# Euclid's *Elements*

## From Number to the Regular Solids

Aris Skliros

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*Euclid's Elements Book I*
*Presentation and proof that it is complete*

Consolidated illustrated edition

# Contents

Part One

# The Question

# How Do You Know It Is There?

Put a finger on any mark in a Euclidean diagram and ask the question that gives this book its title. How do you know it is there?

There are only two kinds of answer. The first is: *because I can see it*. The two circles of Proposition I.1 plainly cross; the point of crossing is plainly on the page; anyone with eyes can check. The second is: *because a permitted action put it there, and I can name the permission*. These answers sound close. They are not close at all. The first makes geometry a report on drawings. The second makes it a demonstrative science, and it is the only answer Euclid ever gives.

The difference is easiest to feel at the place where it is usually missed. A commentator objects that Euclid never proved that the two circles of I.1 meet. The objection assumes that the meeting point is a *thing* whose existence must be separately certified. But in the reconstruction followed here the point is not a thing that was found. It is the outcome of two executions of a granted command, in a plane whose completeness the postulates have already determined, judged by a criterion of coincidence the Common Notions already supply. Asking for a further certificate is asking the diagram to vouch for the argument, which is precisely the inversion that makes the *Elements* look defective.

## 0.1 Book I is not the first book

One correction has to be made before the ledger can be stated honestly, and it changes what the opening page of Book I is entitled to assume.

Book I is not the beginning of the *Elements*. It is the beginning of the *geometry*, which is a different thing. Before it stands the layer set out in **Book 0**: the primitive terms Euclid uses and never defines — *meros*, *peras*, *klisis*, *pēlikotēs*, *mēkos*, *platos*, *bathos* — then the five Common Notions, which are principles of quantity in general and not of figures, and then the two abstract quantitative sciences that depend on no diagram at all: Books VII–IX, the science of *how many*, and Book V, the science of *how great*.

The reason is given in full in Book 0 and can be put in one line here. Definition I.1 defines the point by *meros*, and *meros* is defined nowhere in Book I. Its two formal definitions are VII.3 and V.1, and they agree: *a part is the lesser that measures the greater*. A reader who opens at Book I is therefore using, in the first sentence of the science, a term the science has not yet determined.

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What Book 0 hands over before Proposition I.1

`B0` The prior layer — the seven primitive terms; the five Common Notions as general quantitative principles; and from Books VII and V, the determination of **part**, **multiple**, **equality of wholes and remainders**, and the general theory of **magnitude**.

In particular: a *part* is the lesser magnitude that measures the greater; a *half* is the part that measures its whole exactly twice; a *double* is that whole. These are not geometrical notions and they are not assumed by Book I — they are handed to it.

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This matters to the ledger in a specific and checkable way. It is not a licence to draw anything: `[B0]` adds no action, and the hand is still restricted to the three postulated commands. What it adds is **quantitative warrant** — the right to reason about parts, multiples, halves and doubles of magnitudes that the three commands have produced.

And it settles the single oldest complaint against Book I. At **Proposition I.37** the reader is shown two triangles, each the half of one of two equal parallelograms, and asked to conclude that the triangles are equal. Nothing in Common Notions 1–5 mentions a half. The manuscript tradition duly supplied the want: extra Common Notions about doubles and halves, including *halves of equals are equal*, entered the text early — before Theon — and Heiberg marked them as interpolations.

They were never needed. Once part and multiple stand first, *half* and *double* are names of particular parts and multiples, and their equality properties follow from what those words already mean. The floorboard at I.37 does not give way. It was never loose; the reader had simply been brought into the room through the wrong door.

That argument is worked out where it belongs, at I.37 below, and the same step is discharged again at I.38 and, inverted, at I.41 — which is what Proposition I.47 then spends.

## 0.2 The postulates are permissions, not merely assertions

The five postulates are customarily read as assertions: statements that certain things are true of space. Read them again with an ear for their grammar. The Greek opens with Ἠιτήσθω — *let it have been postulated*, a perfect imperative. Euclid is not describing space. He is asking for something to be *granted*, and what he asks to be granted is a licence to act.

This matters because a licence has an edge. If you are granted three actions, you are granted three, and not a fourth. Everything that appears in the first proposition must be produced by those three, or it does not belong there. The postulates therefore do two jobs at once: they say what exists, and they say *what you may do*. The second job is the one this edition keeps in view on every page.

