

EUCLID'S ELEMENTS

BOOK IX

Prime Numbers, Parity, Continued Proportion, and Perfect
Numbers

36 Propositions

Contents

Book IX

PROPOSITION 1

If two similar plane numbers make some (number by) multiplying one another then the created number will be square.

Proof

Action 1. Let A and B be two similar plane numbers, and let A make C (by) multiplying B.

Action 2. I say that C is square.

Action 3. For let A make D (by) multiplying itself.

Action 4. D is thus square.

Action 5. Therefore, since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D is to C *Prop. VII.17*

Action 6. And since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion *Prop. VIII.18*

Action 7. And if some numbers fall between two numbers in continued proportion then, as many numbers as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) *Prop. VIII.8*

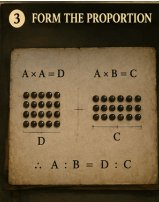
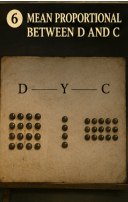
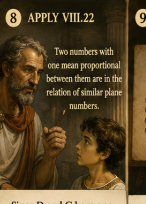
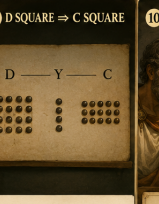
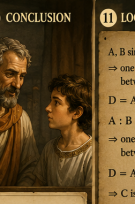
Action 8. And hence one number falls (between) D and C in mean proportion.

Action 9. And D is square.

Action 10. Thus, C is also square *Prop. VIII.22.*

Therefore, If two similar plane numbers make some (number by) multiplying one another then the created number will be square.

Which is the very thing it was required to show.

1 GIVEN 	2 CONSTRUCT $D = A^2$ 	3 FORM THE PROPORTION 	4 FROM VIII.18 	5 FROM VIII.8 	6 MEAN PROPORTIONAL BETWEEN D AND C 
Let A and B be two similar plane numbers, and let $A \times B = C$. To show: C is square.	Multiply A by itself to form D. Therefore D is square.	Since $A \times A = D$ and $A \times B = C$, it follows that $A : B = D : C$.	Therefore, by Book VIII Prop. 18, there is one mean proportional between A and B.	Therefore, by Book VIII Prop. 8, the same number of mean proportionals must lie between D and C.	Thus there is one mean proportional between D and C.
7 D IS SQUARE 	8 APPLY VIII.22 	9 D SQUARE $\Rightarrow$ C SQUARE 	10 CONCLUSION 	11 LOGICAL SKELETON A, B similar plane numbers $\Rightarrow$ one mean proportional between A and B. $D = A^2, C = AB$. $A : B = D : C$. $\Rightarrow$ one mean proportional between D and C. $D = A^2$ is square. $\Rightarrow$ C is also square.	12 Q.E.D. 
But D is already known to be square.	Since D and C have one mean proportional between them, by Book VIII Prop. 22...	...and since the first, D, is square, the second, C, is also square.	Therefore C is a square number.		Which is the very thing it was required to show.

PROPOSITION 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.

Proof

Action 1. Let A and B be two numbers, and let A make the square number C (by) multiplying B.

Action 2. I say that A and B are similar plane numbers.

Action 3. For let A make D (by) multiplying itself.

Action 4. Thus, D is square.

Action 5. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D is to C *Prop. VII.17*

Action 6. And since D is square, and C is also, D and C are thus similar plane numbers.

Action 7. Thus, one number falls (between) D and C in mean proportion *Prop. VIII.18*

Action 8. And as D is to C, so A is to B.

Action 9. Thus, one number also falls (between) A and B in mean proportion *Prop. VIII.8*

Action 10. And if one number falls (between) two numbers in mean proportion then [the] numbers are similar plane numbers *Prop. VIII.20*

Action 11. Thus, A and B are similar plane numbers.

Therefore, If two numbers make a square (number by) multiplying one another then they are similar plane numbers.

Which is the very thing it was required to show.



1. GIVEN
Let two numbers make a square
by multiplying one another.
That is, $A \times B = C$, where C is a square number.

**CONSTRUCT
SIMILAR NUMBERS**

$A = n$ $C = A \times B = n^2$

$B = n$





$D = A^2$
 $= A \times A$

3. Multiply A by itself to form D .
Thus $D = A^2$, and D is square number.

**4. Since $A \times B = C$
and $A^2 = D$,
we have
 $A : B = D : C$.**

**4. Therefore the ratio of A to B
is the same as the ratio of D to C .**

**5. Since A and B are similar plane numbers,
there is one mean proportional number X
between A and B . And since $A : B = D : C$,
there is one mean proportional number Y
between D and C .**

$A \quad X \quad B$
 $D \quad Y \quad C$

Thus $A : X = X : B$ and $D : Y = Y : C$.

**6. Since D is a square
and D and C have one
mean proportional
between them, it follows
(from Book VIII
Prop. 22) that C is
also square.**

$\therefore C$ is square.
Which is the very thing
it was required to show.

PROPOSITION 3

If a cube number makes some (number by) multiplying itself then the created number will be cube.

Proof

Action 1. For let the cube number A make B (by) multiplying itself.

Action 2. I say that B is cube.

Action 3. For let the side C of A have been taken.

Action 4. And let C make D by multiplying itself.

Action 5. So it is clear that C has made A (by) multiplying D.

Action 6. And since C has made D (by) multiplying itself, C thus measures D according to the units in it *Def. 7.15*

Action 7. But, in fact, a unit also measures C according to the units in it *Def. 7.20*

Action 8. Thus, as a unit is to C, so C is to D.

Action 9. Again, since C has made A (by) multiplying D, D thus measures A according to the units in C.

Action 10. And a unit also measures C according to the units in it.

Action 11. Thus, as a unit is to C, so D is to A.

Action 12. But, as a unit is to C, so C is to D.

Action 13. And thus as a unit is to C, so C is to D, and D to A.

Action 14. Thus, two numbers, C and D, have fallen (between) a unit and the number A in continued mean proportion.

Action 15. Again, since A has made B (by) multiplying itself, A thus measures B according to the units in it.

Action 16. And a unit also measures A according to the units in it.

Action 17. Thus, as a unit is to A, so A is to B.

Action 18. And two numbers have fallen (between) a unit and A in mean proportion.

Action 19. Thus two numbers will also fall (between) A and B in mean proportion *Prop. VIII.8*

Action 20. And if two numbers fall (between) two numbers in mean proportion, and the first number is cube, then the second will also be cube *Prop. VIII.23*

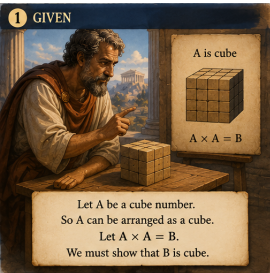
Action 21. And A is cube.

Action 22. Thus, B is also cube.

Therefore, If a cube number makes some (number by) multiplying itself then the created number will be cube.

Which is the very thing it was required to show.

1 GIVEN

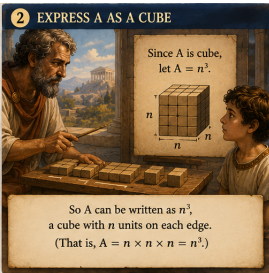


A is cube

$A \times A = B$

Let A be a cube number.
So A can be arranged as a cube.
Let $A \times A = B$.
We must show that B is cube.

2 EXPRESS A AS A CUBE



Since A is cube,
let $A = n^3$.

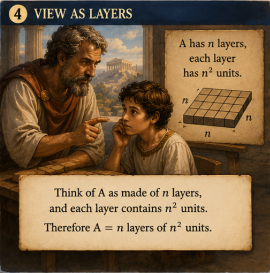
So A can be written as n^3 ,
a cube with n units on each edge.
(That is, $A = n \times n \times n = n^3$.)

3 MULTIPLY A BY ITSELF



Multiply A by itself.
That is, $n^3 \times n^3 = B$.
We will form B by multiplying
the cube A with itself.

4 VIEW AS LAYERS



A has n layers,
each layer
has n^2 units.

Think of A as made of n layers,
and each layer contains n^2 units.
Therefore $A = n$ layers of n^2 units.

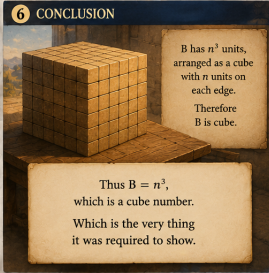
5 MULTIPLY THE LAYERS



$A \times A =$
 $(n \text{ layers of } n^2)$
 $\times (n \text{ layers of } n^2)$
 $= n \text{ layers of } n^2$
placed on each layer
of A .

Multiplying A by itself means placing n copies of A
one on another. Hence B has n layers,
and each layer has n^2 units.
So B has $n \times n^2 = n^3$ units in all.

6 CONCLUSION



B has n^3 units,
arranged as a cube
with n units on
each edge.
Therefore
 B is cube.

Thus $B = n^3$,
which is a cube number.
Which is the very thing
it was required to show.

PROPOSITION 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created number will be cube.

Proof

Action 1. For let the cube number A make C (by) multiplying the cube number B.

Action 2. I say that C is cube.

Action 3. For let A make D (by) multiplying itself.

Action 4. Thus, D is cube *Prop. IX.3*

Action 5. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D is to C *Prop. VII.17*

Action 6. And since A and B are cube, A and B are similar solid numbers.

Action 7. Thus, two numbers fall (between) A and B in mean proportion *Prop. VIII.19*

Action 8. Hence, two numbers will also fall (between) D and C in mean proportion *Prop. VIII.8*

Action 9. And D is cube.

Action 10. Thus, C is also cube *Prop. VIII.23*.

Therefore, If a cube number makes some (number by) multiplying a(nother) cube number then the created number will be cube.

Which is the very thing it was required to show.

1 GIVEN

Let A and B be cube numbers.
Let $A \times B = C$.
To show: C is cube.

A (cube) $\times$ B (cube) = C

2 CONSTRUCT $D = A \times A$

Let A multiply itself to produce D .
 $A \times A = D$.

A $\times$ A = D

So D is formed.
Since A is cube, by Prop. IX.3, D is cube.

3 COMPARE THE PRODUCTS

From $A \times A = D$
and $A \times B = C$,
it follows that
 $A : B = D : C$.

A B D C

4 A AND B ARE CUBES

Since A and B are cube numbers,
they are similar solid numbers.
Therefore there are two mean proportionals between them.

A X Y B

$A : X = X : Y = Y : B$

5 SAME RATIO FOR D AND C

Since $A : B = D : C$,
the same number of mean proportionals must lie between D and C .

D U V C

$D : U = U : V = V : C$

6 D IS CUBE

D was formed by A multiplying itself.
Hence, by Prop. IX.3, D is cube.

Therefore D is cube.

7 APPLY BOOK VIII PROP. 23

In a continued proportion, if the first term is cube, then the last term is also cube.

D U V C

Therefore, since D is cube,
 C is also cube (by Book VIII Prop. 23).

8 LOGICAL SKELETON (SUMMARY)

A, B are cubes
 $A \times B = C$
Construct $D = A \times A$
 D is cube (IX.3)
 $A : B = D : C$

A X Y B
(two means)
 D U V C
(two means)
 D is cube
 $\Rightarrow C$ is cube

9 VISUAL VIEW - CUBIC LAYERS

A is cube B is cube $C = A \times B$

Multiplying one cube by another produces a larger cube.

10 $D = A \times A$ IS CUBE

A A D

$A \times A = D$
By Prop. IX.3, D is cube.

11 CONCLUSION

Hence C is cube.
Which is the very thing it was required to show.

12 Q.E.D.

C is cube.
Which is the very thing it was required to show.
Q.E.D.

PROPOSITION 5

If a cube number makes a(nother) cube number (by) multiplying some number then the number multiplied will also be cube.

Proof

Action 1. For let the cube number A make the cube number C (by) multiplying some number B.

Action 2. I say that B is cube.

Action 3. For let A make D (by) multiplying itself.

Action 4. D is thus cube *Prop. IX.3*

Action 5. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D is to C *Prop. VII.17*

Action 6. And since D and C are (both) cube, they are similar solid numbers.

Action 7. Thus, two numbers fall (between) D and C in mean proportion *Prop. VIII.19*

Action 8. And as D is to C, so A is to B.

Action 9. Thus, two numbers also fall (between) A and B in mean proportion *Prop. VIII.8*

Action 10. And A is cube.

Action 11. Thus, B is also cube *Prop. VIII.23*.

Therefore, If a cube number makes a(nother) cube number (by) multiplying some number then the number multiplied will also be cube.

Which is the very thing it was required to show.

1 GIVEN Let A be a cube number. Let B be some number. Suppose $A \times B = C$, and C is a cube number. To show: B is cube.  A B C	2 CONSTRUCT $D = A \times A$ Let A multiply itself to produce D. $A \times A = D$.  A A D	3 D IS CUBE (BY IX.3) Since A is cube, by Proposition IX.3, D is also cube.  D	4 COMPARE THE PRODUCTS We have $A \times A = D$ and $A \times B = C$. Therefore, $A : B = D : C$.  A : B = D : C
5 D AND C ARE CUBES D is cube and C is cube. Therefore they are similar solid numbers.  D C	6 TWO MEAN PROPORTIONALS BETWEEN D AND C Since D and C are similar solid numbers, two mean proportionals lie between them. $D : X : Y : C$  D X Y C	7 SAME NUMBER BETWEEN A AND B Since $A : B = D : C$, the same number of mean proportionals must lie between A and B. 	8 TWO MEAN PROPORTIONALS BETWEEN A AND B Therefore, $A : U : V : B$  A U V B
9 A IS CUBE The first term A is cube.  A	10 RULE (BY BOOK VIII PROP.23) When two mean proportionals lie between two numbers, and the first is cube, then the last is also cube. 	11 CONCLUSION Since A is cube and two mean proportionals lie between A and B, therefore the last term B is also cube. Thus, B is cube.  B	12 Q.E.D. B is cube. Which is the very thing it was required to show. Q.E.D. 

