</p> <p>Draw <math>FA</math>, <math>FB</math> and <math>FC</math>.</p>                      | <p><b>7</b> Compare triangles <math>ADF</math> and <math>BDF</math>.</p> <p>We have:</p> <ul style="list-style-type: none"> <li><math>AD = DB</math></li> <li><math>DF</math> is common</li> <li><math>\angle ADF = \angle BDF = 90^\circ</math></li> </ul>  | <p><b>8</b> Conclude <math>FA = FB</math>.</p> <p>Therefore the two triangles are equal by Prop. I.4, and hence <math>FA = FB</math>.</p>                            |                                                                                                                                                                                       |
| <p><b>9</b> Compare triangles <math>AEF</math> and <math>CEF</math>.</p> <p>Similarly:</p> <ul style="list-style-type: none"> <li><math>AE = EC</math></li> <li><math>EF</math> is common</li> <li><math>\angle AEF = \angle CEF = 90^\circ</math></li> </ul>                                                                                                                                                                                 | <p><b>10</b> Conclude <math>FA = FC</math>.</p> <p>Therefore the two triangles are equal by Prop. I.4, and hence <math>FA = FC</math>.</p>  | <p><b>11</b> Combine the equalities.</p> <p>We have <math>FA = FB = FC</math>. Thus <math>A, B, C</math> are all equally distant from <math>F</math>.</p>                                                                                                    | <p><b>12</b> Use <math>F</math> as the centre.</p> <p>Draw a circle with centre <math>F</math> and radius <math>FA</math>.</p>                                       |                                                                                                                                                                                       |
| <p><b>13</b> The circle passes through <math>B</math>.</p> <p>Since <math>FB = FA</math>, point <math>B</math> lies on the circle.</p>                                                                                                                                                                                                                                                                                                        | <p><b>14</b> The circle passes through <math>C</math>.</p> <p>Since <math>FC = FA</math>, point <math>C</math> lies on the circle.</p>      | <p><b>15</b> Therefore the circle passes through all three vertices.</p> <p>The circumference passes through <math>A, B, C</math>.</p>                                                                                                                       | <p><b>16</b> Conclusion.</p> <p>The circle with centre <math>F</math> has therefore been circumscribed about the given triangle <math>ABC</math>.</p> <p>Q.E.D.</p>  |                                                                                                                                                                                       |
| <p><b>CONSTRUCTION SUMMARY</b></p> <ul style="list-style-type: none"> <li>Bisect <math>AB</math> at <math>D</math> and <math>AC</math> at <math>E</math>.</li> <li>Erect perpendiculars at <math>D</math> and <math>E</math>.</li> <li>Their intersection is <math>F</math>.</li> <li>Draw circle with centre <math>F</math> and radius <math>FA</math>.</li> <li>The circle passes through <math>A, B</math> and <math>C</math>.</li> </ul>  |                                                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                       | <p><b>Key Idea:</b></p> <p>The point <math>F</math> is equally distant from <math>A, B</math> and <math>C</math>, so a circle centred at <math>F</math> passes through all three.</p> |

## Proposition 6

*To inscribe a square in a given circle.*

Let  $ABCD$  be the given circle, with centre  $E$ .

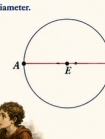
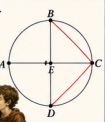
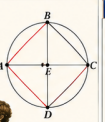
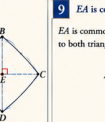
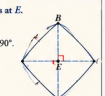
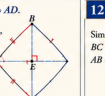
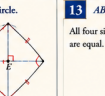
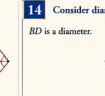
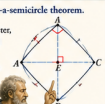
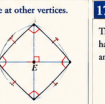
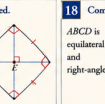
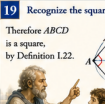
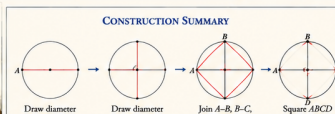
1. Begin with the given circle.
2. Draw diameter  $AC$  through the centre  $E$ .
3. Through  $E$ , draw diameter  $BD$  perpendicular to  $AC$ .
4. Join  $A$  to  $B$ .
5. Join  $B$  to  $C$ .
6. Join  $C$  to  $D$ .
7. Join  $D$  to  $A$ .
8. Compare triangles  $AEB$  and  $AED$ ; since  $E$  is the centre,  $EB = ED$ .
9. The side  $EA$  is common to the two triangles.
10. Because the diameters are perpendicular,  $\angle AEB = \angle AED = 90^\circ$ .
11. By Proposition I.4, conclude  $AB = AD$ .
12. Repeating the same reasoning around the circle gives  $AB = BC = CD = DA$ .
13. Therefore quadrilateral  $ABCD$  is equilateral.
14. Now consider diameter  $BD$ .
15. By Proposition III.31,  $\angle BAD$  is a right angle.
16. By the same theorem, the other three angles of the quadrilateral are right angles.
17. Therefore  $ABCD$  is right-angled.
18. Thus  $ABCD$  is both equilateral and right-angled.
19. By Definition I.22,  $ABCD$  is a square.

- 20.** Since its four vertices lie on the circumference, the square is inscribed in the given circle.

$$AB = BC = CD = DA, \quad \angle A = \angle B = \angle C = \angle D = 90^\circ$$

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 6: To inscribe a square in a given circle.

|                                                                                                                                                                                                                                                                                                         |                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                        |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given circle.<br/>Let <math>ABCD</math> be the given circle.</p>                                                                                                                            | <p><b>2</b> Draw one diameter.<br/>Draw diameter <math>AC</math> through the centre <math>E</math>.</p>                                                                                          | <p><b>3</b> Draw a second diameter perpendicular to the first.<br/>Through <math>E</math>, draw diameter <math>BD</math> so that <math>AC \perp BD</math>.</p>  | <p><b>4</b> Join <math>A</math> to <math>B</math>.<br/>Draw chord <math>AB</math>.</p>                                                |
| <p><b>5</b> Join <math>B</math> to <math>C</math>.<br/>Draw chord <math>BC</math>.</p>                                                                                                                                 | <p><b>6</b> Join <math>C</math> to <math>D</math>.<br/>Draw chord <math>CD</math>.</p>                                                                                                           | <p><b>7</b> Join <math>D</math> to <math>A</math>.<br/>Draw chord <math>DA</math>.<br/>Quadrilateral <math>ABCD</math> is formed.</p>                           | <p><b>8</b> Compare triangles <math>AEB</math> and <math>AED</math>.<br/><math>EB = ED</math><br/>(<math>E</math> is the centre)</p>  |
| <p><b>10</b> Right angles at <math>E</math>.<br/>Since <math>AC \perp BD</math>,<br/><math>\angle AEB = \angle AED = 90^\circ</math>.</p>                                                                              | <p><b>11</b> Conclude <math>AB = AD</math>.<br/>Therefore, by Prop. I.4,<br/><math>AB = AD</math>.</p>                                                                                           | <p><b>12</b> Repeat around the circle.<br/>Similarly,<br/><math>BC = CD =</math><br/><math>AB = AD</math>.</p>                                                  | <p><b>13</b> <math>ABCD</math> is equilateral.<br/>All four sides are equal.</p>                                                      |
| <p><b>15</b> Use angle-in-a-semicircle theorem.<br/>Since <math>BD</math> is a diameter,<br/>the angle subtended by it at <math>A</math> is a right angle (Prop. III.31).<br/><math>\angle BAD = 90^\circ</math>.</p>  | <p><b>16</b> Apply the same at other vertices.<br/>Similarly,<br/><math>\angle ABC = 90^\circ</math>,<br/><math>\angle BCD = 90^\circ</math>,<br/><math>\angle CDA = 90^\circ</math>.</p>        | <p><b>17</b> <math>ABCD</math> is right-angled.<br/>The quadrilateral has four right angles.</p>                                                                | <p><b>18</b> Combine both results.<br/><math>ABCD</math> is equilateral and right-angled.</p>                                         |
| <p><b>20</b> Conclude.<br/>Since <math>A, B, C, D</math> all lie on the circumference of the given circle, square <math>ABCD</math> is inscribed in the circle.</p>                                                    | <p>CONSTRUCTION SUMMARY</p>  <p>Draw diameter <math>AC</math>.<br/>Draw diameter <math>BD \perp AC</math>.<br/>Join <math>A-B, B-C, C-D, D-A</math>.<br/>Square <math>ABCD</math> inscribed.</p> | <p>Therefore, a square has been inscribed in the given circle.<br/><b>Q.E.D.</b></p>                                                                            |                                                                                                                                                                                                                        |

## Proposition 7

*To circumscribe a square about a given circle.*

Let  $ABCD$  be the given circle, with centre  $E$ .