## 0.3 Three verbs, and nothing else

At the moment before Proposition I.1 begins, the entire permitted repertoire is this:

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The initial grant — everything you may do at the outset

P1 Draw a straight line from any point to any point.

P2 Produce a finite straight line continuously in a straight line.

P3 Describe a circle with any centre and any radius.

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Three verbs: **join**, **extend**, **sweep**. That is the whole of it. You may not yet bisect anything. You may not yet carry a length from one place to another. You may not yet drop a perpendicular, copy an angle, or draw a parallel. Every one of those powers has to be *earned*, and Book I is the record of earning them.

Postulates 4 and 5 are of a different kind, and the distinction is worth drawing sharply, because collapsing it is what makes the parallel postulate look like an intruder. Neither tells you to do anything. Postulate 4 declares all right angles equal; Postulate 5 states the condition under which two straight lines, produced, must meet. They are not commands but **constraints** — determinations of the plane in which the three commands are executed. You cannot *perform* Postulate 4. You can only rely on it.

## 0.4 Two ledgers

Because the postulates do two jobs, the growth of Book I has to be tracked in two columns, and this edition keeps both.

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THE TWO LEDGERS

**The ledger of actions.** What you may now *draw*. It begins with P1, P2, P3 and grows only at the fourteen propositions of Book I that are genuine constructions — problems, in the old vocabulary, closing with *which is the very thing it was required to do*.

**The ledger of warrants.** What you may now *conclude* without proving it again. It grows at every proposition, including the theorems, which close with *which is the very thing it was required to show*.

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The distinction is not decorative. It explains the shape of the book. Of the forty-eight propositions, only fourteen enlarge what your hand may do: I.1, I.2, I.3, I.9, I.10, I.11, I.12, I.22, I.23, I.31, I.42, I.44, I.45 and I.46. The other thirty-four enlarge only what your reason may assert. A reader who tracks just the first ledger sees Book I as a slow accumulation of instruments, from the equilateral triangle to the square on a given line — exactly the instruments Book II will require.

## 0.5 What the annotations mean

Every constructive step in what follows carries, in the right margin, the permission that authorises it: `[P1]`, `[P3]`, `[I.2]`, and so on. The rule is strict and admits no exceptions. *If a step cannot be tagged, it is not a step.* A line that appears with no permission beside it is a line that was seen rather than drawn, and it has no place in the demonstration.

Three recurring panels carry the argument:

- **Tools in hand** lists the repertoire immediately before the proposition. Read it as a constraint: this, and nothing else.
- **How do you know it is there?** takes each object the construction introduces — especially the intersections, which are where the alleged gaps always lie — and answers the title question for it.
- **Added to the repertoire** records the delta: the new action, or the new warrant, that the proposition has just paid for.

## 0.6 Why Definition 4 is the hinge

One definition does more work in this reading than any other, and it is the one most often dismissed as empty. Definition 4 says that a straight line is that which lies evenly with the points on itself. Taken as a description of a line's own points it says nothing, which is why commentators have called it vacuous and noted that Euclid appears never to use it.

Read as this reconstruction reads it — κεῖται expressing relation to something *else*, so that the points are those of a reference line laid against the candidate — Definition 4 becomes a **criterion of coincidence**. Two straight lines that agree at two points agree everywhere. And that is exactly the instrument needed at Proposition I.4, where the tradition reaches instead for superposition and has been apologising for it ever since. In the treatment below, I.4 moves no triangle. It constructs the second triangle onto the first with two applications of I.3, and then lets Definition 4 force the bases together. No motion, no new postulate, and no gap. The hinge of Book I is a definition everybody had written off.

## 0.7 What this edition claims, and what it does not

It does not claim that Euclid's Book I satisfies Hilbert. Pasch's axiom, a continuity principle and an explicit treatment of betweenness are genuine modern additions, and rearranging Euclid does not turn them into theorems.

It claims something narrower and internal. *Every object introduced in Book I can be traced to a permission held at the moment it was introduced.* The alleged gaps are not holes in the reasoning. They are the places where a reader, having stopped asking which permission is in force, has begun reading the diagram instead of the argument.

The rest of this book is that trace, carried out forty-eight times.