PROPOSITION 6

If a number makes a cube (number by) multiplying itself then it itself will also be cube.

Proof

- Action 1.** For let the number A make the cube number B (by) multiplying itself.
- Action 2.** I say that A is also cube.
- Action 3.** For let A make C (by) multiplying B.
- Action 4.** Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube.
- Action 5.** And since A has made B (by) multiplying itself, A thus measures B according to the units in (A).
- Action 6.** And a unit also measures A according to the units in it.
- Action 7.** Thus, as a unit is to A, so A is to B.
- Action 8.** And since A has made C (by) multiplying B, B thus measures C according to the units in A.
- Action 9.** And a unit also measures A according to the units in it.
- Action 10.** Thus, as a unit is to A, so B is to C.
- Action 11.** But, as a unit is to A, so A is to B.
- Action 12.** And thus as A is to B, (so) B is to C.
- Action 13.** And since B and C are cube, they are similar solid numbers.
- Action 14.** Thus, there exist two numbers in mean proportion (between) B and C *Prop. VIII.19*
- Action 15.** And as B is to C, (so) A is to B.

Action 16. Thus, there also exist two numbers in mean proportion (between) A and B *Prop. VIII.8*

Action 17. And B is cube.

Action 18. Thus, A is also cube *Prop. VIII.23*.

Therefore, If a number makes a cube (number by) multiplying itself then it itself will also be cube.

Which is the very thing it was required to show.



PROPOSITION 7

If a composite number makes some (number by) multiplying some (other) number then the created number will be solid.

Proof

Action 1. For let the composite number A make C (by) multiplying some number B.

Action 2. I say that C is solid.

Action 3. For since A is a composite number, it will be measured by some number.

Action 4. Let it be measured by D.

Action 5. And, as many times as D measures A, so many units let there be in E.

Action 6. Therefore, since D measures A according to the units in E, E has thus made A (by) multiplying D *Def. 7.15*

Action 7. And since A has made C (by) multiplying B, and A is the (number created) from (multiplying) D, E, the (number created) from (multiplying) D, E has thus made C (by) multiplying B.

Action 8. Thus, C is solid, and its sides are D, E, B.

Therefore, If a composite number makes some (number by) multiplying some (other) number then the created number will be solid.

Which is the very thing it was required to show.

1 GIVEN

Let A be a composite number.
Let A multiply some number B to make C.
To show: C is solid.

A (composite) × B = C

2 A IS COMPOSITE

Since A is composite, it is measured by some number. Let that number be D.

A (composite) D (measuring number)

3 D MEASURES A ACCORDING TO E

Let E be the number according to which D measures A.
Thus, $E \times D = A$.

D × E = A (composite)

4 SUBSTITUTION IN $A \times B = C$

But we are given that $A \times B = C$.
Substituting $A = D \times E$,

D × E = A × B = C

5 C IS PRODUCED BY THREE NUMBERS

So, $C = (D \times E) \times B$.
Therefore C is produced by multiplying three numbers.

D × E × B = C

6 DEFINITION OF SOLID NUMBER

A solid number is a number produced by multiplying three numbers.

side 1 side 2 side 3

7 IDENTIFY THE THREE SIDES OF C

Hence C has three numerical "sides":
D, E, B.

D (side 1) E (side 2) B (side 3) C

8 CONCLUSION

Therefore C is a solid number.

C (solid)

9 Q.E.D.

C is solid.
Which is the very thing it was required to show.
Q.E.D.

C (solid)

17

PROPOSITION 8

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one number, and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two numbers, and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five numbers.

Proof

Action 1. Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit.

Action 2. I say that the third from the unit, B, is square, and all those (numbers after that) which leave an interval of one number.

Action 3. And the fourth (from the unit), C, is cube, and all those (numbers after that) which leave an interval of two numbers.

Action 4. And the seventh (from the unit), F, is both cube and square, and all those (numbers after that) which leave an interval of five numbers.

Action 5. For since as the unit is to A, so A is to B, the unit thus measures the number A the same number of times as A (measures) B *Def. 7.20*

Action 6. And the unit measures the number A according to the units in it.

Action 7. Thus, A also measures B according to the units in A.

Action 8. A has thus made B (by) multiplying itself *Def. 7.15*

Action 9. Thus, B is square.

Action 10. And since B, C, D are continuously proportional, and B is square, D is thus also square *Prop. VIII.22*

Action 11. So, for the same (reasons), F is also square.

Action 12. So, similarly, we can also show that all those (numbers after that) which leave an interval of one number are square.

Action 13. So I also say that the fourth number from the unit, C, is cube, and all those (numbers after that) which leave an interval of two numbers.

Action 14. For since as the unit is to A, so B is to C, the unit thus measures the number A the same number of times that B (measures) C.

Action 15. And the unit measures the number A according to the units in A.

Action 16. And thus B measures C according to the units in A.

Action 17. A has thus made C (by) multiplying B.

Action 18. Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube.

Action 19. And since C, D, E, F are continuously proportional, and C is cube, F is thus also cube *Prop. VIII.23*

Action 20. And it was also shown (to be) square.

Action 21. Thus, the seventh number from the unit is (both) cube and square.

Action 22. So, similarly, we can show that all those (numbers after that) which leave an interval of five numbers are (both) cube and square.

Therefore, If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one number, and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two numbers, and the seventh

(will be) both cube and square, and (all) those (numbers after that) which leave an interval of five numbers.

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION

Let any multitude of numbers be in continuous proportion, starting from a unit.

$$1 : A = A : B = B : C = C : D = D : E = E : F = \dots$$

2 THE THIRD NUMBER IS SQUARE

From $1 : A = A : B$,
A squared makes B.
 $A \times A = B$.
Therefore B is square.

3 EVERY SECOND TERM AFTER IS ALSO SQUARE

The same reasoning holds again and again.
So the terms B, D, F, ... are square.

4 POSITIONS OF SQUARE NUMBERS

Counting from the unit (1) as the first term, the square numbers are the 3rd, 5th, 7th, ...

Positions	1st	2nd	3rd	4th	5th	6th	7th	...
	1	A	B	C	D	E	F	...
			square		square		square	

5 THE FOURTH NUMBER IS CUBE

From $1 : A = A : B = B : C$,
 $A \times B = C$.
But $B = A \times A$.
Hence $C = A \times (A \times A) = A^3$.
Therefore C is cube.

6 EVERY THIRD TERM AFTER IS ALSO CUBE

The same reasoning holds again and again.
So the terms C, F, ... are cube.

7 POSITIONS OF CUBE NUMBERS

Counting from the unit (1) as the first term, the cube numbers are the 4th, 7th, 10th, ...

Positions	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	...
	1	A	B	C	D	E	F	G	H	I	J
				cube			cube			cube	

8 THE SEVENTH NUMBER

The seventh number from the unit is F.
F is in the square sequence (B, D, F, ...), so F is square.
F is in the cube sequence (C, F, ...), so F is cube.

9 THE SEVENTH NUMBER IS BOTH SQUARE AND CUBE

Therefore the seventh number, F, is both square and cube.

10 THE PATTERN REPEATS EVERY SIX TERMS

The pattern of square and cube numbers repeats every six terms.

Positions	1	3	4	5	6	8	9	10	11	13	...
Numbers	A	B	C	D	F	G	H	J	K	M	...
	square	square	square	square	square	square	square	square	square	square	
			cube		cube		cube		cube		

Every 6th term (7th, 13th, 19th, ...) is both square and cube.

11 THE MULTIPLICATIVE VIEW

Let the second term be A.
Then the terms are:

$A^2, A^4, A^6, \dots$ are squares.
 $A^3, A^6, A^9, \dots$ are cubes.
 A^6 is both square and cube.

12 Q.E.D.

Thus, starting from a unit in a continued proportion:

- The 3rd, 5th, 7th, ... terms are square.
- The 4th, 7th, 10th, ... terms are cube.
- The 7th, 13th, 19th, ... terms are both square and cube.

Q.E.D.

PROPOSITION 9

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (num , . ber) after the unit is square, then all the remaining numbers will also be square. And if the number after the unit is cube, then all the remaining numbers will also be cube.

Proof

Action 1. Let any multitude whatsoever of numbers, A, B, C, D, E, F , be continuously proportional, (starting) from a unit.

Action 2. And let the number after the unit, A, be square.

Action 3. I say that all the remaining numbers will also be square.

Action 4. In fact, it has (already) been shown that the third number from the unit, B, is square, and all those (numbers after that) which leave an interval of one number *Prop. IX.8*

So I say that all the remaining numbers are also square.

Action 5. For since A, B, C are continuously proportional, and A is square, C is [thus] also square *Prop. VIII.22*

Action 6. Again, since B, C, D are [also] continuously proportional, and B is square, D is [thus] also square *Prop. VIII.22*

Action 7. So, similarly, we can show that all the remaining numbers are also square.

Action 8. And so let A be cube.

Action 9. I say that all the remaining numbers are also cube.

Action 10. In fact, it has (already) been shown that the fourth number from the unit, C, is cube, and all those (numbers after that) which leave an interval of two numbers *Prop. IX.8*

So I say that all the remaining numbers are also cube.

Action 11. For since as the unit is to A, so A is to B, the unit thus measures A the same number of times as A (measures) B.

Action 12. And the unit measures A according to the units in it.

Action 13. Thus, A also measures B according to the units in (A).

Action 14. A has thus made B (by) multiplying itself.

Action 15. And A is cube.

Action 16. And if a cube number makes some (number by) multiplying itself then the created number is cube *Prop. IX.3*

Action 17. Thus, B is also cube.

Action 18. And since the four numbers A, B, C, D are continuously proportional, and A is cube, D is thus also cube *Prop. VIII.23*

Action 19. So, for the same (reasons), E is also cube, and, similarly, all the remaining numbers are cube.

Therefore, If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (num , ber) after the unit is square, then all the remaining numbers will also be square. And if the number after the unit is cube, then all the remaining numbers will also be cube.

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION

Let many numbers be in continuous proportion, starting from a unit.

$$1 : A = A : B = B : C = C : D = D : E = E : F = \dots$$

We will show two things.

1

A

B

C

D

E

F

...

2 CASE 1 – SUPPOSE A IS SQUARE

Suppose the number after the unit, A, is square.

To show: all the remaining numbers are square.

A is square.

A

3 BY PROPOSITION IX.8

From Proposition IX.8, the 3rd, 5th, 7th, ... terms from the unit are square.

1

A

B

C

D

E

F

...

square

square

square

4 FILLING THE GAPS – $A : B = B : C$

Since $A : B = B : C$ and A is square, by Book VIII, Prop. 22, C is also square.

1

A

B

C

D

E

F

...

C is square.

5 NEXT GAP – $B : C = C : D$

Since $B : C = C : D$ and B is square, D is also square.

1

A

B

C

D

E

F

...

D is square.

6 REPEAT THE SAME REASONING

Continuing in the same way, every remaining term is square.

All are square.

1

A

B

C

D

E

F

G

...

7 CASE 2 – SUPPOSE A IS CUBE

Now suppose the number after the unit, A, is cube.

To show: all the remaining numbers are cube.

A is cube.

A

8 FIRST CONSEQUENCE – $1 : A = A : B$

Since $1 : A = A : B$, $A : A = B$. But A is cube; by Prop. IX.3, B is also cube.

1

A

B

C

D

E

F

...

B is cube.

9 FOUR TERMS STARTING WITH A

The four numbers A, B, C, D are in continued proportion, starting with the cube number A.

A

B

C

D

10 APPLY BOOK VIII, PROP. 23

Therefore, by Book VIII, Prop. 23, the fourth term, D, is also cube.

A

B

C

D

D is cube.

11 PROPAGATE THE CUBE PROPERTY

The same reasoning applies successively to the following terms.

A

B

C

D

E

F

G

...

All are cube.

12 CONCLUSION

Therefore,

- If the number after the unit is square, all the remaining numbers are square.
- If the number after the unit is cube, all the remaining numbers are cube.

Q.E.D.

SUMMARY OF CASE 1 (A IS SQUARE)

Positions: 1 2 3 4 5 6 7 ...

Terms: 1 A B C D E F ...

Result:

1 A B C D E F

 All terms are square.

SUMMARY OF CASE 2 (A IS CUBE)

Positions: 1 2 3 4 5 6 7 ...

Terms: 1 A B C D E F ...

Result:

1 A B C D E F

 All terms are cube.

24

PROPOSITION 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the number after the unit is not square, then no other number will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one number. And if the number after the unit is not cube, then no other number will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two numbers.

Proof

Action 1. Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit.

Action 2. And let the number after the unit, A, not be square.

Action 3. I say that no other number will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one number]

Action 4. For, if possible, let C be square.

Action 5. And B is also square *Prop. IX.8*

Action 6. Thus, B and C have to one another (the) ratio which some square number (has) to (some other) square number.

Action 7. And as B is to C, (so) A is to B.

Action 8. Thus, A and B have to one another (the) ratio which some square number has to (some other) square number.

Action 9. Hence, A and B are similar plane numbers *Prop. VIII.26*

Action 10. And B is square.

Action 11. Thus, A is also square.

Action 12. The very opposite thing was assumed.

Action 13. C is thus not square.

Action 14. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one number.

Action 15. And so let A not be cube.

Action 16. I say that no other number will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two numbers.

Action 17. For, if possible, let D be cube.

Action 18. And C is also cube *Prop. IX.8*

Action 19. For it is the fourth number from the unit.

Action 20. And as C is to D, (so) B is to C.

Action 21. And B thus has to C the ratio which some cube (number has) to (some other) cube number.