1. Begin with the given circle.
2. Draw diameter  $AC$ .
3. Through  $E$ , draw diameter  $BD$  perpendicular to  $AC$ .
4. Through  $A$ , draw tangent  $FG$ .
5. Through  $B$ , draw tangent  $GH$ .
6. Through  $C$ , draw tangent  $HK$ .
7. Through  $D$ , draw tangent  $KF$ , forming quadrilateral  $FGHK$ .
8. By Proposition III.18,  $EA \perp FG$ .
9. Likewise  $EB \perp GH$ ,  $EC \perp HK$ , and  $ED \perp KF$ .
10. Since  $GH \perp BD$  and  $AC \perp BD$ ,  $GH \parallel AC$ .
11. Similarly  $AC \parallel FK$ .
12. Hence  $GH \parallel FK$  by Proposition I.30.
13. In the same way,  $FG \parallel HK$ .
14. The regions formed with the diameters and tangent lines are parallelograms.
15. By Proposition I.34,  $FG = HK$  and  $GH = FK$ .
16. Since  $AC$  and  $BD$  are diameters of the same circle,  $AC = BD$ .
17. From the parallelograms,  $GH = AC$ ,  $FK = AC$ ,  $FG = BD$ , and  $HK = BD$ .
18. Therefore  $FG = GH = HK = KF$ .
19. Hence  $FGHK$  is equilateral.
20. Consider the parallelogram formed by  $A, E, B, G$ .

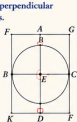
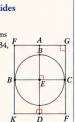
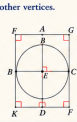
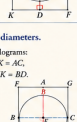
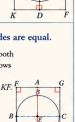
21. Because  $AC \perp BD$ ,  $\angle AEB = 90^\circ$ .
22. By Proposition I.34, the opposite angle at  $G$  is also right.
23. Repeating the argument gives right angles at  $H, K, F$ .
24. Therefore  $FGHK$  is right-angled.
25. Thus  $FGHK$  is both equilateral and right-angled.
26. Therefore  $FGHK$  is a square.
27. Since each side is tangent to the circle, the square has been circumscribed about the given circle.

$$\boxed{FG = GH = HK = KF}$$

*Which is the very thing it was required to do.*



BOOK IV – PROPOSITION 7: To circumscribe a square about a given circle.

| PANEL 1 – CONSTRUCTION                                                                                                                                                                                                          |                                                                                                                                                                                                  | PANEL 2 – PROVE EQUAL SIDES                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                                                                                                                                   | PANEL 3 – PROVE RIGHT ANGLES AND CONCLUDE                                                                                                                                                                                                                  |                                                                                                                                                                                                                                       |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given circle.</p> <p>Let <math>ABCD</math> be the given circle.</p>                                                 | <p><b>2</b> Draw one diameter.</p> <p>Draw diameter <math>AC</math> through the centre <math>E</math>.</p>      | <p><b>8</b> Radii are perpendicular to tangents.</p> <p><math>EA \perp FG</math>,<br/><math>EB \perp GH</math>,<br/><math>EC \perp HE</math>,<br/><math>ED \perp FK</math><br/>(Prop. III.18).</p>  | <p><b>9</b> <math>GH \parallel AC</math> and <math>FK \parallel AC</math>.</p> <p>Since <math>GH \perp BD</math> and <math>AC \perp BD</math>, <math>GH \parallel AC</math>. Similarly, <math>FK \parallel AC</math>. Therefore, <math>GH \parallel FK</math> (Prop. I.30).</p>  | <p><b>14</b> Use a parallelogram.</p> <p>In parallelogram <math>AEBG</math>, <math>\angle AEB = 90^\circ</math> since <math>AC \perp BD</math>.</p>                       | <p><b>15</b> Opposite angle is right.</p> <p>By Prop. I.34 (opposite angles of a parallelogram are equal), <math>\angle AGB = 90^\circ</math>.</p>  |
| <p><b>3</b> Draw a perpendicular diameter.</p> <p>Through <math>E</math>, draw diameter <math>BD</math> so that <math>AC \perp BD</math>.</p>  | <p><b>4</b> Draw a tangent at <math>A</math>.</p> <p>Through <math>A</math>, draw tangent <math>FG</math>.</p>  | <p><b>10</b> <math>FG \parallel HK</math>.</p> <p>By the same reasoning, <math>FG \parallel HK</math>.</p>                                                                                          | <p><b>11</b> Opposite sides are equal.</p> <p>In the parallelograms formed, by Prop. I.34, <math>FG = HK</math> and <math>GH = FK</math>.</p>                                                                                                                                    | <p><b>16</b> Repeat at other vertices.</p> <p>Similarly, <math>\angle BHK = 90^\circ</math>, <math>\angle HKF = 90^\circ</math>, <math>\angle KFG = 90^\circ</math>.</p>  | <p><b>17</b> All angles are right.</p> <p>Quadrilateral <math>FGHK</math> has four right angles.</p>                                                |
| <p><b>5</b> Draw a tangent at <math>B</math>.</p> <p>Through <math>B</math>, draw tangent <math>GH</math>.</p>                                 | <p><b>6</b> Draw a tangent at <math>C</math>.</p> <p>Through <math>C</math>, draw tangent <math>HK</math>.</p>  | <p><b>12</b> Relate to diameters.</p> <p>From the parallelograms: <math>GH = AC</math> and <math>FK = AC</math>, <math>FG = BD</math> and <math>HK = BD</math>.</p>                                 | <p><b>13</b> All four sides are equal.</p> <p>Since <math>AC = BD</math> (both diameters), it follows that <math>FG = GH = HK = FK</math>.</p>                                                                                                                                   | <p><b>18</b> Combine both results.</p> <p><math>FGHK</math> is both equilateral (Panel 2) and right-angled (Panel 3).</p>                                                 | <p><b>19</b> Recognize the square.</p> <p>Therefore, by Definition I.22, <math>FGHK</math> is a square.</p>                                         |
| <p><b>7</b> Draw a tangent at <math>D</math>.</p> <p>Through <math>D</math>, draw tangent <math>FK</math>.</p>                                 | <p>The four tangent lines form the quadrilateral <math>FGHK</math> about the given circle.</p>                  | <p>Therefore, quadrilateral <math>FGHK</math> is equilateral (all four sides are equal).</p>                                                                                                        | <p><b>20</b> Conclusion.</p> <p>The square <math>FGHK</math> has been circumscribed about the given circle <math>ABCD</math>.</p>                                                                                                                                                | <p><b>Q.E.D.</b> A square has been circumscribed about the given circle.</p>                                                                                             |                                                                                                                                                                                                                                       |

## Proposition 8

*To inscribe a circle in a given square.*

Let  $ABCD$  be the given square.

1. Begin with the given square  $ABCD$ .
2. Bisect side  $AD$  at  $E$ .
3. Bisect side  $AB$  at  $F$ .
4. Through  $E$ , draw  $EH \parallel AB$  and  $CD$ .
5. Through  $F$ , draw  $FK \parallel AD$  and  $BC$ .
6. Let  $EH$  and  $FK$  meet at  $G$ .
7. Recognize the parallelograms formed by these two lines.
8. Since  $ABCD$  is a square,  $AD = AB$ .
9. Since  $E, F$  are midpoints,  $AE = \frac{1}{2}AD$  and  $AF = \frac{1}{2}AB$ .
10. Therefore  $AE = AF$ .
11. By equality of opposite sides in the parallelograms,  $GE = GF$ .
12. Similarly  $GH = GK$ , and these lengths equal  $GE$  and  $GF$ .
13. Hence  $GE = GF = GH = GK$ .
14. With centre  $G$  and radius  $GE$ , draw a circle.
15. The same circle passes through  $F, H, K$ .
16. Since  $EH \parallel AB$  and  $AB \perp AD$ ,  $GE \perp AD$ .
17. Since  $FK \parallel AD$  and  $AD \perp AB$ ,  $GF \perp AB$ .
18. Similarly  $GH \perp BC$  and  $GK \perp CD$ .
19. Therefore the circle touches  $AD$  at  $E$ .
20. It touches  $AB$  at  $F$ .
21. It touches  $BC$  at  $H$ .

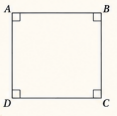
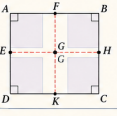
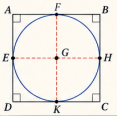
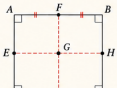
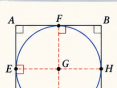
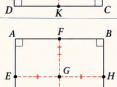
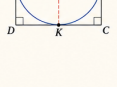
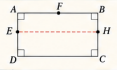
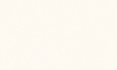
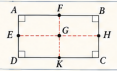
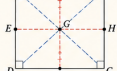
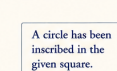
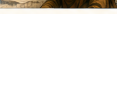
**22.** It touches  $CD$  at  $K$ .