Action 22. And C is cube.

Action 23. Thus, B is also cube [Props. 7.13, 8.25]

Action 24. And since as the unit is to A, (so) A is to B, and the unit measures A according to the units in it, A thus also measures B according to the units in (A).

Action 25. Thus, A has made the cube number B (by) multiplying itself.

Action 26. And if a number makes a cube (number by) multiplying itself then it itself will be cube *Prop. IX.6*

Action 27. Thus, A is also cube.

Action 28. The very opposite thing was assumed.

Action 29. Thus, D is not cube.

Action 30. So, similarly, we can show that no other number is cube either, apart from the fourth from the unit, and (all) those

(numbers after that) which leave an interval of two numbers.

Therefore, If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the number after the unit is not square, then no other number will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one number. And if the number after the unit is not cube, then no other number will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two numbers.

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION

Let many numbers be in continuous proportion, starting from a unit.

$$1 : A = A : B = B : C = C : D = D : E = E : F \dots$$

2 PART I – SUPPOSE A IS NOT SQUARE

Suppose the number after the unit, A, is not square.

A is not square.

3 BY PROPOSITION IX.8

From Proposition IX.8, the terms B, D, F, ... (third from the unit and every alternate term) are square.

4 SUPPOSE C WERE SQUARE (TO CONTRADICT)

Suppose, contrary to the required conclusion, that C is square.

5 B AND C WOULD BOTH BE SQUARE

Then B and C would both be square numbers.

6 A : B = B : C

But $A : B = B : C$. Therefore A and B have the same ratio as two square numbers.

7 A WOULD BE SQUARE (CONTRADICTION)

Since B is square, it follows that A is square. But this contradicts our supposition.

8 THEREFORE C IS NOT SQUARE

Therefore C is not square. By the same reasoning, none of the intervening terms are square.

9 CONCLUSION OF PART I

Hence, the only square numbers are B, D, F, ... (the 3rd from the unit and every alternate term).

10 PART II – SUPPOSE A IS NOT CUBE

Now suppose the number after the unit, A, is not cube.

A is not cube.

11 BY PROPOSITION IX.8

From Proposition IX.8, the terms C, E, ... (fourth from the unit and every third term) are cube.

12 SUPPOSE D WERE CUBE (TO CONTRADICT)

Suppose, contrary to the required conclusion, that D is cube.

13 THEN B WOULD BE CUBE

Since C and D are both cube numbers and $B : C = C : D$, it follows that B is cube.

14 A : A = A : B, AND B IS CUBE

But $1 : A = A : B$, therefore $A : A = A : B$. So A, multiplied by itself, makes the cube number B.

$A \times A = B$

15 BY PROPOSITION IX.A, A WOULD BE CUBE

By Proposition IX.A, if a number makes a cube by multiplying itself, then it itself is cube. This contradicts our supposition.

16 CONCLUSION OF PART II

Therefore D is not cube, nor any intervening term. The only cube numbers are C, E, ... (the 4th from the unit and every third term).

17 Q.E.D.

Thus:

- If the number after the unit is not square, the only square numbers are B, D, F, ...
- If the number after the unit is not cube, the only cube numbers are C, E, ...

Which is the very thing it was required to show.

Q.E.D.

PROPOSITION 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser number measures a greater according to some existing number among the proportional numbers.

Proof

Action 1. Let any multitude whatsoever of numbers, B, C, D, E, be continuously proportional, (starting) from the unit A.

Action 2. I say that, for B, C, D, E, the least number, B, measures E according to some one of C, D.

Action 3. For since as the unit A is to B, so D is to E, the unit A thus measures the number B the same number of times as D (measures) E.

Action 4. Thus, alternately, the unit A measures D the same number of times as B (measures) E *Prop. VII.15*

Action 5. And the unit A measures D according to the units in it.

Action 6. Thus, B also measures E according to the units in D.

Action 7. Hence, the lesser number B measures the greater E according to some existing number among the proportional numbers (namely, D). *Pirisma*.

Action 8. Corollary And (it is) clear that what(ever relative) place the measuring number has from the unit, the number according to which it measures has the same (relative) place from the measured number, in (the direction of the number) before it.

Therefore, If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser number measures a greater according to some existing number among the proportional numbers.

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION

Let many numbers be in continuous proportion, starting from a unit.

$$A : B = B : C = C : D = D : E$$

2 TAKE THE LEAST AND A GREATER NUMBER

Let B be the least of the proportional numbers, and E a greater number.

We must show that B measures E according to some one of the numbers already existing.

3 USE THE CONTINUOUS PROPORTION

In a continued proportion,

$$A : B = D : E$$

$$\frac{A}{B} = \frac{D}{E}$$

4 ALTERNATE THE MEANS

Therefore, alternately,

$$\frac{A}{D} = \frac{B}{E}$$

$$A : D = B : E$$

5 INTERPRET THE NEW PROPORTION

$A : D = B : E$ means

As many times as the unit A is contained in D, so many times is B contained in E.

6 THE UNIT MEASURES D

But the unit A measures D according to the number of units contained in D.

7 THEREFORE B MEASURES E ACCORDING TO D

Therefore, B measures E according to D.

8 GENERAL FORM

Thus, a lesser number measures a greater according to some number already existing among the proportional numbers.

Earlier term $\times$ Intermediate term = Later term

(lesser) $\times$ (already existing) = (greater)

9 COROLLARY – SYMMETRIC POSITION

Whatever relative position the measuring number has from the unit, the number according to which it measures has the same relative position backward from the measured number.

1 A B C D

1 1 1 1

1 step 1 step

If A measures D (1 step from the unit), then C (1 step before D) is the measuring number.

PROPOSITION 12

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then however many prime numbers the last number is measured by, the number next to the unit will also be measured by the same (prime numbers).

Proof

Action 1. Let any multitude whatsoever of numbers, A, B, C, D, be (continuously) proportional, (starting) from a unit.

Action 2. I say that however many prime numbers D is measured by, A will also be measured by the same (prime numbers).

Action 3. For let D be measured by some prime number E.

Action 4. I say that E measures A.

Action 5. For (suppose it does) not.

Action 6. E is prime, and every prime number is prime to every number which it does not measure *Prop. VII.29*

Action 7. Thus, E and A are prime to one another.

Action 8. And since E measures D, let it measure it according to F .

Action 9. Thus, E has made D (by) multiplying F .

Action 10. Again, since A measures D according to the units in C [Prop. 9.11 corr.], A has thus made D (by) multiplying C.

Action 11. But, in fact, E has also made D (by) multiplying F .

Action 12. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F .

Action 13. Thus, as A is to E, (so) F is to C *Prop. VII.19*

Action 14. And A and E are prime (to one another), and numbers prime (to one another are) also the least (of those numbers having the same ratio as them) *Prop. VII.21*, and the least numbers measure those numbers having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following *Prop. VII.20*

Action 15. Thus, E measures C.

Action 16. Let it measure it according to G.

Action 17. Thus, E has made C (by) multiplying G.

Action 18. But, in fact, via the (proposition) before this, A has also made C (by) multiplying B [Prop. 9.11 corr.]

Action 19. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) E, G.

Action 20. Thus, as A is to E, (so) G is to B *Prop. VII.19*

Action 21. And A and E are prime (to one another), and numbers prime (to one another are) also the least (of those numbers having the same ratio as them) *Prop. VII.21*, and the least numbers measure those numbers having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following *Prop. VII.20*

Action 22. Thus, E measures B.

Action 23. Let it measure it according to H.

Action 24. Thus, E has made B (by) multiplying H.

Action 25. But, in fact, A has also made B (by) multiplying itself *Prop. IX.8*

Action 26. Thus, the (number created) from (multiplying) E, H is equal to the (square) on A.

Action 27. Thus, as E is to A, (so) A is to H *Prop. VII.19*

Action 28. And A and E are prime (to one another), and numbers prime (to one another are) also the least (of those numbers having the same ratio as them) *Prop. VII.21*, and the least numbers measure those numbers having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following *Prop. VII.20*

Action 29. Thus, E measures A, as the leading (measuring the) leading.

Action 30. But, in fact, (E) also does not measure (A).

Action 31. The very thing is impossible.

Action 32. Thus, E and A are not prime to one another.

Action 33. Thus, (they are) composite (to one another).

Action 34. And numbers composite (to one another) are (both) measured by some [prime] number *Def. 7.14*

Action 35. And since E is assumed (to be) prime, and a prime number is not measured by another number (other) than itself *Def. 7.11*, E thus measures (both) A and E.

Action 36. Hence, E measures A.

Action 37. And it also measures D.

Action 38. Thus, E measures (both) A and D.

Action 39. So, similarly, we can show that however many prime numbers D is measured by, A will also be measured by the same (prime numbers).

Therefore, If any multitude whatsoever of numbers is continuous, and mutually proportional, (starting) from a unit, then however many prime numbers the last number is measured by, the number next to the unit will also be measured by the same (prime numbers).

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION

Let a multitude of numbers be in continuous proportion, starting from a unit.

$$1 : A = A : B = B : C = C : D$$

2 TAKE A PRIME NUMBER E THAT MEASURES D

Let E be a prime number that measures the last number D.

$$E \mid D$$

$D = E \times E \times \dots \times E$ (It is prime)

3 WE MUST SHOW THAT E MEASURES A

We must show that the same prime E also measures the number next to the unit, namely A.

$$E \mid A$$

4 ASSUME THE CONTRARY

Suppose, contrary to what we want, that E does not measure A.

$$E \nmid A$$

5 E IS PRIME, SO E IS PRIME TO A

Since E is prime, if E does not measure A, then E and A are prime to one another.

$$(E, A) = 1$$

$E \nmid D$ (It is prime)

6 THE PROPORTION IS GENERATED FROM A

By the nature of a continuous proportion starting from 1, every term is formed from A by successive multiplication. So the numbers are:

1, A, A², A³, A⁴
1, A, B, C, D

7 A PRIME NOT IN A DOES NOT APPEAR IN ITS POWERS

If E is prime to A, then E cannot divide any power of A.

$$A, A^2, A^3, A^4, \dots$$

$E \nmid A^2$
 $E \nmid A^3$
 $E \nmid A^4 (= D)$

8 BUT E MEASURES D (BY ASSUMPTION)

But this contradicts our original assumption that E measures D.

$$E \mid D$$

$D = E \times E \times \dots \times E$

9 CONTRADICTION

We have a contradiction: From E prime to A, we concluded $E \nmid D$, but we assumed $E \mid D$.

IMPOSSIBLE

10 THE ASSUMPTION WAS FALSE

Therefore our supposition was false.

$$E \mid A$$

$E \mid A$
 $E \mid B$
 $E \mid C$
 $E \mid D$

A is measured by E

11 THE SAME FOR EVERY PRIME DIVISOR OF D

The argument holds for every prime number that measures D. So all prime divisors of D also measure A.

$E \mid D \Rightarrow E \mid A$
 $F \mid D \Rightarrow F \mid A$
 $G \mid D \Rightarrow G \mid A$

12 CONCLUSION (Q.E.D.)

Thus: If a multitude of numbers is in continuous proportion, starting from a unit, then however many prime numbers the last number is measured by, the number next to the unit will also be measured by the same prime numbers.

Q.E.D.

PROPOSITION 13

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (num [] ber) after the unit is prime, then the greatest number will be measured by no [other] numbers except (num E bers) existing among the proportional numbers.

Proof

Action 1. Let any multitude whatsoever of numbers, A, B, C, Δ , be continuously proportional, (starting) from a unit.

Action 2. And let the number after the unit, A, be prime.

Action 3. I say that the greatest of them, D, will be measured by no other numbers except A, B, C.

Action 4. For, if possible, let it be measured by E, and let E not be the same as one of A, B, C.

Action 5. So it is clear that E is not prime.

Action 6. For if E is prime, and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it *Prop. IX.12*

Action 7. The very thing is impossible.

Action 8. Thus, E is not prime.

Action 9. Thus, (it is) composite.

Action 10. And every composite number is measured by some prime number *Prop. VII.31*

Action 11. Thus, E is measured by some prime number.

Action 12. So I say that it will be measured by no other prime number than A.

Action 13. For if E is measured by another (prime number), and E measures D, then this (prime number) will thus also measure D.

Action 14. Hence, it will also measure A, (despite A) being prime (and) not being the same as it *Prop. IX.12*

Action 15. The very thing is impossible.

Action 16. Thus, A measures E.

Action 17. And since E measures D, let it measure it according to F .

Action 18. I say that F is not the same as one of A, B, C.

Action 19. For if F is the same as one of A, B, C, and measures D according to E, then one of A, B, C thus also measures D according to E.

Action 20. But one of A, B, C (only) measures D according to some one of A, B, C *Prop. IX.11*

Action 21. And thus E is the same as one of A, B, C.

Action 22. The very opposite thing was assumed.

Action 23. Thus, F is not the same as one of A, B, C.

Action 24. Similarly, we can show that F is measured by A, (by) again showing that F is not prime.

Action 25. For if (F is prime), and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it *Prop. IX.12*

Action 26. The very thing is impossible.

Action 27. Thus, F is not prime.

Action 28. Thus, (it is) composite.

Action 29. And every composite number is measured by some prime number *Prop. VII.31*

Action 30. Thus, F is measured by some prime number.

Action 31. So I say that it will be measured by no other prime number than A.

Action 32. For if some other prime number measures F , and F measures D, then this (prime number) will thus also measure D.

Action 33. Hence, it will also measure A, (despite A) being prime (and) not being the same as it *Prop. IX.12*

Action 34. The very thing is impossible.

Action 35. Thus, A measures F .

Action 36. And since E measures D according to F , E has thus made D (by) multiplying F .

Action 37. But, in fact, A has also made D (by) multiplying C [Prop. 9.11 corr.]

Action 38. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F .

Action 39. Thus, proportionally, as A is to E, so F is to C *Prop. VII.19*

Action 40. And A measures E.

Action 41. Thus, F also measures C.

Action 42. Let it measure it according to G.

Action 43. So, similarly, we can show that G is not the same as one of A, B, and that it is measured by A.

Action 44. And since F measures C according to G, F has thus made C (by) multiplying G.

Action 45. But, in fact, A has also made C (by) multiplying B [Prop. 9.11 corr.]

Action 46. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) F , G.

Action 47. Thus, proportionally, as A is to F , so G is to B *Prop. VII.19*

Action 48. And A measures F .

Action 49. Thus, G also measures B.

Action 50. Let it measure it according to H.

Action 51. So, similarly, we can show that H is not the same as A.

Action 52. And since G measures B according to H, G has thus made B (by) multiplying H.

Action 53. But, in fact, A has also made B (by) multiplying itself *Prop. IX.8*

Action 54. Thus, the (number created) from (multiplying) H, G is equal to the square on A.

Action 55. Thus, as H is to A, (so) A is to G *Prop. VII.19*

Action 56. And A measures G.

Action 57. Thus, H also measures A, (despite A) being prime (and) not being the same as it.

Action 58. The very thing is absurd.

Action 59. Thus, the greatest number D cannot be measured by another number except (one of) A, B, C.

Therefore, If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (num [] ber) after the unit is prime, then the greatest number will be measured by no [other] numbers except (num E bers) existing among the proportional numbers.

Which is the very thing it was required to show.

1 GIVEN - A CONTINUOUS PROPORTION

Let a multitude of numbers be in continuous proportion, starting from a unit.

$$1 : A = A : B = B : C = C : D$$

and let A be prime.

2 WHAT WE MUST PROVE

We must prove that D is measured by no numbers except A, B, and C (which are among the proportional numbers).

Divisors of D are only A, B, C.

A, B, C are allowed. Others are not.

3 ASSUME THE CONTRARY

Suppose some number E (not A, B, or C) measures D.

$$E \mid D$$

$$E \neq A, E \neq B, E \neq C$$

4 E CANNOT BE PRIME

If E were prime and $E \mid D$, by Proposition IX.12, E would also divide A.

Impossible! (A is prime)

Therefore, E is not prime.

5 E IS COMPOSITE

Since E is composite, it has a prime divisor. That prime divisor also divides D.

By IX.12, that prime divides A.

Therefore $A \mid E$.

6 WRITE $D = E \times F$

Let $E \times F = D$.

7 A DIVIDES F AS WELL

By the same reasoning as before, we also have $A \mid F$.

A divides both E and F.

8 COMPARE WITH $D = A \times C$

By Proposition IX.11, $D = A \times C$.

9 FROM $E \times F = A \times C$

Since $E \times F = D = A \times C$, we have

$$A : E = F : C$$

and since $A \mid E$, it follows that $F \mid C$.

10 LET F MEASURE C

Let F measure C according to another number G.

$$C = F \times G$$

11 A DIVIDES G

By the same reasoning again, $A \mid G$.

12 COMPARE WITH $C = A \times B$

By IX.11, $C = A \times B$.

13 FROM $F \times G = A \times B$

Since $F \times G = C = A \times B$, we have

$$A : F = G : B$$

and since $A \mid F$, it follows that $G \mid B$.

14 LET G MEASURE B

Let G measure B according to another number H.

$$B = G \times H$$

15 COMPARE WITH $B = A \times A$

By Proposition IX.9, $B = A \times A$.

16 CONTRADICTION

Since $G \times H = B = A \times A$, we obtain $H \mid A$. But A is prime, and H is not A. Therefore H cannot divide A.

Impossible!

16 CONCLUSION (Q.E.D.)

Therefore our assumption was false. No number outside of A, B, or C can measure D. The greatest term D is measured only by A, B, and C.

Q.E.D.

PROPOSITION 14

If a least number is measured by some prime numbers then it will not be measured by any other prime number except (one of) the original measuring numbers.

Proof

Action 1. For let A be the least number measured by the prime numbers B, C, D.

Action 2. I say that A will not be measured by any other prime number except (one of) B, C, D.

Action 3. For, if possible, let it be measured by the prime number E.

Action 4. And let E not be the same as one of B, C, D.

Action 5. And since E measures A, let it measure it according to F .

Action 6. Thus, E has made A (by) multiplying F .

Action 7. And A is measured by the prime numbers B, C, D.

Action 8. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original numbers *Prop. VII.30*

Action 9. Thus, B, C, D will measure one of E, F .

Action 10. In fact, they do not measure E.

Action 11. For E is prime, and not the same as one of B, C, D.

Action 12. Thus, they (all) measure F , which is less than A.

Action 13. The very thing is impossible.

Action 14. For A was assumed (to be) the least number measured by B, C, D.

Action 15. Thus, no prime number can measure A except (one of) B, C, D.

Therefore, If a least number is measured by some prime numbers then it will not be measured by any other prime number except (one of) the original measuring numbers.

Which is the very thing it was required to show.

1 GIVEN – THE LEAST NUMBER A

Let A be the least number measured by the prime numbers B, C, and D.

A
(least number)

2 ASSUME ANOTHER PRIME E ALSO MEASURES A

Assume, for contradiction, that another prime number E, distinct from B, C, D, also measures A.

E A

3 LET E MEASURE A ACCORDING TO F

Let E measure A according to a number F.

$$A = E \times F$$

A F

4 B, C, D ARE PRIMES THAT MEASURE A

By assumption, B, C, and D are prime numbers that measure A.

A

5 BY BOOK VII.30: A PRIME DIVIDES A FACTOR

Book VII, Prop. 30: If a prime number measures a product of two numbers, it must measure one of those numbers.

$$A = E \times F$$

A E F

6 SO B, C, D DIVIDE E OR F

Therefore each of the primes B, C, and D must divide either E or F.

E or F

7 THEY CANNOT DIVIDE E

But E is a prime number and is distinct from B, C, and D.

Hence none of B, C, and D can divide E.

E B C D F

8 THEREFORE B, C, D DIVIDE F

Therefore B, C, and D must divide F.

F

9 BUT F IS SMALLER THAN A

Since $A = E \times F$ and $E > 1$, it follows that $F < A$.

F < A

10 F IS MEASURED BY B, C, D

Thus we have found a number F smaller than A that is measured by the same primes B, C, and D.

F < A

11 CONTRADICTION

But A was assumed to be the least number measured by B, C, and D.

Yet F is smaller than A and is measured by the same primes.

F < A

12 OUR ASSUMPTION IS FALSE

This is a contradiction. Therefore our assumption that E exists is false.

E

13 NO OTHER PRIME DIVIDES A

Hence no prime number other than B, C, and D can divide A.

A

14 CONCLUSION

Therefore A is measured by no other prime number except B, C, and D.

A

15 SUMMARY OF THE LOGIC

Assume E (prime) | A.

$A = E \times F$.

By VII.30, B, C, D divide E or F.

They cannot divide E.

Therefore B, C, D divide F.

But $F < A$.

Contradiction (A was least).

So E cannot exist.

A = E x F

B C D F < A

16 Q.E.D.

If a least number is measured by some prime numbers, then it will not be measured by any other prime number except one of the original measuring numbers.

Q.E.D.

A

43

PROPOSITION 15

If three continuously proportional numbers are the least of those numbers having the same ratio as them then two (of them) added together in any way are prime to the remaining one.

Proof

Action 1. Let A, B, C be three continuously proportional numbers (which are) the least of those numbers having the same ratio as them.

Action 2. I say that two of A, B, C added together in any way are prime to the remaining one, (that is) A and B (prime) to C, B and C to A, and, further, A and C to B.

Action 3. Let the two least numbers, DE and EF, having the same ratio as A, B, C, have been taken *Prop. VIII.2*

Action 4. So it is clear that DE has made A (by) multiplying itself, and has made B (by) multiplying EF, and, further, EF has made C (by) multiplying itself *Prop. VIII.2*

Action 5. And since DE, EF are the least (of those numbers having the same ratio as them), they are prime to one another *Prop. VII.22*

Action 6. And if two numbers are prime to one another then the sum (of them) is also prime to each *Prop. VII.28*

Action 7. Thus, DF is also prime to each of DE, EF.

Action 8. But, in fact, DE is also prime to EF.

Action 9. Thus, DF, DE are (both) prime to EF.

Action 10. And if two numbers are (both) prime to some number then the number created from (multiplying) them is also prime to the remaining number *Prop. VII.24*

Action 11. Hence, the (number created) from (multiplying) F D, DE is prime to EF.

Action 12. Hence, the (number created) from (multiplying) F D, DE is also prime to the (square) on EF *Prop. VII.25*

the remaining number. But the (number created) from (multiplying) F D, DE is the (square) on DE plus the (number created) from (multiplying) DE, EF *Prop. 2.3*

Action 13. Thus, the (square) on DE plus the (number created) from (multiplying) DE, EF is prime to the (square) on EF .

Action 14. And the (square) on DE is A, and the (number created) from (multiplying) DE, EF is B, and the (square) on EF is C.

Action 15. Thus, A, B summed is prime to C.

Action 16. So, similarly, we can show that B, C (summed) is also prime to A.

Action 17. So I say that A, C (summed) is also prime to B.

Action 18. For since ΔE A, ΔE , EZ B, DF is prime to each of DE, EF then the (square) on DF EZ Γ . A, Γ B $\cdot$ is also prime to the (number created) from (multiplying) DE, EF *Prop. VII.25*

Action 19. But, the (sum of the squares) on DE, EF plus twice the (number created) from (multiplying) DE, EF is equal to the (square) on DF *Prop. 2.4*

Action 20. And thus the (sum of the squares) on DE, EF plus twice the (rectangle contained) by DE, EF [is] prime to the (rectangle contained) by DE, EF .

Action 21. By separation, the (sum of the squares) on DE, EF plus once the (rectangle contained) by DE, EF is prime to the (rectangle contained) by DE, EF .

Therefore, If three continuously proportional numbers are the least of those numbers having the same ratio as them then two (of them) added together in any way are prime to the remaining one.

Which is the very thing it was required to show.

1 GIVEN – THREE LEAST PROPORTIONAL NUMBERS

Let three numbers A, B, C be continuously proportional, and let them be the least numbers having the same ratio:

$$A : B = B : C$$

We must show: $A + B \perp C$, $B + C \perp A$, $A + C \perp B$

2 LET THE TWO LEAST NUMBERS IN THE RATIO BE

By Book VII, Prop. 2, let the least numbers in the same ratio be DE and EF. Then:

$$DE = x, EF = y$$

Then the three proportional numbers are:

$$A = DE^2 = x^2, B = DE \cdot EF = xy, C = EF^2 = y^2$$

3 DE AND EF ARE PRIME TO ONE ANOTHER

Since DE and EF are the least numbers in their ratio, they are prime to one another.

$$x \perp y$$

4 LET DF = DE + EF

Let DF be the sum of them.

$$DF = DE + EF = x + y$$

Since $x \perp y$, by Book VII, Prop. 28:

$$x + y \perp x, x + y \perp y$$

5 FIRST RESULT: A + B IS PRIME TO C

$(x + y)$ is prime to x , and x is prime to y . Therefore $(x + y) \times x$ is prime to y^2 , and also prime to y^2 .

$$(x + y) \times x = x^2 + xy = A + B$$

$$y^2 = C$$

Therefore, $A + B$ is prime to C .

6 SECOND RESULT: B + C IS PRIME TO A

By the same reasoning (interchanging the roles of x and y):

$$(x + y) \times y = xy + y^2 = B + C$$

$$x^2 = A$$

Therefore, $B + C$ is prime to A .

7 FOR THE THIRD RESULT, SQUARE DF

Square DF:

$$DF^2 = (x + y)^2$$

So, $(x + y)^2 = x^2 + 2xy + y^2$

8 DF^2 IS PRIME TO xy

Since $(x + y)$ is prime to both x and y , its square is prime to the product xy . Therefore $x^2 + 2xy + y^2$ is prime to xy .

9 REMOVE ONE COPY OF xy

From $x^2 + 2xy + y^2$, subtract one xy . Coprimality is preserved.

Result: $x^2 + xy + y^2$ is prime to xy .

10 REMOVE THE SECOND COPY OF xy

Subtract the remaining xy . Coprimality is still preserved.

Result: $x^2 + y^2$ is prime to xy .

11 THIRD RESULT: A + C IS PRIME TO B

But $x^2 = A$, $y^2 = C$, $xy = B$.

$$A + C \perp B$$

Therefore, $A + C$ is prime to B .

12 CONCLUSION

In a continued proportion A, B, C that are the least numbers in their ratio, any two added together are prime to the remaining one.

$$A + B \perp C$$

$$B + C \perp A$$

$$A + C \perp B$$

Which is the very thing it was required to show.

PROPOSITION 16

*If two numbers are prime to one another then as the
 $\frac{A}{B}$, first is to the second, so the second will
 not (be) to some other number.*

Proof

Action 1. For let the two numbers A and B be prime to one another.

Action 2. I say that as A is to B, so B is not to some other number.

Action 3. For, if possible, let it be that as A is to B, (so) B is to C.

Action 4. And A and B are prime (to one another).

Action 5. And numbers prime (to one another are) also the least
 (of those numbers having the same ratio as them) *Prop.*
VII.21

Action 6. And the least numbers measure those numbers having the
 same ratio (as them) an equal number of times, the lead-
 ing (measuring) the leading, and the following the follow-
 ing *Prop. VII.20*

Action 7. Thus, A measures B, as the leading (measuring) the lead-
 ing.

Action 8. And (A) also measures itself.

Action 9. Thus, A measures A and B, which are prime to one an-
 other.