**23.** Thus the circle centred at  $G$  touches all four sides and has been inscribed in square  $ABCD$ .

$$\boxed{GE = GF = GH = GK}$$

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 8: To inscribe a circle in a given square.

| PANEL 1 – CONSTRUCTION                                                                                                                                                                                           | PANEL 2 – EQUAL DISTANCES FROM THE CENTRE                                                                                                                                                                                           | PANEL 3 – TANGENCY AND CONCLUSION                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given square.<br/>Let <math>ABCD</math> be the given square.</p>                                     | <p><b>7</b> The construction forms parallelograms.<br/>Opposite sides in a parallelogram are equal.</p>                                            | <p><b>14</b> With centre <math>G</math>, draw a circle of radius <math>GE</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                              |
| <p><b>2</b> Bisect side <math>AD</math>.<br/>Let <math>E</math> be the midpoint of <math>AD</math>.</p>                         | <p><b>8</b> In the square, <math>AD = AB</math>.</p>                                                                                               | <p><b>15</b> The circle passes through <math>E, F, H, K</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                |
| <p><b>3</b> Bisect side <math>AB</math>.<br/>Let <math>F</math> be the midpoint of <math>AB</math>.</p>                         | <p><b>9</b> Since <math>E</math> and <math>F</math> are midpoints,<br/><math>AE = \frac{1}{2} AD</math> and <math>AF = \frac{1}{2} AB</math>.</p>  | <p><b>16</b> <math>GE \perp AD</math>, because<br/><math>EH \parallel AB</math> and <math>AB \perp AD</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                  |
| <p><b>4</b> Through <math>E</math>, draw <math>EH</math> parallel to <math>AB</math> and <math>CD</math>.</p>                   | <p><b>10</b> Therefore, <math>AE = AF</math>.</p>                                                                                                  | <p><b>17</b> <math>GF \perp AB</math>, because<br/><math>FK \parallel AD</math> and <math>AD \perp AB</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                  |
| <p><b>5</b> Through <math>F</math>, draw <math>FK</math> parallel to <math>AD</math> and <math>BC</math>.</p>                   | <p><b>11</b> By equality of opposite sides in the parallelograms,<br/><math>GE = GF</math>.</p>                                                    | <p><b>18</b> Similarly,<br/><math>GH \perp BC</math> and<br/><math>GK \perp CD</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                         |
| <p><b>6</b> Let the two lines meet at <math>G</math>.<br/><math>EH</math> and <math>FK</math> intersect at <math>G</math>.</p>  | <p><b>12</b> Similarly,<br/><math>GH = GK</math> and these are equal to <math>GE</math> and <math>GF</math>.</p>                                   | <p><b>19</b> Therefore the circle touches <math>AD</math> at <math>E</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                                   |
| <p><b>7</b> Hence<br/><math>GE = GF = GH = GK</math>.<br/>All four distances from <math>G</math> are equal.</p>                 | <p>Therefore, <math>G</math> is the centre of the required circle.</p>                                                                             | <p><b>20</b> It touches <math>AB</math> at <math>F</math>.</p> <p><b>21</b> It touches <math>BC</math> at <math>H</math>.</p> <p><b>22</b> It touches <math>CD</math> at <math>K</math>.</p> <p><b>23</b> Thus the circle centred at <math>G</math>, passing through <math>E, F, H, K</math>, touches all four sides of the square <math>ABCD</math>.</p> <div style="border: 1px solid black; padding: 5px; width: fit-content;"> <p>A circle has been inscribed in the given square.</p> </div>  |

## Proposition 9

*To circumscribe a circle about a given square.*

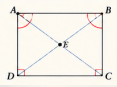
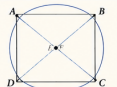
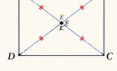
Let  $ABCD$  be the given square.

1. Begin with the given square  $ABCD$ .
2. Draw diagonal  $AC$ .
3. Draw diagonal  $BD$ .
4. Let the diagonals meet at  $E$ .
5. Compare triangles  $DAC$  and  $BAC$ :  $DA = AB$ ,  $AC$  is common, and  $DC = BC$ .
6. By Proposition I.8,  $\angle DAC = \angle BAC$ .
7. Hence  $AC$  bisects  $\angle DAB$ .
8. Similarly, the diagonals bisect the other angles of the square.
9. Since  $\angle DAB = \angle ABC$ , their halves are equal.
10. Therefore  $\angle EAB = \angle EBA$ .
11. By Proposition I.6,  $EA = EB$ .
12. Repeating the same reasoning gives  $EA = EC$  and  $EA = ED$ .
13. Hence  $EA = EB = EC = ED$ .
14. With centre  $E$  and radius  $EA$ , draw a circle.
15. Since  $EB = EA$ , the circle passes through  $B$ .
16. Since  $EC = EA$ , the circle passes through  $C$ .
17. Since  $ED = EA$ , the circle passes through  $D$ .
18. Thus the circle passes through  $A, B, C, D$  and is circumscribed about the given square.

$$EA = EB = EC = ED$$

*Which is the very thing it was required to do.*

**BOOK IV – PROPOSITION 9:** To circumscribe a circle about a given square.

| PANEL 1 – CONSTRUCTION AND ANGLE BISECTION                                                                                                                                                                                                                                                                                                        | PANEL 2 – EQUAL DISTANCES FROM THE CENTRE                                                                                                                                                                                  | PANEL 3 – THE CIRCUMCIRCLE                                                                                                                                                      |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Begin with the given square. Let <math>ABCD</math> be the given square.</p>                                                                                                                                                                          | <p><b>8</b> Since <math>\angle DAB = \angle ABC</math> (right angles of the square), their halves are equal.</p>                          | <p><b>13</b> With centre <math>E</math>, draw a circle of radius <math>EA</math>.</p>          |
| <p><b>2</b> Draw diagonal <math>AC</math>.</p>                                                                                                                                                                                                                   | <p><b>9</b> Hence <math>\angle EAB = \angle EBA</math>, because <math>AC</math> and <math>BD</math> bisect those angles.</p>              | <p><b>14</b> Since <math>EB = EA</math>, point <math>B</math> lies on the circle.</p>          |
| <p><b>3</b> Draw diagonal <math>BD</math>.</p>                                                                                                                                                                                                                   | <p><b>10</b> In triangle <math>AEB</math>, <math>\angle EAB = \angle EBA</math>. By Prop. 1.6, <math>EA = EB</math>.</p>                  | <p><b>15</b> Since <math>EC = EA</math>, point <math>C</math> lies on the circle.</p>          |
| <p><b>4</b> Let the diagonals meet at <math>E</math>.</p>                                                                                                                                                                                                        | <p><b>11</b> Similarly, <math>EA = EC</math> and <math>EA = ED</math>.</p>                                                                | <p><b>16</b> Since <math>ED = EA</math>, point <math>D</math> lies on the circle.</p>          |
| <p><b>5</b> In triangles <math>DAC</math> and <math>BAC</math>, <math>DA = AB</math> (sides of a square) <math>AC = AC</math> (common) <math>DC = BC</math> (sides of a square) <math>\therefore</math> By Prop. 1.8, <math>\angle DAC = \angle BAC</math>.</p>  | <p><b>12</b> Therefore <math>EA = EB = EC = ED</math>. All four vertices are equally distant from <math>E</math>.</p>                     | <p><b>17</b> The circle centred at <math>E</math> passes through <math>A, B, C, D</math>.</p>  |
| <p><b>6</b> Therefore diagonal <math>AC</math> bisects <math>\angle DAB</math>.</p>                                                                                                                                                                              | <p>Thus <math>E</math> is the centre of a circle through <math>A, B, C, D</math>, since all are equally distant from <math>E</math>.</p>  | <p>Therefore, a circle has been circumscribed about the given square <math>ABCD</math>.</p>    |
| <p><b>7</b> Similarly, the diagonals bisect the other angles of the square.</p>                                                                                                                                                                                  |                                                                                                                                           |                                                                                                |

## Proposition 10

*To construct an isosceles triangle having each of the angles at the base double the remaining angle.*

1. Begin with a straight-line  $AB$ .
2. Cut  $AB$  at  $C$  so that  $AB \cdot BC = AC^2$ , by Proposition II.11.
3. With centre  $A$  and radius  $AB$ , draw circle  $BDE$ .
4. In the circle, insert chord  $BD$  such that  $BD = AC$ , by Proposition IV.1.
5. Join  $A$  to  $D$ .
6. Join  $D$  to  $C$ .
7. Circumscribe a circle about triangle  $ACD$ , by Proposition IV.5.
8. From the original construction,  $AB \cdot BC = AC^2$ .
9. Since  $BD = AC$ , conclude  $AB \cdot BC = BD^2$ .
10. By Proposition III.37,  $BD$  is tangent to the circumcircle of triangle  $ACD$  at  $D$ .
11. By Proposition III.32,  $\angle BDC = \angle DAC$ .
12. Add  $\angle CDA$  to both sides:  $\angle BDA = \angle CDA + \angle DAC$ .
13. By the exterior-angle theorem at  $C$ ,  $\angle BCD = \angle CDA + \angle DAC$ , hence  $\angle BDA = \angle BCD$ .
14. Since  $AB = AD$ , triangle  $ABD$  is isosceles and  $\angle BDA = \angle DBA$ .
15. Thus  $\angle BDA = \angle DBA = \angle BCD$ .
16. In triangle  $BCD$ ,  $\angle DBC = \angle BCD$ , so by Proposition I.6,  $BD = DC$ .
17. Since  $BD = AC$ , it follows that  $AC = CD$ .
18. Therefore triangle  $ACD$  is isosceles and  $\angle CDA = \angle DAC$ .
19. Hence  $\angle CDA + \angle DAC = 2\angle DAC$ .
20. But  $\angle BCD = \angle CDA + \angle DAC$ , so  $\angle BCD = 2\angle DAC$ .

21. Since  $\angle BDA = \angle BCD$  and  $\angle DBA = \angle BCD$ , each base angle of triangle  $ABD$  is twice the angle at  $A$ .
22. Thus triangle  $ABD$  is the required isosceles triangle.
23. Therefore an isosceles triangle has been constructed with each base angle double the remaining angle.

$$AB \cdot BC = AC^2 = BD^2$$

*Which is the very thing it was required to do.*



BOOK IV – PROPOSITION 10: To construct an isosceles triangle having each of the angles at the base double the remaining angle.

**PANEL 1 – CONSTRUCTION**

- 1 Begin with a straight-line  $AB$ .
- 2 Cut  $AB$  at  $C$  so that  $AB \times BC = AC^2$  (Prop. II.11).
- 3 With centre  $A$  and radius  $AB$ , draw circle  $BDE$ .
- 4 In the circle, draw chord  $BD$  such that  $BD = AC$  (Prop. IV.1).
- 5 Join  $A$  to  $D$ .
- 6 Join  $D$  to  $C$ .
- 7 Circumscribe a circle about triangle  $ACD$  (Prop. IV.5).

**PANEL 2 – PROOF: TANGENT AND ANGLES**

- 8 From the construction,  $AB \times BC = AC^2$ .
- 9 Since  $BD = AC$ ,  $AB \times BC = BD^2$ .
- 10 By Prop. III.37 (converse of tangent-secant),  $BD$  is tangent to the circumcircle of triangle  $ACD$  at  $D$ .
- 11 By Prop. III.32 (tangent-chord),  $\angle BDC = \angle DAC$ .
- 12 Add  $\angle CDA$  to both sides:  $\angle BDA = \angle CDA + \angle DAC$ .
- 13 By exterior angle theorem at  $C$ ,  $\angle BCD = \angle CDA + \angle DAC$ . Hence  $\angle BDA = \angle BCD$ .
- 14 Since  $AB = AD$  (radii of the circle  $BDE$ ), by Prop. I.5,  $\angle BDA = \angle DBA$ .
- 15 Therefore  $\angle BDA = \angle DBA = \angle BCD$ .
- 16 In triangle  $BCD$ , since  $\angle DBC = \angle BCD$ , by Prop. I.6,  $BD = DC$ .
- 17 But  $BD = AC$  (construction). Therefore  $AC = CD$ .
- 18 In triangle  $ACD$ ,  $AC = CD$ , by Prop. I.5,  $\angle CDA = \angle DAC$ .

Key results so far:

- $BD$  is tangent at  $D$ .
- $AC = CD$ .
- $\angle BDA = \angle DBA = \angle BCD$ .

**PANEL 3 – EQUAL ANGLES AND CONCLUSION**

- 19 Since  $\angle CDA = \angle DAC$ , add them:  $\angle CDA + \angle DAC = 2\angle DAC$ .
- 20 But  $\angle BCD = \angle CDA + \angle DAC$ . Therefore  $\angle BCD = 2\angle DAC$ .
- 21 Since  $\angle BDA = \angle BCD$  and  $\angle DBA = \angle BCD$ , we have  $\angle BDA = \angle DBA = 2\angle DAC$ .
- 22 Thus in triangle  $ABD$ , the two angles at the base (at  $B$  and  $D$ ) are each double the remaining angle (at  $A$ ).
- 23 Therefore the required isosceles triangle has been constructed.

**Q.E.D.**  
An isosceles triangle in which each of the angles at the base is double the remaining angle has been constructed.

## Proposition 11

*To inscribe an equilateral and equiangular pentagon in a given circle.*

Let  $ABCDE$  be the given circle.

1. Begin with the given circle.
2. Construct the special isosceles triangle  $FGH$  of Proposition IV.10.
3. By Proposition IV.2, inscribe triangle  $ACD$  equiangular with  $FGH$ .
4. Hence  $\angle ACD = 2\angle CAD$  and  $\angle CDA = 2\angle CAD$ .
5. Bisect  $\angle ACD$  with  $CE$ .
6. Bisect  $\angle CDA$  with  $DB$ .
7. Extend  $CE$  and  $DB$  to meet the circumference at  $E$  and  $B$ .
8. The five angles  $\angle CAD$ ,  $\angle ACE$ ,  $\angle ECD$ ,  $\angle CDB$ ,  $\angle BDA$  are equal.
9. By Proposition III.26, equal inscribed angles cut off equal arcs.
10.  $\angle CAD$  subtends arc  $CD$ .
11.  $\angle ACE$  subtends arc  $AE$ .
12.  $\angle ECD$  subtends arc  $ED$ .
13.  $\angle CDB$  subtends arc  $CB$ .
14.  $\angle BDA$  subtends arc  $BA$ .
15. Therefore arcs  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal.
16. By Proposition III.29, equal arcs are subtended by equal chords.
17. Hence  $AB = BC = CD = DE = EA$ .
18. Therefore pentagon  $ABCDE$  is equilateral.
19. Join  $A$  to  $C$ .
20. Join  $B$  to  $E$ .

21. Angle  $ABC$  stands on the arc from  $A$  to  $C$  not containing  $B$ , namely  $AE + ED + DC$ .
22. Angle  $CDE$  stands on the arc from  $C$  to  $E$  not containing  $D$ , namely  $CB + BA + AE$ .
23. Since all five component arcs are equal, these two three-arc sums are equal.
24. Therefore  $\angle ABC = \angle CDE$  by Proposition III.27.
25. Repeat the same comparison around the pentagon.
26. Hence all five interior angles are equal.
27. Therefore the pentagon is equiangular.
28. The pentagon is both equilateral and equiangular.
29. Therefore an equilateral and equiangular pentagon has been inscribed in the given circle.

$$\boxed{AB = BC = CD = DE = EA}$$

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 11: To inscribe an equilateral and equiangular pentagon in a given circle.

**PANEL 1 – CONSTRUCTION**

1 Let  $ABCDE$  be the given circle.

2 Construct isosceles triangle  $FGH$  (Prop. I.10) such that each base angle is double the vertex angle.

3 By Prop. IV.2, inscribe in the circle a triangle  $ACD$  equiangular with  $FGH$ .  
 $\angle CAD = \angle F$ ,  
 $\angle ACD = \angle G$ ,  
 $\angle CDA = \angle H$ .

4 Therefore  $\angle ACD$  and  $\angle CDA$  are each double  $\angle CAD$ .

5 Bisect  $\angle ACD$ ; draw  $CE$ .

6 Bisect  $\angle CDA$ ; draw  $DB$ .

7 Extend  $CE$  and  $DB$  to the circle. Points  $E$  and  $B$  lie on the circumference.

**PANEL 2 – EQUAL SIDES (EQUILATERAL)**

8  $CE$  and  $DB$  bisect the two double angles.  
Hence the five highlighted angles are equal.  
 $\angle CAD = \angle ACE = \angle ECD = \angle CDB = \angle BDA$ .

9 Equal angles standing on the circumference cut off equal arcs (Prop. III.26).

10 Therefore the five arcs are equal.  
 $\text{arc } AB = \text{arc } BC = \text{arc } CD = \text{arc } DE = \text{arc } EA$ .

11 Equal arcs of the same circle are subtended by equal straight-lines (Prop. III.29).

12 Therefore all five sides are equal.  
 $AB = BC = CD = DE = EA$ .

Hence  $ABCDE$  is an equilateral pentagon.