Action 10. The very thing is absurd.

Action 11. Thus, as A is to B, so B cannot be to C.

Therefore, If two numbers are prime to one another then as the
 $\frac{A}{B}$, first is to the second, so the second
 will not (be) to some other number.

Which is the very thing it was required to show.

1 GIVEN – A AND B ARE PRIME TO ONE ANOTHER Let the two numbers be A and B, and suppose that A and B are prime to one another. $A \nmid B$	2 ASSUME, TO THE CONTRARY, THAT $A : B = B : C$ Assume, for contradiction, that there exists a number C such that $A : B = B : C$. $A : B = B : C$	3 SINCE A AND B ARE PRIME TO ONE ANOTHER, THEY ARE THE LEAST NUMBERS IN THEIR RATIO (Book VII, Prop. 21) Since A and B are prime to one another, they are the <u>least numbers</u> having the ratio $A : B$. least in the ratio $A : B$	4 BUT B AND C HAVE THE SAME RATIO But, by our assumption, $A : B = B : C$. So B and C also have the same ratio $A : B$. same ratio as A and B
5 THE LEAST NUMBERS MEASURE CORRESPONDING NUMBERS AN EQUAL NUMBER OF TIMES (Book VII, Prop. 20) The least numbers having a given ratio measure the corresponding numbers having the same ratio an <u>equal number of times</u>.	6 THEREFORE A MEASURES B Therefore the leading term A must measure the leading term B. $A \mid B$	7 OF COURSE A MEASURES A And, of course, A measures A. $A \mid A$	8 SO A MEASURES BOTH A AND B Thus A measures both A and B. $A \mid A$ $A \mid B$
9 BUT THIS CONTRADICTS THAT A AND B ARE PRIME TO ONE ANOTHER But if A measures B, then A and B are not prime to one another. $A \nmid B$	10 IMPOSSIBLE! This contradicts our original assumption that A and B are prime to one another.	11 THEREFORE OUR ASSUMPTION WAS FALSE Therefore our supposition that there exists a number C such that $A : B = B : C$ is false. $A : B \neq B : C$	12 CONCLUSION (Q.E.D.) Hence, if two numbers A and B are prime to one another, then as the first is to the second, so the second will not be to some other number. Q.E.D.

PROPOSITION 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first is to the second, so the last will not be to some other number.

Proof

Action 1. Let A, B, C, D be any multitude whatsoever of continuously proportional numbers.

Action 2. And let the outermost of them, A and D, be prime to one another.

Action 3. I say that as A is to B, so D is not to some other number.

Action 4. For, if possible, let it be that as A is to B, so D is to E.

Action 5. Thus, alternately, as A is to D, (so) B is to E *Prop. VII.13*

Action 6. And A and D are prime (to one another).

Action 7. And numbers prime (to one another are) also the least (of those numbers having the same ratio as them) *Prop. VII.21*

Action 8. And the least numbers measure those numbers having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following *Prop. VII.20*

Action 9. Thus, A measures B.

Action 10. And as A is to B, (so) B is to C.

Action 11. Thus, B also measures C.

Action 12. And hence A measures C *Def. 7.20*

Action 13. And since as B is to C, (so) C is to D, and B measures C, C thus also measures D *Def. 7.20*

Action 14. But, A was (found to be) measuring C.

Action 15. And hence A also measures D.

Action 16. And (A) also measures itself.

Action 17. Thus, A measures A and D, which are prime to one another.

Action 18. The very thing is impossible.

Action 19. Thus, as A is to B, so D cannot be to some other number.

Therefore, If any multitude whatsoever of numbers is continuous, and the outermost of them are prime to one another, then as the first is to the second, so the last will not be to some other number.

Which is the very thing it was required to show.

1 GIVEN – A CONTINUOUS PROPORTION Let four numbers A, B, C, D be continuously proportional: $A : B = B : C = C : D$	2 OUTERMOST NUMBERS ARE PRIME TO ONE ANOTHER Let the outermost numbers A and D be prime to one another. $A \perp D$	3 ASSUME, TO THE CONTRARY Assume there exists a number E such that $A : B = D : E$	4 BY ALTERNATION Therefore, $A : D = B : E$
5 A AND D ARE LEAST IN THEIR RATIO Since A and D are prime to one another, they are the least numbers having the ratio A : D. <i>(Book VII, Prop. 21)</i> Least numbers in the ratio A : D	6 SO A MEASURES B From A : D = B : E and the fact that A and D are the least in their ratio, they must measure B and E respectively. $A \mid B$	7 THEN B MEASURES C From the continued proportion A : B = B : C and since A B, it follows that $B \mid C$	8 SO A MEASURES C Because A B and B C, it follows that $A \mid C$
9 THEN C MEASURES D From the continued proportion B : C = C : D and since B C, it follows that $C \mid D$	10 THEREFORE A MEASURES D Since A C and C D, it follows that $A \mid D$	11 BUT A ALSO MEASURES ITSELF Of course, $A \mid A$	12 A MEASURES BOTH OUTERMOST NUMBERS Thus A measures both outermost numbers: $A \mid A$ and $A \mid D$
13 CONTRADICTION But A and D were assumed to be prime to one another. $A \perp D$ This is impossible!	14 SO OUR ASSUMPTION WAS FALSE Therefore our assumption that there exists a number E such that $A : B = D : E$ is false.	15 THERE IS NO SUCH NUMBER E Hence, there is no number E such that $A : B = D : E$ for any number E.	16 Q.E.D. Therefore, if a multitude of numbers is continuously proportional and the outermost numbers are prime to one another, then as the first is to the second, so the last will not be to some other number. Q.E.D.

PROPOSITION 18

For two given numbers, to investigate whether it is possible to find a third number proportional to them.

Proof

- Action 1.** Let A and B be the two given numbers.
- Action 2.** And let it be required to investigate whether it is possible to find a third number proportional to them.
- Action 3.** So A and B are either prime to one another, or not.
- Action 4.** And if they are prime to one another then it has (already) been show that it is impossible to find a third number proportional to them *Prop. IX.16*
- Action 5.** And so let A and B not be prime to one another.
- Action 6.** And let B make C (by) multiplying itself.
- Action 7.** So A either measures, or does not measure, C.
- Action 8.** Let it first of all measure (C) according to D.
- Action 9.** Thus, A has made C (by) multiplying D.
- Action 10.** But, in fact, B has also made C (by) multiplying itself.
- Action 11.** Thus, the (number created) from (multiplying) A, D is equal to the (square) on B.
- Action 12.** Thus, as A is to B, (so) B is to D *Prop. VII.19*
- Action 13.** Thus, a third number has been found proportional to A, B, (namely) D.
- Action 14.** And so let A not measure C.
- Action 15.** I say that it is impossible to find a third number proportional to A, B.
- Action 16.** For, if possible, let it have been found, (and let it be) D.

Action 17. Thus, the (number created) from (multiplying) A, D is equal to the (square) on B *Prop. VII.19*

Action 18. And the (square) on B is C.

Action 19. Thus, the (number created) from (multiplying) A, D is equal to C.

Action 20. Hence, A has made C (by) multiplying D.

Action 21. Thus, A measures C according to D.

Action 22. But (A) was, in fact, also assumed (to be) not measuring (C).

Action 23. The very thing is absurd.

Action 24. Thus, it is not possible to find a third number proportional to A, B when A does not measure C.

Therefore, For two given numbers, to investigate whether it is possible to find a third number proportional to them.

Which is the very thing it was required to show.

1 GIVEN TWO NUMBERS A AND B

Let the two given numbers be A and B.

2 OUR PROBLEM

We want to know whether there exists a number D such that $A : B :: B : D$.

3 FIRST QUESTION: ARE A AND B PRIME TO ONE ANOTHER?

First, ask whether A and B are prime to one another.

4A CASE 1: A AND B ARE PRIME TO ONE ANOTHER

By Proposition IX.16, no third proportional can exist.

NO third proportional exists.

4B CASE 2: A AND B ARE NOT PRIME TO ONE ANOTHER

Let B multiply itself, producing C.

So $C = B^2$.

5 SECOND QUESTION: DOES A MEASURE $C = B^2$?

Now ask whether A measures $C = B^2$.

$A \mid C$?

$C = B^2$

6A CASE 2A: A MEASURES $C = B^2$

Suppose A measures C according to D.

Thus $A \times D = B^2$.

7A THEN D IS THE THIRD PROPORTIONAL

By Book VII, Prop. 19, from $A \times D = B^2$ it follows that $A : B :: B : D$.

So $D = B^2 / A$ is the third proportional. A third proportional EXISTS.

6B CASE 2B: A DOES NOT MEASURE $C = B^2$

Suppose A does not measure C.

$A \nmid C$

$C = B^2$

7B ASSUME, TO THE CONTRARY, THAT A THIRD PROPORTIONAL D EXISTS

Assume, contrary to what we want, that there exists a number D such that $A : B :: B : D$.

$A : B :: B : D$

8B THEN $A \times D = B^2 = C$

By Book VII, Prop. 19, from $A : B :: B : D$ it follows that $A \times D = B^2$.

Hence A measures C.

9B CONTRADICTION!

But we supposed A does not measure C. This is a contradiction.

IMPOSSIBLE!

Therefore no such D can exist.

10 CONCLUSION OF CASE 2B

If A does not measure B^2 , then no third proportional exists.

NO third proportional exists.

11 COMPLETE RULE (THE TEST)

For two given numbers A and B:

12 SUMMARY

A third proportional D to A and B exists exactly when A measures the square of B. Then $D = B^2 / A$.

$D = \frac{B^2}{A}$

15 Q. E. D.

Which is the very thing it was required to show.

Q. E. D.

PROPOSITION 19

For three given numbers, to investigate when it is possible to find a fourth number proportional to them.

Proof

Action 1. Let A, B, C be the three given numbers.

Action 2. And let it be required to investigate when it is possible to find a fourth number proportional to them.

Action 3. In fact, (A, B, C) are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

Action 4. In fact, if A, B, C are continuously proportional, and the outermost of them, A and C , are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them *Prop. IX.17*

Action 5. So let A, B, C not be continuously proportional, (with) the outermost of them again being prime to one another.

Action 6. I say that, in this case, it is also impossible to find a fourth number proportional to them.

Action 7. For, if possible, let it have been found, (and let it be) D .

Action 8. Hence, it will be that as A is to B , (so) C is to D .

Action 9. And let it be contrived that as B is to C , (so) D is to E .

Action 10. And since as A is to B , (so) C is to D , and as B is to C , (so) D is to E , thus, via equality, as A is to C , (so) C is to E *Prop. VII.14*

Action 11. And A and C are prime (to one another).

- Action 12.** And numbers prime (to one another are) also the least (numbers having the same ratio as them) *Prop. VII.21*
- Action 13.** And the least numbers measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following *Prop. VII.20*
- Action 14.** Thus, A measures C, (as) the leading (measuring) the leading.
- Action 15.** And it also measures itself.
- Action 16.** Thus, A measures A and C, which are prime to one another.
- Action 17.** The very thing is impossible.
- Action 18.** Thus, it is not possible to find a fourth number proportional to A, B, And so let A, B, C again be continuously proportional, and let A and C not be prime to one another.
- Action 19.** I say that it is possible to find a fourth number proportional to them.
- Action 20.** For let B make D (by) multiplying C.
- Action 21.** Thus, A either measures or does not measure D.
- Action 22.** Let it, first of all, measure (D) according to E.
- Action 23.** Thus, A has made D (by) multiplying E.
- Action 24.** But, in fact, B has also made D (by) multiplying C.
- Action 25.** Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C.
- Action 26.** Thus, proportionally, as A [is] to B, (so) C is to E *Prop. VII.19*
- Action 27.** Thus, a fourth number proportional to A, B, C has been found, (namely) E.

Action 28. And so let A not measure D.

Action 29. I say that it is impossible to find a fourth number proportional to A, B, C.

Action 30. For, if possible, let it have been found, (and let it be) E.

Action 31. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C.

Action 32. But, the (number created) from (multiplying) B, C is D.

Action 33. And thus the (number created) from (multiplying) A, E is equal to D.

Action 34. Thus, A has made D (by) multiplying E.

Action 35. Thus, A measures D according to E.

Action 36. Hence, A measures D.

Action 37. But, it also does not measure (D).

Action 38. The very thing is absurd.

Action 39. Thus, it is not possible to find a fourth number proportional to A, B, C when A does not measure D.

Action 40. And so (let) A, B, C (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another.

Action 41. And let B make D (by) multiplying C.

Action 42. So, similarly, it can be show that if A measures D then it is possible to find a fourth number proportional to (A, B, C), and impossible if (A) does not measure (D).

Therefore, For three given numbers, to investigate when it is possible to find a fourth number proportional to them.

Scholarly note. The source edition marks Proposition IX.19 with a dagger and observes that one transmitted case of Euclid's proof is defective. The proposition is retained here in the form transmitted by the source, with the note kept separate from the proof.

Which is the very thing it was required to show.

1 GIVEN – THREE NUMBERS A, B, C

Let the three given numbers be A, B, C. We want to know when there exists a fourth number D such that $A : B = C : D$.

2 TWO QUESTIONS TO CONSIDER

First, are A, B, C continuously proportional? Second, are the outermost numbers A and C prime to one another?

3A CASE 1 – CONTINUOUSLY PROPORTIONAL AND A AND C PRIME TO ONE ANOTHER

Suppose $A : B = B : C$ and $A \perp C$.

By Proposition IX.17, the ratio A : B cannot be continued from C to another number.

Therefore, no fourth proportional exists.

3B CASE 2 – NOT CONTINUOUSLY PROPORTIONAL BUT A AND C ARE PRIME TO ONE ANOTHER

Suppose A, B, C are not continuously proportional, but $A \perp C$. Suppose, contrary to fact, that there exists a number D such that $A : B = C : D$.

3B(a) CONSTRUCT ANOTHER PROPORTION

Let there be a number E such that $B : C = D : E$.

3B(b) COMBINE THE TWO PROPORTIONS

From $A : B = C : D$ and $B : C = D : E$, $A : C = C : E$.

Since $A \perp C$, A must measure C.

3B(c) CONTRADICTION

But A also measures A. So A would be a common measure of A and C, which are prime to one another.

IMPOSSIBLE!

Therefore no fourth proportional exists.

4 CASE 3 – CONTINUOUSLY PROPORTIONAL BUT A AND C NOT PRIME TO ONE ANOTHER

Suppose $A : B = B : C$, but A and C are not prime to one another.

Form D by multiplying B and C.

5 FORM $D = B \times C$

6 ASK – DOES A MEASURE D?

Does A measure D? (that is, does A divide D?)

7A IF YES: A MEASURES D

Suppose A measures D. Let E be such that $A \times E = D$.

Thus $A \times E = B \times C$.

7B CONCLUDE FOURTH PROPORTIONAL EXISTS

By Book VII, Prop. 19, from $A \times E = B \times C$, we get $A : B = C : E$.

E is the required fourth proportional.

8A IF NO: A DOES NOT MEASURE D

Suppose A does not measure D.

14 ASSUME CONTRARY: SUPPOSE E EXISTS

Suppose, contrary to fact, that a fourth proportional E exists such that $A : B = C : E$.

8C REACH CONTRADICTION

Then, by Book VII, Prop. 19, $A \times E = B \times C = D$.

But this means A measures D, contrary to our assumption.

9 CASE 4 – NEITHER CONTINUOUS PROPORTION NOR OUTERMOST PRIME

If A, B, C are not continuously proportional and A and C are not prime to one another, form $D = B \times C$ and ask again: Does A measure D?

- IF YES $\rightarrow$ fourth proportional exists.
- IF NO $\rightarrow$ fourth proportional does not exist.

This completes the investigation. Q.E.D.

PROPOSITION 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.

Proof

Action 1. Let A, B, C be the assigned prime numbers.

Action 2. I say that the (set of all) primes numbers is more numerous than A, For let the least number measured by A, B, C have been taken, and let it be DE *Prop. VII.36*

Action 3. And let the unit DF have been added to DE.

Action 4. So EF is either prime, or not.

Action 5. Let it, first of all, be prime.

Action 6. Thus, the (set of) prime numbers A, B, C, EF , (which is) more numerous than A, B, C, has been found.

Action 7. And so let EF not be prime.

Action 8. Thus, it is measured by some prime number *Prop. VII.31*

Action 9. Let it be measured by the prime number G.

Action 10. I say that G is not the same as any of A, B, C.

Action 11. For, if possible, let it be (the same).

Action 12. And A, B, C (all) measure DE.

Action 13. Thus, G will also measure DE.

Action 14. And it also measures EF .

Action 15. (So) G will also measure the remainder, unit DF , (despite) being a number *Prop. VII.28*

Action 16. The very thing is absurd.

Action 17. Thus, G is not the same as one of A, B, C.

Action 18. And it was assumed (to be) prime.

Action 19. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found.

Therefore, The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.

Which is the very thing it was required to show.



PROPOSITION 21

If any multitude whatsoever of even numbers is added together then the whole is even.

Proof

Action 1. For let any multitude whatsoever of even numbers, AB, BC, CD, DE, lie together.

Action 2. I say that the whole, AE, is even.

Action 3. For since everyone of AB, BC, CD, DE is even, it has a half part *Def. 7.6*

Action 4. And hence the whole AE has a half part.

Action 5. And an even number is one (which can be) divided in half *Def. 7.6*

Action 6. Thus, AE is even.

Therefore, If any multitude whatsoever of even numbers is added together then the whole is even.

Which is the very thing it was required to show.

1 GIVEN – SEVERAL EVEN NUMBERS

Let any multitude of even numbers be taken together.
Let the even numbers be AB, BC, CD, DE.

Each of these numbers is even.

2 EACH EVEN NUMBER HAS A HALF

Since each is even,
each has a half part.

3 ADD ALL THE EVEN NUMBERS TOGETHER

Add all the even numbers together.
Let their whole sum be AE.

4 ADD THEIR HALF PARTS TOGETHER

Then their half parts
can also be added together.

5 THE SUM HAS A HALF PART

Therefore the whole number AE
has a half part.

This is a half of AE.

6 DEFINITION OF EVEN NUMBER

But an even number is a number
which can be divided into
two equal parts.

7 THEREFORE AE IS EVEN

Since AE can be divided
into two equal parts,
AE is even.

8 GENERAL FOR ANY MULTITUDE

This reasoning works for any number
of even numbers whatsoever.

The sum of any multitude of even numbers is even.

9 Q.E.D.

Which is the very thing
it was required to show.

Q.E.D.

PROPOSITION 22

If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.

Proof

Action 1. For let any even multitude whatsoever of odd numbers, AB, BC, CD, DE, lie together.

Action 2. I say that the whole, AE, is even.

Action 3. For since everyone of AB, BC, CD, DE is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even *Def. 7.7*

Action 4. And hence the sum of them will be even *Prop. IX.21*

Action 5. And the multitude of the units is even.

Action 6. Thus, the whole AE is also even *Prop. IX.21*.

Therefore, If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.

Which is the very thing it was required to show.

1 GIVEN – AN EVEN MULTITUDE OF ODD NUMBERS

Let there be any even number of odd numbers.
For example, take four odd numbers:
3, 5, 7, 9

3 (odd) 5 (odd) 7 (odd) 9 (odd)

There are four odd numbers – an even multitude.

2 EACH ODD NUMBER IS OF THE FORM $2k + 1$

Each odd number can be written as $2k + 1$.
 $3 = 2 \times 1 + 1$ $5 = 2 \times 2 + 1$
 $7 = 2 \times 3 + 1$ $9 = 2 \times 4 + 1$

$2 \times 1 + 1$ $2 \times 2 + 1$ $2 \times 3 + 1$ $2 \times 4 + 1$

Odd numbers are one more than an even number.

3 PAIR THEM TWO BY TWO

Since the number of odd numbers is even, we can pair them two by two.

(3 with 5) and (7 with 9).

4 THE SUM OF EACH PAIR IS EVEN

The sum of two odd numbers is always even.
 $\text{odd} + \text{odd} = \text{even}$

$3 + 5 = 8$ (even) $7 + 9 = 16$ (even)

Both pairs give even numbers.

5 SO WE GET A MULTITUDE OF EVEN NUMBERS

Thus the even multitude of odd numbers becomes a multitude of even numbers.

8 (even) 16 (even)

We now have even numbers: 8 and 16.

6 BY PROPOSITION IX.21

By Proposition IX.21:
The sum of any multitude of even numbers is even.

24

Therefore, their sum is even.

7 THE SUM OF THE EVEN NUMBERS

Add the even numbers:
 $8 + 16 = 24$

24

The whole sum is 24.

8 24 CAN BE DIVIDED INTO TWO EQUAL PARTS

Since 24 can be divided into two equal parts, 24 is even.

12 12

$24 = 12 + 12$

9 CONCLUSION

Therefore, if any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.
Q.E.D.

PROPOSITION 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.

Proof

Action 1. For let any multitude whatsoever of odd numbers, AB, BC, CD, lie together, and let the multitude of them be odd.

Action 2. I say that the whole, AD, is also odd.

Action 3. For let the unit DE have been subtracted from CD.

Action 4. The remainder CE is thus even *Def. 7.7*

Action 5. And CA is also even *Prop. IX.22*

Action 6. Thus, the whole AE is also even *Prop. IX.21*

Action 7. And DE is a unit.

Action 8. Thus, AD is odd *Def. 7.7*.

Therefore, If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.

Which is the very thing it was required to show.

1 GIVEN – AN ODD MULTITUDE OF ODD NUMBERS

Let an odd number of odd numbers be taken together.
For example, take these odd numbers.

3 (odd) 5 (odd) 7 (odd)

There are three odd numbers – an odd multitude.

2 REMOVE ONE UNIT FROM THE LAST ODD NUMBER

Take away one unit from the last odd number.

3 (odd) 5 (odd) 6 (even) 1

The remaining part (6) is even.

3 THE REMAINING ODD NUMBERS ARE EVEN IN NUMBER

The remaining odd numbers are now two in number, which is even.

Two odd numbers (even in number)

4 BY PROPOSITION IX.22 – THEIR SUM IS EVEN

By Proposition IX.22:
The sum of an even multitude of odd numbers is even.

8 (even)

So $3 + 5 = 8$, which is even.

5 ADD THE EVEN REMAINDER (THE 6)

Add the even remainder (6) to this even sum (8).

8 (even) 6 (even) 14 (even)

By Proposition IX.21:
The sum of even numbers is even.

6 THEREFORE AE IS EVEN

Therefore the whole of the numbers except the removed unit is even.
Call this total AE.

AE = 14 (even)

7 RESTORE THE REMOVED UNIT

Restore the removed unit (1) to AE.

AE = 14 (even) 1

8 EVEN PLUS ONE IS ODD

AE is even, and DE = 1.
Therefore,
 $AD = AE + DE = \text{even} + 1$.

AD = 15 (odd)

9 CONCLUSION

Therefore if any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.
Q.E.D.

Summary of the reasoning:

Odd multitude of odd numbers → Remove one unit from one odd → Remaining odds are even in number → Their sum is even (IX.22) → Add the even remainder (IX.21) → Even + 1 = odd

PROPOSITION 24

If an even number is subtracted from an(other) even number then the remainder will be even.

Proof

Action 1. For let the even number BC have been subtracted from the even number AB.

Action 2. I say that the remainder CA = AB - BC, is even.

Action 3. For since AB is even, it has a half part Def. 7.6

Action 4. So, $AF = \frac{1}{2} AB$ for the same (reasons), BC also has a half part.

Action 5. And hence the remainder [CA has a half part]

Thus, AC is even.

Therefore, If an even number is subtracted from an(other) even number then the remainder will be even.

Which is the very thing it was required to show.

1

GIVEN - TWO EVEN NUMBERS

Let the even number AB be subtracted from the even number BC . Let the remainder be CA .

AB (even)
 BC (even)
Same length

2

EACH EVEN NUMBER HAS A HALF

Since AB is even, it has a half part.

half of AB half of AB

Since BC is even, it also has a half part.

half of BC half of BC

3

REMOVE BC FROM AB

Remove the even number BC from the even number AB .

4

THE REMAINDER IS CA

The remainder left is CA .

CA (remainder)

5

CORRESPONDING HALVES ARE REMOVED

We remove one half of BC from one half of AB ...

... and the other half from the other half.

6

THE REMAINDER STILL HAS A HALF

The remainder CA still has a half part.

half of CA half of CA

7

A NUMBER WITH A HALF IS EVEN

By definition, a number that has a half part is even.

half half

Therefore CA is even.

8

ALGEBRAIC VIEW (OPTIONAL)

Let $AB = 2m$ and $BC = 2n$.
 $CA = AB - BC = 2m - 2n = 2(m - n)$.
So CA has a half part.

Hence CA is even.

9

CONCLUSION

Therefore, if an even number is subtracted from another even number, then the remainder will be even.

Q.E.D.

PROPOSITION 25

If an odd number is subtracted from an even number then the remainder will be odd.

Proof

Action 1. For let the odd number BC have been subtracted from the even number AB.

Action 2. I say that the remainder CA is odd.

Action 3. For let the unit CD have been subtracted from BC.

Action 4. DB is thus even *Def. 7.7*

Action 5. And AB is also even.

Action 6. And thus the remainder AD is even *Prop. IX.24*

Action 7. And CD is a unit.

Action 8. Thus, CA is odd *Def. 7.7*.

Therefore, If an odd number is subtracted from an even number then the remainder will be odd.

Which is the very thing it was required to show.

1

GIVEN – AN EVEN AND AN ODD NUMBER

Let the even number AB be subtracted from the odd number BC . Let the remainder be CA .

We want to show that CA is odd.

2

REMOVE ONE UNIT FROM THE ODD NUMBER BC

Remove one unit CD from BC .

Since BC is odd, DB is even.

3

BOTH AB AND DB ARE EVEN

AB is even, and DB is even.

4

SUBTRACT THE EVEN NUMBER DB FROM AB

Subtract the even number DB from the even number AB .

5

THE REMAINDER AD IS EVEN (BY IX.24)

By Proposition IX.24, the remainder AD is even.

6

THE ORIGINAL REMAINDER CA CONTAINS AD AND CD

The original remainder CA contains AD together with the unit CD .

7

$CA = AD + CD = \text{EVEN} + 1$

Thus $CA = AD + CD$. Since AD is even and $CD = 1$, $CA = \text{even} + 1$.

8

AN EVEN NUMBER PLUS ONE IS ODD

By definition, an even number plus one is an odd number. Therefore CA is odd.

9

CONCLUSION

Therefore, if an odd number is subtracted from an even number, then the remainder will be odd.

Q.E.D.

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PROPOSITION 26

If an odd number is subtracted from an odd number then the remainder will be even.

Proof

Action 1. For let the odd number BC have been subtracted from the odd number AB.

Action 2. I say that the remainder CA is even.

Action 3. For since AB is odd, let the unit BD have been subtracted (from it).

Action 4. Thus, the remainder AD is even *Def. 7.7*

Action 5. So, for the same (reasons), CD is also even.

Action 6. And hence the remainder CA is even *Prop. IX.24*.

Therefore, If an odd number is subtracted from an odd number then the remainder will be even.

Which is the very thing it was required to show.

1

GIVEN – TWO ODD NUMBERS

Let the odd number AB be subtracted from the odd number BC. Let the remainder be CA.