**PANEL 3 – EQUAL ANGLES (EQUIANGULAR)**

13 Draw diagonals  $AC$  and  $BE$ .

14  $\angle ABC$  stands on arc  $AEC$  (not containing  $D$ ).  
 $\angle AEC$  stands on arc  $ABC$  (not containing  $D$ ).  
Since all five arcs are equal,  
 $\angle AEC = \angle ABC$ .

15 By Prop. III.27, equal arcs subtend equal angles.  
Therefore  $\angle ABC = \angle CDE$ .

16 By the same reasoning around the pentagon,  
 $\angle BCD = \angle DEA = \angle EAB$ .

17 Hence all five interior angles are equal.

18 Therefore  $ABCDE$  is an equilateral (all sides equal) and equiangular (all angles equal) pentagon inscribed in the given circle.

## Proposition 12

*To circumscribe an equilateral and equiangular pentagon about a given circle.*

Let  $ABCDE$  denote the five equal divisions of the given circle.

1. Begin with the given circle.
2. Take five points  $A, B, C, D, E$  so that arcs  $AB, BC, CD, DE, EA$  are equal.
3. Draw the tangent at  $A$ .
4. Draw the tangent at  $B$ .
5. Draw the tangent at  $C$ .
6. Draw the tangent at  $D$ .
7. Draw the tangent at  $E$ ; let the five tangents form pentagon  $GHKLM$ .
8. Find the centre  $F$  of the circle.
9. Join  $FB, FC, FD, FK, FL$  as required by the proof.
10. Since  $KL$  touches the circle at  $C$ ,  $FC \perp KL$ .
11. Likewise the corresponding radii at  $B$  and  $D$  are perpendicular to their tangents.
12. In right triangle  $FCK$ ,  $FK^2 = FC^2 + CK^2$ .
13. In right triangle  $FBK$ ,  $FK^2 = FB^2 + BK^2$ .
14. Since  $FB = FC$ , their squares are equal.
15. Subtracting equals gives  $BK^2 = CK^2$ .
16. Therefore  $BK = CK$ .
17. Compare triangles  $BFK$  and  $CFK$ :  $FB = FC$ ,  $FK$  is common, and  $BK = CK$ .
18. By Proposition I.8,  $\angle BFK = \angle KFC$  and  $\angle BKF = \angle FKC$ .
19. Thus  $FK$  bisects the corresponding central and outer angles.

20. Repeat the same reasoning at the adjacent outer vertex  $L$ .
21. Since arcs  $BC$  and  $CD$  are equal, central angles  $\angle BFC$  and  $\angle CFD$  are equal.
22. Hence the corresponding half-angles  $\angle KFC$  and  $\angle LFC$  are equal.
23. Compare triangles  $FKC$  and  $FLC$ .
24. By Proposition I.26,  $KC = CL$  and the remaining corresponding angles are equal.
25. Therefore  $KL = 2KC$ .
26. Similarly  $HK = 2BK$ .
27. Since  $BK = KC$ , conclude  $HK = KL$ .
28. Repeat around the pentagon to obtain  $GH = HK = KL = LM = MG$ .
29. Therefore the outer pentagon is equilateral.
30. The small angles at  $K$  and  $L$  are equal.
31. Since the full pentagon angles there are twice these equal small angles,  $\angle HKL = \angle KLM$ .
32. Repeat this reasoning around all five vertices.
33. Therefore all five interior angles are equal.
34. Hence  $GHKLM$  is equiangular and, as already shown, equilateral.
35. Since every side is tangent to the circle, the required equilateral and equiangular pentagon has been circumscribed about the given circle.

$$\boxed{GH = HK = KL = LM = MG}$$

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 12: To circumscribe an equilateral and equiangular pentagon about a given circle.

| PANEL 1 – CONSTRUCTION<br>(Figures 1–9)                                                                                              |       |       | PANEL 2 – RIGHT ANGLES AND<br>PYTHAGOREAN EQUALITIES<br>(Figures 10–18)                            |        | PANEL 3 – EQUAL TANGENT SEGMENTS<br>(Figures 19–27)                                                          |        |        | PANEL 4 – EQUAL SIDES AND EQUAL ANGLES<br>(Figures 28–36)                                                                                                                                             |        |        |        |        |        |        |        |        |
|--------------------------------------------------------------------------------------------------------------------------------------|-------|-------|----------------------------------------------------------------------------------------------------|--------|--------------------------------------------------------------------------------------------------------------|--------|--------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1<br>                                                                                                                                | 2<br> | 3<br> | 10<br>                                                                                             | 11<br> | 19<br>                                                                                                       | 20<br> | 21<br> | 28<br>                                                                                                                                                                                                | 29<br> | 30<br> | 31<br> | 32<br> | 33<br> | 34<br> | 35<br> | 36<br> |
| <p>Let ABCDE be the given circle.</p> <p>Take five points so that arcs AB, BC, CD, DE, EA are equal.</p> <p>Draw a tangent at A.</p> |       |       | <p>Since KL touches the circle at C, FC ⊥ KL.</p> <p>Similarly, FB ⊥ GH at B and FD ⊥ LM at D.</p> |        | <p>Repeat at L. FL bisects ∠BFD and ∠ELE.</p> <p>Also BC and CD are equal. By Prop. III.22, ∠BFC = ∠CFD.</p> |        |        | <p>By the same reasoning all around, GH = HK = KL = LM = MG.</p> <p>Therefore the pentagon GHKLM is equilateral.</p>                                                                                  |        |        |        |        |        |        |        |        |
| <p>Draw a tangent at B.</p> <p>Draw a tangent at C.</p> <p>Draw a tangent at D.</p>                                                  |       |       | <p>In ΔFBK, ∠FBK is right. By Pythagoras, <math>FK^2 = FB^2 + BK^2</math>.</p>                     |        | <p>In ΔFKC and ΔFLC, ∠KFC = ∠LFC, ∠FCK = ∠FCL = 90°, FC common.</p> <p>Therefore KC = CL.</p>                |        |        | <p>The interior angle at K is twice the small angle; the interior angle at L is twice the small angle.</p> <p>Therefore ∠HKL = ∠KLM.</p> <p>Similarly around, all five interior angles are equal.</p> |        |        |        |        |        |        |        |        |
| <p>Draw a tangent at E.</p> <p>The five tangents form pentagon GHKLM.</p> <p>Let F be the centre. Join FI, FC, FD, FK, FL.</p>       |       |       | <p>Therefore BK = CK.</p> <p>Subtracting equals from equals, <math>BK^2 = CK^2</math>.</p>         |        | <p>Thus KL = 2 · KC.</p> <p>Similarly, HK = 2 · BK.</p> <p>Since BK = KC, HK = KL.</p>                       |        |        | <p>Each side of the pentagon touches the given circle.</p> <p>Therefore GHKLM is equiangular.</p> <p>Hence an equilateral and equiangular pentagon has been circumscribed about the given circle.</p> |        |        |        |        |        |        |        |        |

Blue circle: given circle    Pink lines: tangents    Red square: right angle (radius ⊥ tangent)    Equal angles are shown with the same colour.    Equal lengths are shown with the same marks.

## Proposition 13

*To inscribe a circle in a given pentagon which is equilateral and equiangular.*

Let  $ABCDE$  be the given equilateral and equiangular pentagon.

1. Begin with the given regular pentagon.
2. Bisect  $\angle BCD$  with  $CF$ .
3. Bisect  $\angle CDE$  with  $DF$ .
4. Let the two angle bisectors meet at  $F$ .
5. Join  $F$  to  $B$ .
6. Join  $F$  to  $A$ .
7. Join  $F$  to  $E$ .
8. Compare triangles  $BCF$  and  $DCF$ :  $BC = CD$ ,  $CF$  is common, and  $\angle BCF = \angle FCD$ .
9. By Proposition I.4, the triangles are congruent.
10. Hence  $BF = DF$ .
11. Also  $\angle CBF = \angle CDF$ .
12. Since the pentagon is equiangular,  $\angle ABC = \angle CDE$ .
13. Since  $DF$  bisects  $\angle CDE$ ,  $\angle CDE = 2\angle CDF$ .
14. Since  $\angle CDF = \angle CBF$ ,  $\angle ABC = 2\angle CBF$ .
15. Therefore  $FB$  bisects  $\angle ABC$ .
16. Similarly  $FA$  bisects  $\angle BAE$  and  $FE$  bisects  $\angle AED$ .
17. Drop  $FG \perp AB$ .
18. Drop  $FH \perp BC$ .
19. Drop  $FK \perp CD$ .
20. Drop  $FL \perp DE$ .