AB (odd)
BC (odd)

We want to show that CA is even.

2

REMOVE ONE UNIT FROM AB

Remove one unit BD from AB.

AD (even)
BD = 1

Since AB is odd, $AD = AB - 1$ is even.

3

REMOVE ONE UNIT FROM BC

Remove one unit from BC in the same way.

CD (even)
DC = 1

Since BC is odd, $CD = BC - 1$ is even.

4

WE NOW HAVE TWO EVEN NUMBERS

AD is even and CD is even.

AD (even)
CD (even)

Both numbers have halves.

5

THE REMAINDER IS AD MINUS CD

The required remainder CA is the difference of these two even numbers.

$CA = AD - CD$

6

BY PROPOSITION IX.24

By Proposition IX.24: If an even number is subtracted from an even number, the remainder is even.

CA is even.

7

VISUAL HALF ARGUMENT

Both AD and CD can be split into two equal parts.

half of AD
half of AD
half of CD
half of CD

Subtract equal halves from equal halves.

8

THE REMAINDER ALSO HAS A HALF

The remainder CA also has a half.

half of CA
half of CA

A number that has a half is even.

9

CONCLUSION

Therefore, if an odd number is subtracted from an odd number, the remainder will be even.

Q.E.D.

CA (even)

PROPOSITION 27

If an even number is subtracted from an odd number then the remainder will be odd.

Proof

Action 1. For let the even number BC have been subtracted from the odd number AB.

Action 2. I say that the remainder CA A [] AΔ · ΔB . is odd.

For let the unit AD have been subtracted (from AB).

Action 3. DB is thus even *Def. 7.7*

Action 4. And BC is also even.

Action 5. Thus, the remainder CD is also even *Prop. IX.24*

Action 6. CA is thus odd *Def. 7.7*.

Therefore, If an even number is subtracted from an odd number then the remainder will be odd.

Which is the very thing it was required to show.

1 GIVEN – AN ODD AND AN EVEN NUMBER

Let the odd number AB be subtracted from the even number BC. Let the remainder be CA.

We want to show that CA is odd.

2 REMOVE ONE UNIT FROM THE ODD NUMBER AB

Remove one unit AD from AB.

Since AB is odd, $DB = AB - AD$ is even.

3 DB IS EVEN (AFTER REMOVING ONE UNIT)

After removing AD, DB is even.

4 SUBTRACT THE EVEN NUMBER BC FROM DB

Subtract the even number BC from the even number DB.

5 THE DIFFERENCE CD IS EVEN (BY PROPOSITION IX.24)

By Proposition IX.24, $DB - BC = CD$ is even.

6 THE ORIGINAL REMAINDER CA CONTAINS CD AND AD

The original remainder CA consists of CD together with the removed unit AD.

7 ADD BACK THE REMOVED UNIT AD

Add back the unit AD to CD.

8 EVEN PLUS ONE IS ODD

Since CD is even and $AD = 1$, $CA = CD + AD = \text{even} + 1$, which is odd.

9 CONCLUSION

Therefore, if an even number is subtracted from an odd number, then the remainder will be odd.

Q.E.D.

PROPOSITION 28

If an odd number makes some (number by) multiplying an even number then the created number will be even.

Proof

Action 1. For let the odd number A make C (by) multiplying the even number B.

Action 2. I say that C is even.

Action 3. For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A *Def. 7.15*

Action 4. And B is even.

Action 5. Thus, C is composed out of even numbers.

Action 6. And if any multitude whatsoever of even numbers is added together then the whole is even *Prop. IX.21*

Action 7. Thus, C is even.

Therefore, If an odd number makes some (number by) multiplying an even number then the created number will be even.

Which is the very thing it was required to show.

1

GIVEN – AN ODD AND AN EVEN NUMBER

Let A be an odd number and B be an even number. Let A multiply B to make C.

A = 5 (odd)

B = 6 (even)

2

MULTIPLICATION MEANS REPEATED ADDITION

Since A multiplies B, C is made of as many copies of B as there are units in A.

1st B 2nd B 3rd B 4th B 5th B

A = 5

C = B + B + B + B + B (five copies of B)

3

WRITE THE MULTIPLICATION AS A SUM

Therefore,

$C = B + B + B + B + B$

4

EACH COPY OF B IS EVEN

But B is even. Therefore every copy of B in this sum is even.

1st B → even

2nd B → even

3rd B → even

4th B → even

5th B → even

5

C IS A SUM OF EVEN NUMBERS

Hence C is a sum of even numbers.

C =

6

BY PROPOSITION IX.21

By Proposition IX.21: Any multitude of even numbers added together has an even sum.

sum of even numbers is even

7

THEREFORE C IS EVEN

Therefore, C is even.

C (even)

8

VISUAL HALF – C SPLITS INTO TWO EQUAL PARTS

Since C is even, it has a half. We can split C into two equal parts.

half of C half of C

9

CONCLUSION

Therefore, if an odd number multiplies an even number, the number made will be even.

Q.E.D.

PROPOSITION 29

If an odd number makes some (number by) multiplying an odd number then the created number will be odd.

Proof

Action 1. For let the odd number A make C (by) multiplying the odd number B.

Action 2. I say that C is odd.

Action 3. For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A *Def. 7.15*

Action 4. And each of A, B is odd.

Action 5. Thus, C is composed out of odd numbers, (and) the multitude of them is odd.

Action 6. Hence C is odd *Prop. IX.23*.

Therefore, If an odd number makes some (number by) multiplying an odd number then the created number will be odd.

Which is the very thing it was required to show.

1

GIVEN – TWO ODD NUMBERS

Let A be an odd number
and B be an odd number.
Let A multiply B
to make C.

A = 5 (odd)

B = 7 (odd)

We want to show that C is odd.

2

MULTIPLICATION MEANS REPEATED ADDITION

Since A multiplies B,
C is made of as many copies
of B as there are units in A.

1st B 2nd B 3rd B 4th B 5th B

$C = B + B + B + B + B$ (five copies of B)

A has 5 units, so there are 5 copies of B.

3

B IS ODD

B has an odd number of units.
Here each copy of B has 7 units,
which is odd.

one copy of B

7 (odd)

Therefore every copy of B is odd.

4

THE NUMBER OF COPIES IS ODD

A is odd, so the number
of copies of B is odd.

1st 2nd 3rd 4th 5th

There are 5 copies of B, which is an odd number.

5

C IS A SUM OF AN ODD MULTITUDE OF ODD NUMBERS

C is made by adding
an odd multitude (5) of odd
numbers (each copy of B).

1st 2nd 3rd 4th 5th

$C = B + B + B + B + B$

Thus C is a sum of an odd multitude of odd numbers.

6

BY PROPOSITION IX.23

By Proposition IX.23:
An odd multitude of odd numbers
added together has an odd sum.

PROP. IX.23
An odd multitude of
odd numbers has
an odd sum.

7

THEREFORE C IS ODD

Therefore C, which is
the sum we have formed,
is odd.

1st 2nd 3rd 4th 5th

C

C is an odd number.

8

VISUAL CONFIRMATION (SAME PARITY)

We can pair the units
of C into twos, and one
unit will remain.

Pairs of two One left over

Having one left over shows C is odd.

9

CONCLUSION

Therefore, if an odd number
multiplies an odd number,
the number made will be odd.

Q.E.D.

PROPOSITION 30

If an odd number measures an even number then it will also measure one half of it.

Proof

Action 1. For let the odd number A measure the even number B.

Action 2. I say that (A) will also measure one half of (B).

Action 3. For since A measures B, let it measure it according to C.

Action 4. I say that C is not odd.

Action 5. For, if possible, let it be (odd).

Action 6. And since A measures B according to C, A has thus made B (by) multiplying C.

Action 7. Thus, B is composed out of odd numbers, (and) the multitude of them is odd.

Action 8. B is thus odd *Prop. IX.23*

Action 9. The very thing is absurd.

Action 10. For (B) was assumed (to be) even.

Action 11. Thus, C is not odd.

Action 12. Thus, C is even.

Action 13. Hence, A measures B an even number of times.

Action 14. So, on account of this, (A) will also measure one half of (B).

Therefore, If an odd number measures an even number then it will also measure one half of it.

Which is the very thing it was required to show.

1

GIVEN – A ODD MEASURES AN EVEN NUMBER.

Let A be an odd number and B be an even number. Suppose A measures B.

A (odd)

B (even)

A measures B exactly C times.

$B = A \times C$

2

ASSUME, FOR CONTRADICTION, THAT C IS ODD

Let A measure B according to C. Suppose C is odd.

$B = A \times C$ (suppose C is odd)

Then C is an odd number of copies of A.

A (odd)

C copies (odd number)

3

ODD TIMES ODD WOULD GIVE AN ODD TOTAL.

But A is odd, and (by the assumption) C is odd. So the product $A \times C$ is a product of two odd numbers.

By Proposition IX.29, A product of two odd numbers is odd. Hence B would be odd.

4

CONTRADICTION

However, B was given to be even.

So we have a contradiction.

5

THEREFORE C CANNOT BE ODD

Therefore C is not odd. Hence C is even.

$B = A \times C$ and C is even.

C copies (even number)

6

WRITE C AS TWO TIMES ANOTHER NUMBER

Since C is even, let $C = 2D$ for some number D.

$C = 2D$

D copies of A

D copies of A

Total $C = 2D$ copies of A

7

B IS TWICE A MULTIPLE OF A

$B = A \times C = A \times (2D) = 2(AD)$

AD

AD

B

$B = 2(AD)$

8

THEREFORE A MEASURES HALF OF B

One half of B is AD.

$\frac{B}{2} = AD = A \times D$

Thus A measures one half of B.

AD = $A \times D$

A

A

A

D copies of A

9

CONCLUSION

Therefore, if an odd number measures an even number, then it will also measure one half of it.

Q.E.D.

PROPOSITION 31

If an odd number is prime to some number then it will also be prime to its double.

Proof

Action 1. For let the odd number A be prime to some number B.

Action 2. And let C be double B.

Action 3. I say that A is [also] prime to For if [A and C] are not prime (to one another) then some number will measure them.

Action 4. Let it measure (them), and let it be D.

Action 5. And A is odd.

Action 6. Thus, D is also odd.

Action 7. And since D, which is odd, measures C, and C is even, [D] will thus also measure half of C *Prop. IX.30*

Action 8. And B is half of C.

Action 9. Thus, D measures B.

Action 10. And it also measures A.

Action 11. Thus, D measures (both) A and B, (despite) them being prime to one another.

Action 12. The very thing is impossible.

Action 13. Thus, A is not unprime to C.

Action 14. Thus, A and C are prime to one another.

Therefore, If an odd number is prime to some number then it will also be prime to its double.

Which is the very thing it was required to show.

1

GIVEN – AN ODD NUMBER PRIME TO SOME NUMBER

Let A be an odd number and B a number such that A is prime to B.
(They have no common measure.)

2

DOUBLE THE NUMBER B

Let C be double of B.
So, $C = 2B$.

3

TO PROVE – A IS PRIME TO C

We must show that A is also prime to C.
That is, $\gcd(A, C) = 1$.

4

ASSUME, FOR CONTRADICTION, THAT A AND C ARE NOT PRIME

Suppose A and C are not prime to one another.
Then some number D measures both A and C.

5

D IS ODD BECAUSE IT MEASURES ODD A

Since D measures the odd number A, D must be odd.

6

D MEASURES EVEN C, SO IT MEASURES HALF OF C

Now D (odd) measures C (even).
By Proposition IX.30, D also measures one half of C.

7

THEREFORE D MEASURES B

But $C/2 = B$.
Therefore, D measures B.

8

CONTRADICTION

Thus D measures both A and B.
 $D \mid A$ and $D \mid B$.

9

CONCLUSION

Therefore our assumption was false.
A and C are prime to one another.
That is, $\gcd(A, C) = 1$.

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PROPOSITION 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even number only.

Proof

Action 1. For let any multitude of numbers whatsoever, B, C, D, have been (continually) doubled, (starting) from the dyad A.

Action 2. I say that B, C, D are even-times-even numbers only.

Action 3. In fact, (it is) clear that each [of B, C, D] is an even-times-even number.

Action 4. For it is doubled from a dyad *Def. 7.8*

Action 5. I also say that (they are even-times-even numbers) only.

Action 6. For let a unit be laid down.

Action 7. Therefore, since any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the number A after the unit is prime, the greatest of A, B, C, D, (namely) D, will not be measured by any other numbers except A, B, C *Prop. IX.13*

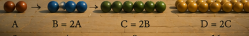
Action 8. And each of A, B, C is even.

Action 9. Thus, D is an even-time-even number only *Def. 7.8*

Action 10. So, similarly, we can show that each of B, C is [also] an eventime-even number only.

Therefore, Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even number only.

Which is the very thing it was required to show.

1 START WITH THE DYAD Let A be the dyad, that is, $A = 2$.  $A = 2$	2 CONTINUALLY DOUBLE Double each number to get the next.  A $B = 2A$ $C = 2B$ $D = 2C$ 2 4 8 16	3 THE PROGRESSION BEGINS The numbers formed are 2, 4, 8, 16, each is obtained from the previous by doubling.  2 4 8 16 ...
4 PLACE THE UNIT BEFORE THEM Place the unit before the dyad and write the continued proportion.  1 2 4 8 16 ... $1 : A = A : B = B : C = C : D = \dots$	5 THE NUMBER AFTER THE UNIT IS PRIME $A = 2$ is the number immediately after the unit. Therefore A is prime.  1 2 4 8 16 prime	6 APPLY PROPOSITION IX.13 Since A (2) is prime, by Proposition IX.13 the greatest term D can be measured by no numbers except the earlier terms A, B, C.  1 2 4 8 16 A B C
7 THE ONLY DIVISORS ARE THESE Therefore the only numbers (greater than the unit) that can measure D are 2, 4, and 8.  1 2 4 8 16 ... even ✓ even ✓ even	8 NO ODD DIVISOR GREATER THAN THE UNIT Since every possible divisor greater than the unit is even, D has no odd divisor. Hence D cannot be even-times-odd.  1 2 4 16 even-times-even only	9 CONCLUSION The same reasoning applies to B and C as well. Thus each of the numbers which is continually doubled, starting from a dyad, is even-times-even only.  1 2 4 8 16 32 ... even-times-even
IN MODERN LANGUAGE: $\Rightarrow$ These numbers are the powers of two: $2^2, 2^3, 2^4, 2^5, 2^6, \dots$ $\Rightarrow$ Every divisor greater than 1 is itself even. $\Rightarrow$ Hence each is even-times-even only.		