21. Drop  $FM \perp EA$ .
22. Compare right triangles  $FHC$  and  $FKC$ :  $\angle HCF = \angle FCK$ , both have right angles, and  $FC$  is common.
23. By Proposition I.26,  $FH = FK$ .
24. Repeat the argument around the pentagon.
25. Therefore  $FG = FH = FK = FL = FM$ .
26. With centre  $F$  and radius  $FG$ , draw a circle.
27. Since the five perpendicular distances are equal, the circle passes through  $H, K, L, M$ .
28. The circle touches  $AB$  at  $G$ .
29. It touches  $BC$  at  $H$ .
30. It touches  $CD$  at  $K$ .
31. It touches  $DE$  at  $L$  and  $EA$  at  $M$ .
32. Proposition III.16 excludes the possibility that a side cuts the circle at one of these perpendicular endpoints.
33. Therefore the circle centred at  $F$  touches all five sides and has been inscribed in the given regular pentagon.

$$\boxed{FG = FH = FK = FL = FM}$$

*Which is the very thing it was required to do.*

BOOK IV – PROPOSITION 13: To inscribe a circle in a given equilateral and equiangular pentagon.

| PANEL 1 – CONSTRUCTION (Figures 1–9)                           |                                     |                                   | PANEL 2 – THE POINT F AND ANGLE BISECTIONS (Figures 10–18)                                   |                                                                                                                      |                                                                      | PANEL 3 – PERPENDICULARS AND EQUAL DISTANCES (Figures 19–27) |                                              |                                                                                                                                                  | PANEL 4 – THE INSCRIBED CIRCLE AND TANGENCY (Figures 28–36)                              |                                                     |                                                     |
|----------------------------------------------------------------|-------------------------------------|-----------------------------------|----------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------|--------------------------------------------------------------|----------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|-----------------------------------------------------|-----------------------------------------------------|
|                                                                |                                     |                                   |                                                                                              |                                                                                                                      |                                                                      |                                                              |                                              |                                                                                                                                                  |                                                                                          |                                                     |                                                     |
| Let $ABCDE$ be the given equilateral and equiangular pentagon. | Bisect $\angle BCD$ . Draw $CF$ .   | Bisect $\angle CDE$ . Draw $DF$ . | $BC = CD$ (equilateral). $CF$ common. $\angle BCF = \angle CDF$ .                            | By Prop. I.4, $\triangle BCF \cong \triangle DCF$ (equilateral). Therefore $BF = DF$ and $\angle CBF = \angle CDE$ . | $\angle ABC = \angle CDE$ (equilateral). $\angle CBF = \angle CDE$ . | From $F$ draw $FG \perp AB$ , meeting $AB$ at $G$ .          | Draw $FH \perp BC$ , meeting $BC$ at $H$ .   | Draw $FK \perp CD$ , meeting $CD$ at $K$ .                                                                                                       | With centre $F$ and radius $FG$ , draw the circle.                                       | It passes through $H$ . So it touches $BC$ at $H$ . | It passes through $K$ . So it touches $CD$ at $K$ . |
|                                                                |                                     |                                   |                                                                                              |                                                                                                                      |                                                                      |                                                              |                                              |                                                                                                                                                  |                                                                                          |                                                     |                                                     |
| Let the two bisectors meet at $F$ .                            | Join $BF$ .                         | Join $DF$ .                       | So $\angle CDE = \angle CDF$ . Since $\angle CDE = \angle CDF$ , $\angle ABC = \angle CDE$ . | Hence $BF$ bisects $\angle ABC$ . So $\angle ABF = \angle CBF$ .                                                     | Similarly, $DF$ bisects $\angle BDC$ .                               | Draw $FL \perp DE$ , meeting $DE$ at $L$ .                   | Draw $FM \perp EA$ , meeting $EA$ at $M$ .   | In right triangles $FHE$ and $FGL$ , $\angle HEF = \angle GLF$ , $\angle HFE = \angle GLF$ , $\angle FHE = \angle FGL = 90^\circ$ , $FE$ common. | It passes through $L$ . So it touches $DE$ at $L$ .                                      | It passes through $M$ . So it touches $EA$ at $M$ . | The circle touches all five sides.                  |
|                                                                |                                     |                                   |                                                                                              |                                                                                                                      |                                                                      |                                                              |                                              |                                                                                                                                                  |                                                                                          |                                                     |                                                     |
| Join $FE$ .                                                    | $F$ is joined to all five vertices. | Construction complete.            | Similarly, $FE$ bisects $\angle AED$ .                                                       | $F$ lies on all five angle bisectors.                                                                                | Thus $F$ is the centre of the required circle.                       | By Prop. I.26, $FG = FH$ .                                   | By the same reasoning, $FG = FH = FL = FN$ . | All perpendicular distances from $F$ are equal.                                                                                                  | Therefore a circle has been inscribed in the given equilateral and equiangular pentagon. |                                                     |                                                     |

$ABCDE$ : given equilateral and equiangular pentagon.   
 - - - - : lines drawn.   
 : right angle (perpendicular).   
 : equal angles (same colour).   
 + + : equal sides (same marks).   
 : required incircle.

BOOK IV – PROPOSITION 13: To inscribe a circle in a given pentagon which is equilateral and equiangular.

| PANEL 1 – CONSTRUCTION OF F (Figures 1–9)                      |                                       |                                                    | PANEL 2 – EQUAL DISTANCES (Figures 10–18)                                                                       |                                            |                                                    | PANEL 3 – FIVE PERPENDICULARS (Figures 19–27)     |                                                      |                                                | PANEL 4 – THE INCIRCLE AND CONCLUSION (Figures 28–36)                                                                                        |                                  |                                    |
|----------------------------------------------------------------|---------------------------------------|----------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|--------------------------------------------|----------------------------------------------------|---------------------------------------------------|------------------------------------------------------|------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------|------------------------------------|
|                                                                |                                       |                                                    |                                                                                                                 |                                            |                                                    |                                                   |                                                      |                                                |                                                                                                                                              |                                  |                                    |
| Let $ABCDE$ be the given equilateral and equiangular pentagon. | Bisect angle $BCD$ . Draw $CF$ .      | Bisect angle $CDE$ . Draw $DF$ .                   | $\angle ABC = \angle CDE$ (pentagon equiangular). $DF$ bisects $\angle CDE$ . Hence $\angle ABC = \angle CDE$ . | Hence $BF$ bisects $\angle ABC$ .          | By the same reasoning, $FA$ bisects $\angle BAE$ . | Drop a perpendicular from $F$ to $CD$ at $K$ .    | Drop a perpendicular from $F$ to $DE$ at $L$ .       | Drop a perpendicular from $F$ to $EA$ at $M$ . | With centre $F$ and radius $FG$ , draw the circle.                                                                                           | The circle touches $BC$ at $H$ . | The circle touches $CD$ at $K$ .   |
|                                                                |                                       |                                                    |                                                                                                                 |                                            |                                                    |                                                   |                                                      |                                                |                                                                                                                                              |                                  |                                    |
| Let the bisectors meet at $F$ .                                | Join $F$ to $B$ .                     | Join $F$ to $A$ .                                  | By the same reasoning, $FA$ bisects $\angle BAE$ .                                                              | Thus $F$ lies on all five angle bisectors. |                                                    | Five perpendiculars: $FG, FH, FK, FL, FM$ .       |                                                      |                                                | The circle touches $BC$ at $H$ .                                                                                                             | The circle touches $CD$ at $K$ . | The circle touches $DE$ at $L$ .   |
|                                                                |                                       |                                                    |                                                                                                                 |                                            |                                                    |                                                   |                                                      |                                                |                                                                                                                                              |                                  |                                    |
| Join $F$ to $E$ .                                              | $BC = CD$ (equilateral). $CF$ common. | $\angle CBF = \angle CDF$ , $BF = DF$ (Prop. I.4). | Drop a perpendicular from $F$ to $AB$ .                                                                         | Drop a perpendicular from $F$ to $BC$ .    |                                                    | By the same reasoning, $FG = FH = FK = FL = FM$ . | All five perpendicular distances from $F$ are equal. |                                                | If any side cut the circle instead of touching it, this would contradict Prop. III.16 (a perpendicular to a diameter outside falls outside). | The circle touches $EA$ at $M$ . | The circle touches all five sides. |

$ABCDE$ : given equilateral and equiangular pentagon.   
 - - - - : lines drawn.   
 : right angle (perpendicular).   
 : equal angles (same colour).   
 + + : equal sides (same marks).   
 : required incircle.

## Proposition 14

*To circumscribe a circle about a given pentagon which is equilateral and equiangular.*

Let  $ABCDE$  be the given regular pentagon.

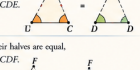
1. Begin with the given equilateral and equiangular pentagon  $ABCDE$ .
2. Bisect  $\angle BCD$  with  $CF$ .
3. Bisect  $\angle CDE$  with  $DF$ .
4. Let the two angle bisectors meet at  $F$ .
5. Join  $F$  to  $B$ .
6. Join  $F$  to  $A$ .
7. Join  $F$  to  $E$ .
8. By the reasoning of Proposition IV.13,  $FB, FA, FE$  bisect the corresponding remaining pentagon angles.
9. Since the pentagon is equiangular,  $\angle BCD = \angle CDE$ .
10. Therefore their halves are equal:  $\angle FCD = \angle CDF$ .
11. By Proposition I.6,  $FC = FD$ .
12. Repeating the same reasoning gives  $FB = FC, FA = FC$ , and  $FE = FC$ .
13. Hence  $FA = FB = FC = FD = FE$ .
14. With centre  $F$  and radius  $FA$ , draw a circle.
15. The circle passes through  $A$ .
16. Since  $FB = FA$ , it passes through  $B$ .
17. Since  $FC = FA$ , it passes through  $C$ .
18. Since  $FD = FA$ , it passes through  $D$ .
19. Since  $FE = FA$ , it passes through  $E$ .

- 20.** Therefore the circle centred at  $F$  has been circumscribed about the given regular pentagon.

$$FA = FB = FC = FD = FE$$

*Which is the very thing it was required to do.*

**BOOK IV – PROPOSITION 14:** To circumscribe a circle about a given pentagon which is equilateral and equiangular.

| PANEL 1 – CONSTRUCTION<br>(Figures 1–5)                                                                                                                              | PANEL 2 – POINT F ON ALL<br>ANGLE BISECTORS<br>(Figures 6–10)                                                                                                                                                                                                                                                                         | PANEL 3 – EQUAL DISTANCES FROM F<br>TO THE VERTICES<br>(Figures 11–15)                                                                                                                                    | PANEL 4 – CONCLUSION<br>(Figures 16–20)                                                                                                                                                                                                     |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>1 Let <math>ABCDE</math> be the given equilateral and equiangular pentagon.</p>  | <p>6 Join <math>FA</math>.</p>                                                                                                                                                                                                                       | <p>11 In triangle <math>CFD</math>, since <math>\angle FCD = \angle CDF</math>, by Prop. 1.6, <math>FC = FD</math>.</p>  | <p>16 Since <math>FB = FA</math>, point <math>B</math> lies on the circle.</p>                                                                            |
| <p>2 Bisect <math>\angle BCD</math>. Draw <math>CF</math>.</p>                      | <p>7 Join <math>FE</math>.</p>                                                                                                                                                                                                                       | <p>12 By the same reasoning, <math>FB = FC</math>, <math>FA = FC</math>, <math>FE = FC</math>.</p>                       | <p>17 Since <math>FC = FA</math>, point <math>C</math> lies on the circle.</p>                                                                            |
| <p>3 Bisect <math>\angle CDE</math>. Draw <math>DF</math>.</p>                      | <p>8 By the same reasoning as in Prop. 13, <math>FB</math>, <math>FA</math> and <math>FE</math> also bisect the angles at <math>B</math>, <math>A</math> and <math>E</math> respectively. Thus <math>F</math> lies on all five angle bisectors.</p>  | <p>13 Therefore <math>FA = FB = FC = FD = FE</math>.</p>                                                                 | <p>18 Since <math>FD = FA</math>, point <math>D</math> lies on the circle.</p>                                                                            |
| <p>4 Let the two bisectors meet at <math>F</math>.</p>                              | <p>9 Since the pentagon is equiangular, <math>\angle BCD = \angle CDE</math>.</p>                                                                                                                                                                    | <p>14 With centre <math>F</math> and radius <math>FA</math>, draw a circle.</p>                                          | <p>19 Since <math>FE = FA</math>, point <math>E</math> lies on the circle.</p>                                                                            |
| <p>5 Join <math>FB</math>.</p>                                                      | <p>10 Therefore their halves are equal, <math>\angle FCD = \angle CDF</math>.</p>                                                                                                                                                                    | <p>15 The circle passes through <math>A</math>, <math>B</math>, <math>C</math>, <math>D</math> and <math>E</math>.</p>   | <p>20 Therefore the circle centred at <math>F</math> has been circumscribed about the given equilateral and equiangular pentagon <math>ABCDE</math>.</p>  |

**Q.E.D.**

A circle has been circumscribed about the given equilateral and equiangular pentagon.

## Proposition 15

*To inscribe an equilateral and equiangular hexagon in a given circle.*

Let  $ABCDEF$  be the given circle, with centre  $G$ .

1. Begin with the given circle.
2. Draw diameter  $AD$ .
3. Find the centre  $G$  of the circle.
4. With centre  $D$  and radius  $DG$ , draw an auxiliary circle.
5. Let the auxiliary circle meet the given circle at  $E$  and  $C$ .
6. Join  $E$  to  $G$ .
7. Join  $C$  to  $G$ .
8. Produce  $EG$  through  $G$  to meet the original circle at  $B$ .
9. Produce  $CG$  through  $G$  to meet the original circle at  $F$ .
10. Join the six consecutive vertices  $AB, BC, CD, DE, EF, FA$ .
11. Since  $G$  is the centre of the original circle,  $GE = GD$ .
12. Since  $D$  is the centre of the auxiliary circle,  $DE = DG$ .
13. Therefore  $GE = GD = DE$ .
14. Hence triangle  $EGD$  is equilateral.
15. Its three angles are equal.
16. By Proposition I.32, each angle of triangle  $EGD$  is one third of two right angles.
17. The same reasoning shows  $\angle DGC$  is one third of two right angles.
18. Since  $E, G, B$  are collinear, the adjacent angles on that straight line sum to two right angles.
19. Therefore  $\angle CGB$  is also one third of two right angles.

20. By vertically opposite angles,  $\angle BGA, \angle AGF, \angle FGE$  equal their opposite central angles.
21. Hence all six central angles are equal.
22. By Proposition III.26, equal central angles stand on equal arcs.
23. Therefore arcs  $AB, BC, CD, DE, EF, FA$  are equal.
24. By Proposition III.29, the corresponding chords are equal.
25. Thus  $AB = BC = CD = DE = EF = FA$ , so the hexagon is equilateral.
26. Compare the equal arcs  $FA$  and  $ED$ .
27. Add the common arc  $ABCD$  to both.
28. The resulting larger arcs are equal.
29. Angles  $FED$  and  $AFE$  stand on these equal arcs.
30. By Proposition III.27,  $\angle AFE = \angle FED$ .
31. Repeating around the hexagon shows all six interior angles are equal.
32. Hence  $ABCDEF$  is equiangular.
33. Therefore the hexagon is both equilateral and equiangular and is inscribed in the given circle.
34. Corollary: a side of the regular hexagon is equal to the radius of the circle.