PROPOSITION 33

If a number has an odd half then it is an even-timeodd number only.

Proof

Action 1. Let A be a number whose half is odd.

Action 2. Because the half of A measures A exactly twice, A is even-times-odd.

Action 3. To prove the word “only,” assume for contradiction that A is also even-times-even.

Action 4. If A is even-times-even, some even number E measures A an even number of times.

Action 5. Since E measures A an even number of times, E also measures one half of A .

Action 6. But by the given condition, one half of A is odd.

Action 7. Thus an even number would measure an odd number, which is impossible.

Action 8. Therefore the assumption is false: A is not even-times-even.

Action 9. Hence A is even-times-odd only.

Therefore, If a number has an odd half then it is an even-timeodd number only.

Which is the very thing it was required to show.

1

GIVEN – A NUMBER WHOSE HALF IS ODD

Let A be a number such that one half of A is odd.

$\frac{A}{2}$ → This half is odd.

2

FIRST CONSEQUENCE

That odd half measures A exactly twice.
So A is even-times-odd.

$\text{odd} \times 2 = A$

$A = 2 \times (\text{odd}) \Rightarrow A$ is even-times-odd.

3

TO PROVE THE WORD "ONLY"

Assume, for contradiction, that A is also even-times-even.

$A = \text{even} \times \text{even}$

Suppose A is even-times-even.

4

THEN SOME EVEN DIVISOR MEASURES A

If A is even-times-even, then some even number (E) measures A an even number of times.

E even number E (repeated an even number of times)

5

THAT EVEN DIVISOR ALSO MEASURES $A/2$

Since E measures A an even number of times, E will also measure half of A .

$E \mid (A/2)$

6

BUT THE HALF OF A IS ODD

But by the given statement, one half of A is odd.

$A/2$ is odd.

7

REACH THE CONTRADICTION

Therefore an even number E would measure an odd number ($A/2$).

E (even) $A/2$ (odd)

Even number measuring an odd number – impossible!

8

REJECT THE FALSE ASSUMPTION

Therefore our assumption is false.
 A cannot be even-times-even.

A is not even-times-even.

9

FINAL CONCLUSION

We have shown A is even-times-odd, and not even-times-even.
Therefore A is even-times-odd only.

$A = \text{even} \times \text{odd}$

A is even-times-odd ONLY.
Q.E.D.

PROPOSITION 34

*If a number is neither one of the numbers doubled
, from a dyad, nor has an odd half, then
it is (both) an even-times-even and an even-times-odd number.*

Proof

Action 1. Let A be a number which is not one of the numbers continually doubled from the dyad, and whose half is not odd.

Action 2. Since half of A is not odd, it is even; hence A is even-times-even.

Action 3. Begin repeatedly halving A : A , $A/2$, $A/4$, $A/8$, and so on.

Action 4. Continuing the halving process, we must eventually arrive at an odd number O .

Action 5. Suppose no odd number appeared; then repeated halving would eventually reach the dyad 2, and reversing the process would show that A arose by continual doubling from 2.

Action 6. This contradicts the hypothesis, so an odd number O must appear before reaching 2.

Action 7. Because O is reached by repeated halving, O measures A .

Action 8. The quotient is a power of two and hence even; therefore the odd number O measures A an even number of times, so A is even-times-odd.

Action 9. Thus A is both even-times-even and even-times-odd.

Therefore, If a number is neither one of the numbers doubled
, from a dyad, nor has an odd half,
then it is (both) an even-times-even and an even-times-odd number.

Which is the very thing it was required to show.

1

THE GIVEN NUMBER A

Let A be a number which is not one of the numbers continually doubled from 2: 2, 4, 8, 16, 32, ...

Also, half of A is not odd.

We must prove that A is both even-times-even and even-times-odd.

2

A IS EVEN-TIMES-EVEN

Since half of A is not odd, it is even.

$A/2 = A/2$

Thus A is measured by an even number ($A/2$), an even number of times.

Therefore, A is even-times-even.

3

BEGIN REPEATEDLY HALVING A

$A \rightarrow A/2 \rightarrow A/4 \rightarrow A/8 \rightarrow \dots$

Bisect A , then bisect its half, and continue.

4

EVENTUALLY WE REACH AN ODD NUMBER

If we continue this process of halving, we must eventually arrive at an odd number, O .

$A \rightarrow A/2 \rightarrow A/4 \rightarrow \dots \rightarrow O \text{ (odd)}$

5

WHY AN ODD NUMBER MUST APPEAR

Suppose we never reach an odd number. Then we will eventually arrive at the dyad (2).

$A \rightarrow 3 \rightarrow 4 \rightarrow 2$

But then, reversing the process, we see that A was obtained by continually doubling 2.

That would make A one of the numbers continually doubled from 2. Contradiction!

6

THEREFORE WE MUST REACH AN ODD NUMBER

Since this is impossible, our halving process must encounter an odd number O before we reach 2.

$A \rightarrow A/2 \rightarrow A/4 \rightarrow \dots \rightarrow O \text{ (odd)}$

Thus, repeated halving of A reveals an odd number O .

7

O MEASURES A

The odd number O was reached by repeated halving.

$A = \underbrace{A}_{O} + \underbrace{O}_{O} + \underbrace{O}_{O} + \dots + O$

2^k times (an even number of times)

Therefore, O measures A .

8

A IS EVEN-TIMES-ODD

Since O is odd and it measures A an even number of times, A is even-times-odd.

$A = O \times 2^k$

O (odd) $\times$ 2^k (even)

A is written as an odd number times an even number.

9

CONCLUSION

From (2), A is even-times-even.

From (8), A is even-times-odd.

A is even-times-even and A is even-times-odd

Therefore, A is both even-times-even and even-times-odd.

Q.E.D.

PROPOSITION 35

If there is any multitude whatsoever of continually proportional numbers, and numbers equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those numbers before it.

Proof

Action 1. Set out continuously proportional numbers A, BC, D, EF, beginning from the least A.

Action 2. Subtract from BC a part BG equal to A, leaving GC; and subtract from EF a part FH equal to A, leaving EH.

Action 3. Construct FK equal to BC and FL equal to D.

Action 4. Since FK equals BC and FH equals BG, subtracting equals from equals gives HK equal to GC.

Action 5. Rewrite the continued proportion as $EF:FL = FL:FK = FK:FH$.

Action 6. By separation, obtain $EL:FL = LK:FK = KH:FH$.

Action 7. Combine corresponding ratios to obtain $KH:FH = (EL+LK+KH):(FL+FK+FH)$.

Action 8. Simplify the sums: $EL+LK+KH = EH$ and $FL+FK+FH = D+BC+A$.

Action 9. Replace KH by GC and FH by A.

Action 10. Therefore $GC:A = EH:(A+BC+D)$.

Therefore, If there is any multitude whatsoever of continually proportional numbers, and numbers equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those numbers before it.

Which is the very thing it was required to show.

1

THE GIVEN CONTINUOUS PROPORTION

Let there be a multitude of continuously proportional numbers, starting from the least.

So that $A : BC = BC : D = D : EF$.

2

SUBTRACT THE FIRST FROM THE SECOND AND THE LAST

From the second (BC), subtract a part (BG) equal to the first (A).
From the last (EF), subtract a part (FH) also equal to the first (A).

Remainders: GC and EH.

3

CONSTRUCT AUXILIARY EQUAL SEGMENTS

On EF, from F take FK equal to BC.
On EF, from F take FL equal to D.

$FK = BC$ and $FL = D$.

4

IDENTIFY AN EQUAL REMAINDER

Since $FK = BC$ and $FH = BG = A$, subtracting equal parts from equal whole gives equal remainders.

Therefore $HK = GC$.

5

USE THE ORIGINAL CONTINUED PROPORTION

Because $EF : D = D : BC = BC : A$, and since $FL = D$, $FK = BC$, $FH = A$, we have:

$EF : FL = FL : FK = FK : FH$.

6

SEPARATE THE RATIOS

Subtract the following term from the leading term in each ratio.

$EL : FL = LK : FK = KH : FH$

These are the separated ratios.

7

COMBINE CORRESPONDING RATIOS

As one leading term is to one following term, so all the leading terms together are to all the following terms together.

$KH : FH = (EL + LK + KH) : (FL + FK + FH)$.

8

SIMPLIFY THE SUMS

The leading remainders add together.

$EL + LK + KH = EH$.

The following terms add together.

$FL + FK + FH = D + BC + A$.

9

REPLACE WITH THE ORIGINAL REMAINDERS

From step 4, $KH = GC$, and we also have $FH = A$.

$GC : A = EH : (D + BC + A)$.

10

REORDER THE SUM

Rearrange the sum of the preceding terms.

$GC : A = EH : (A + BC + D)$.

11

INTERPRETATION

GC is the excess of the second over the first.

$\frac{GC}{A} = \frac{EH}{A + BC + D}$

EH is the excess of the last over the first.

$A + BC + D$ is the sum of all the numbers before the last.

12

CONCLUSION

Therefore, in continuous proportion, subtracting the first from the second and the last produces remainders that are related thus:

$GC : A = EH : (A + BC + D)$.

Which is the very thing it was required to show.

PROPOSITION 36

If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, , until the whole sum added together becomes prime, and the sum multiplied into the last number makes some number, then the (number so) created will be perfect.

Proof

- Action 1.** Starting from the unit, set out numbers in continued double proportion until their whole sum is prime.
- Action 2.** Let E be equal to that prime sum.
- Action 3.** Let E multiply the last term D to make FG; it is required to prove that FG is perfect.
- Action 4.** From E construct a second progression, with the same number of terms, continually doubling: E, HK, L, M.
- Action 5.** Compare the two progressions; via equality obtain $A:D = E:M$, hence $ED = AM = FG$.
- Action 6.** Since A is the dyad, $FG = 2M$, so FG is the next term after M in the second doubling progression.
- Action 7.** Apply Proposition IX.35 to E, HK, L, M, FG by subtracting a quantity equal to E from the second and the last terms.
- Action 8.** Use the resulting proportion to reconstruct FG as the sum of E, HK, L, M together with A, B, C, D and the unit.
- Action 9.** Observe that all of the displayed numbers divide FG.
- Action 10.** Assume for contradiction that some additional divisor P of FG exists outside the displayed list.
- Action 11.** Let Q be its cofactor, so $PQ = FG = ED$, and compare the two factorizations.

Action 12. Apply Proposition IX.13 to the first doubling progression to restrict divisors of D to its preceding terms.

Action 13. Use the ratio relations between the two progressions to force the supposed new divisor back into the displayed list, contradicting its novelty.

Action 14. Therefore FG has no other proper divisors beyond the displayed ones.

Action 15. The sum of all those proper divisors has already been shown equal to FG .

Action 16. By the definition of a perfect number, FG is perfect.

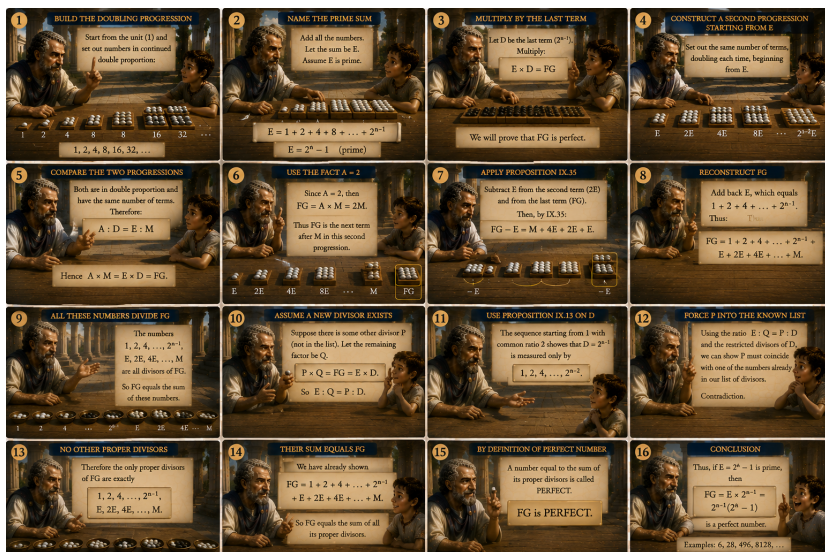
Therefore, If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, , until the whole sum added together becomes prime, and the sum multiplied into the last number makes some number, then the (number so) created will be perfect.

Modern explanatory note. In modern notation, the construction yields

$$2^{n-1}(2^n - 1),$$

which is perfect whenever $2^n - 1$ is prime. The classical examples are 6, 28, 496, 8128.

Which is the very thing it was required to show.



Conclusion of Book IX

Book IX has carried the arithmetic of the *Elements* from continued proportion and the behavior of squares and cubes through divisibility, prime numbers, third and fourth proportionals, the infinitude of primes, parity, the classification of even numbers, the summation of a geometric progression, and finally the construction of perfect numbers.