$$\boxed{AB = BC = CD = DE = EF = FA}$$

$$\boxed{\text{side of regular hexagon} = \text{radius of the circle}}$$

*Which is the very thing it was required to do.*

## BOOK IV – PROPOSITION 15: To inscribe an equilateral and equiangular hexagon in a given circle.

| PANEL 1 – CONSTRUCTION<br>(Figures 1–8)                                                                                                                           |                                                                                                                                                  |                                                                                                                                                     | PANEL 2 – EQUAL CENTRAL ANGLES<br>AND EQUAL SIDES (Figures 9–18)                                                                                                             |                                                                                                                                                                                               |                                                                                                                                                                                                                                   | PANEL 3 – EQUAL INTERIOR ANGLES<br>(Figures 19–27)                                                                                     |                                                                                                                                                                                       |                                                                                                                                                        | PANEL 4 – CONCLUSION AND COROLLARY<br>(Figures 28–34)                                                                                                                                                                                                         |                                                                                                                                             |                                                                                                                                                     |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>1</b><br><br>Let $ABCDEF$ be the given circle.                                | <b>2</b><br><br>Draw a diameter $AD$ .                          | <b>3</b><br><br>Let $G$ be the centre of the circle. (Prop. III.1) | <b>10</b><br><br>In $\triangle EGD$ , $GE = GD = DE$ .                                      | <b>11</b><br><br>Therefore $\triangle EGD$ is equilateral.                                                   | <b>12</b><br><br>Each angle of an equilateral triangle is one third of two right angles. (Prop. I.32)                                            | <b>19</b><br><br>Consider equal arcs $FA$ and $ED$ .  | <b>20</b><br><br>Add the common arc $ABCD$ to both.                                                  | <b>21</b><br><br>Thus arcs $FABCD$ and $EDCBA$ are equal.             | <b>28</b><br><br>$ABCDEF$ is both equilateral and equiangular.                                                                                                               | <b>29</b><br><br>All six vertices lie on the given circle. | <b>30</b><br><br>Therefore the hexagon is inscribed in the circle. |
| <b>4</b><br><br>With centre $D$ and radius $DG$ , draw an auxiliary circle.      | <b>5</b><br><br>Let it meet the given circle at $E$ and $C$ .   | <b>6</b><br><br>Join $EG$ and $CG$ .                               | <b>13</b><br><br>In $\triangle DGC$ , $\angle DGC$ is one third of two right angles.        | <b>14</b><br><br>Since $E, G, B$ are in a straight line, $\angle CGB$ is also one third of two right angles. | <b>15</b><br><br>Vertically opposite angles are equal. Thus $\angle BGA$ , $\angle AGF$ and $\angle FGE$ are each one third of two right angles. | <b>22</b><br><br>$\angle FED$ stands on arc $FABCD$ . | <b>23</b><br><br>Equal arcs subtend equal angles at the circumference. (Prop. III.27)                | <b>24</b><br><br>Therefore $\angle AFE = \angle FED$ .                | <b>31</b><br><br>Thus an equilateral and equiangular hexagon has been inscribed in the given circle.                                                                         |                                                                                                                                             |                                                                                                                                                     |
| <b>7</b><br><br>Produce $EG$ through $G$ to the circle at $B$ .                  | <b>8</b><br><br>Produce $CG$ through $G$ to the circle at $F$ . |                                                                                                                                                     | <b>16</b><br><br>Equal central angles stand on equal arcs. (Prop. III.26)                   | <b>17</b><br><br>arc $AB =$ arc $BC =$ arc $CD =$ arc $DE =$ arc $FA$ .                                      | <b>18</b><br><br>Equal arcs are subtended by equal straight lines. (Prop. III.29)                                                                | <b>25</b><br><br>By the same reasoning.               | <b>26</b><br><br>$\angle FBC = \angle CDA$ , $\angle BCD = \angle DEA$ , $\angle CDE = \angle EAB$ . | <b>27</b><br><br>Hence all six interior angles of $ABCDEF$ are equal. | <b>32</b><br><br>Corollary: A side of the regular hexagon is equal to the radius of the circle.                                                                              | <b>33</b><br><br>$GA = GB = GC = GD = GE = GF = r$ .       |                                                                                                                                                     |
| <b>9</b><br><br>Join $AB, BC, CD, DE, EF$ and $FA$ . Hexagon $ABCDEF$ is formed. |                                                                                                                                                  |                                                                                                                                                     | <b>19</b><br><br>Therefore $AB = BC = CD = DE = EF = FA$ . Hexagon $ABCDEF$ is equilateral. |                                                                                                                                                                                               |                                                                                                                                                                                                                                   | <b>28</b><br><br>Hexagon $ABCDEF$ is equiangular.     |                                                                                                                                                                                       |                                                                                                                                                        | <b>34</b><br><br>Since $AB$ is also a radius (it subtends central angle of one third of two right angles), $AB = r$ . Hence a side of the regular hexagon equals the radius. |                                                                                                                                             |                                                                                                                                                     |



## Proposition 16

*To inscribe an equilateral and equiangular fifteen-sided figure in a given circle.*

Let  $ABCD$  be the given circle.

1. Begin with the given circle.
2. By Proposition IV.2, inscribe chord  $AC$  as a side of an equilateral triangle.
3. By Proposition IV.11, inscribe chord  $AB$  as a side of a regular pentagon.
4. Therefore arc  $ABC$  is one third of the circle.
5. Thus  $\text{arc}(ABC) = \frac{1}{3} = \frac{5}{15}$ .
6. Arc  $AB$  is one fifth of the circle.
7. Thus  $\text{arc}(AB) = \frac{1}{5} = \frac{3}{15}$ .
8. Arc  $BC$  is the difference between arcs  $ABC$  and  $AB$ .
9. Hence  $\text{arc}(BC) = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$ .
10. By Proposition III.30, bisect arc  $BC$  at  $E$ .
11. Therefore  $\text{arc}(BE) = \text{arc}(EC)$ .
12. Since arc  $BC$  was  $\frac{2}{15}$ , each half is  $\frac{1}{15}$  of the circle.
13. Join  $B$  to  $E$ .
14. Join  $E$  to  $C$ .
15. Chords  $BE$  and  $EC$  therefore subtend arcs of exactly one fifteenth of the circumference.
16. By Proposition IV.1, insert successive chords equal to  $BE$  around the circle.
17. Continue until the circumference is divided into fifteen equal arcs.
18. Join consecutive division points.
19. Equal arcs subtend equal chords, so all fifteen sides are equal.

20. Since the corresponding arc spans are equal, all fifteen interior angles are equal.
21. Therefore an equilateral and equiangular fifteen-sided figure has been inscribed in the given circle.

$$\text{arc}(ABC) = \frac{1}{3} = \frac{5}{15}, \quad \text{arc}(AB) = \frac{1}{5} = \frac{3}{15}$$

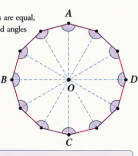
$$\text{arc}(BC) = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$$

|                                                  |
|--------------------------------------------------|
| $\text{arc}(BE) = \text{arc}(EC) = \frac{1}{15}$ |
|--------------------------------------------------|

Euclid further observes that, by analogous constructions, a regular fifteen-sided figure may be circumscribed about the circle, and circles may be inscribed in or circumscribed about such a figure.

*Which is the very thing it was required to do.*

**BOOK IV – PROPOSITION 16:** To inscribe an equilateral and equiangular fifteen-sided figure in a given circle.

| PANEL 1 – PREPARATION<br>(Figures 1–5)                                                                                                                                        | PANEL 2 – FINDING A 1/15 ARC<br>(Figures 6–12)                                                                                                                                                                         | PANEL 3 – COPYING THE ARC AROUND<br>THE CIRCLE (Figures 13–19)                                                                                                                                       | PANEL 4 – EQUAL ANGLES<br>AND CONCLUSION (Figures 20–21)                                                                                                                                                                                                                                  |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>1</b> Let <math>ABCD</math> be the given circle.</p>                                   | <p><b>6</b> Arc <math>ABC = 5/15</math> of the circle.</p> <p><b>7</b> Arc <math>AB = 3/15</math> of the circle.</p> <p><b>8</b> Arc <math>BC</math> is the difference.</p>                                            | <p><b>13</b> Join <math>BE</math>.</p>                                                                              | <p><b>20</b> All fifteen arcs are equal, so the inscribed angles are equal.</p>                                                                                                                         |
| <p><b>2</b> By Prop. IV.2, inscribe <math>AC</math>, a side of an equilateral triangle.</p>  | <p><b>9</b> Therefore arc <math>BC = 5/15 - 3/15 = 2/15</math> of the circle.</p>                                                     | <p><b>14</b> Join <math>EC</math>.</p>                                                                              | <p>Hence all fifteen interior angles of <math>ABCDEE\dots</math> are equal. The figure is equiangular.</p>                                                                                                                                                                                |
| <p><b>3</b> By Prop. IV.11, inscribe <math>AB</math>, a side of a regular pentagon.</p>      | <p><b>10</b> By Prop. III.30, bisect the arc <math>BC</math> at <math>E</math>.</p>                                                   | <p><b>15</b> <math>BE</math> and <math>EC</math> are sides subtending arcs of <math>1/15</math> of the circle.</p>  | <p><b>21</b> Therefore an equilateral and equiangular fifteen-sided figure has been inscribed in the given circle.</p>                                                                                                                                                                    |
| <p><b>4</b> Arc <math>ABC</math> is one third of the circle.</p>                             | <p><b>11</b> Thus arc <math>BE =</math> arc <math>EC</math>.</p> <p><b>12</b> Therefore each is <math>1/15</math> of the circle.</p>  | <p><b>16</b> By Prop. IV.1, insert successively straight-lines equal to <math>BE</math> all around the circle.</p>  | <p><b>FURTHER REMARK</b><br/>It is also possible to circumscribe a regular fifteen-sided figure about the circle, and to inscribe or circumscribe circles in or about such a fifteen-sided figure.</p>  |
| <p><b>5</b> Arc <math>AB</math> is one fifth of the circle.</p>                              | <p><b>19</b> All fifteen sides are equal.</p>                                                                                         | <p><b>17</b> Continue until the circle is divided into fifteen equal arcs.</p>                                      | <p><b>18</b> Join consecutive division points.</p>                                                                                                                                                       |

# End of Book IV

Euclid's Elements – Book IV

*Inscribed and Circumscribed Figures*