Part Two

# The Book Itself

# Introduction

This study reconstructs Book I of Euclid's *Elements* as a cumulative and internally articulated system in which definitions, postulates, common notions, and propositions acquire their full force through dependency on one another. Its principal thesis is that the familiar charges of hidden assumptions—concerning uniqueness, intersection, continuity, order and betweenness, superposition, and the equality of figures—need not be accepted as defects in Euclid's own framework. The reconstruction developed here distinguishes carefully between two questions: what a modern formal axiomatization requires when its primitive terms are deliberately left uninterpreted, and what Euclid's own geometrical language determines when definitions and postulates are read relationally and cumulatively. The result is a proposition-by-proposition account of Book I in which the standard objections are stated in their strongest form and answered from within the architecture of the *Elements*. Particular attention is given to Definition 4, Definition 7, Definition 17, Definition 23, Postulate 5, circle and line intersections, the method traditionally described as superposition in Proposition I.4, order reasoning, the theory of parallels, and the equality of rectilinear figures culminating in Propositions I.47–I.48.

## Preamble

The present treatise aims to restore the internal coherence of Euclid's *Elements*, Book I, by reconstructing the true meaning of its definitions, postulates, and propositions in their original logical and philosophical order. No modern axioms, symbolic logic, or external assumptions are introduced. The reasoning proceeds solely from Euclid's own text, read in its Greek

phrasing, and interpreted through the natural logic of the postulates themselves.

The central thesis is that the *Elements* constitute not a system of empirical geometry but a system of intelligible relations. The postulates are not arbitrary assumptions but determinations of being: each defines what is meant by line, plane, or circle. When these are properly understood, all later propositions follow by necessity, without recourse to external axioms or modern order relations. This work regards the *Elements* not as a static set of state-

ments but as a dynamic sequence of operations. Each definition, postulate, and proposition functions like a call in a program, invoked only after the preceding steps have defined the objects on which it acts. Book I is therefore a process of generation—a living algorithm of thought that begins from the definition of the point and culminates in the Pythagorean theorem. Each successive concept emerges only when the preceding logical conditions make its appearance possible.

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The analysis begins from the basic definitions of the point, the line, and the plane, showing that these notions are not fixed entities but gradually constructed through successive invocations. Definition 4 (“a straight line is that which lies evenly with the points on itself”) introduces the idea of rectitude through comparison with a reference of the same kind—a *criterion of truth*. Definition 7 then extends this relational structure to the plane, defining it as the surface that lies evenly with the straight lines upon it. These are not descriptions of visual form but statements of logical uniformity. The line and plane are determined by equivalence relations, not by sight.

Through subsequent definitions and postulates, these ideas mature into full geometrical being. Definition 17 introduces central symmetry through the diameter of the circle; Definition 23 defines parallelism as uniform direction

and non-intersection; and Postulates 1, 2, 4, and 5 complete the structure by establishing connection, continuation, equality of right angles, and the defining behaviour of lines in the plane. By the time Euclid reaches the theorems on parallelism and the right triangle, the nature of the Euclidean plane has been internally generated from its first principle—the point.

Thus, the text of the *Elements* grows like a living organism. The point is its birth, the line its first motion, the plane its field of being, and the completed propositions its maturity. What modern readers have mistaken for incompleteness or dependence on visual intuition is, in truth, the necessary economy of a system in which each concept is introduced only when its grounds have been logically secured.

## Retrospect

Euclid's *Elements*, though long regarded as the archetype of rigorous reasoning, have often been charged with incompleteness. Critics from Proclus to Hilbert have claimed that Euclid omitted the axioms of order, betweenness, continuity, or area, and that his reasoning rested upon the intuition of diagrams rather than upon logic. Yet these objections arise chiefly from reading Euclid through the categories of later mathematics.

When his text is read through the lens of its own definitions, however, a different picture emerges. The *Elements* are not a manual of constructions, but an ontology of extension. Each definition describes the mode of being of a geometrical entity. The postulates express the relations by which these entities coexist in the plane. From these foundations all of Book I unfolds as a necessary sequence of determinations. Euclid's Book I unfolds as a precise logical sequence. Each element—definition, common notion, postulate, proposition—appears exactly when the preceding steps make it possible. This order mirrors a recursive program of reasoning in which earlier constructions are continuously invoked as subroutines for later theorems.

The first definitions (1–3) introduce the primitives: the point, the line, and the extremities of the line. Definition 4 then sets the criterion of rectitude, appealing not to sight but to an abstract standard of evenness. This leaves

the visible form of the line open to be determined by later reasoning. The meaning of straightness evolves step by step:

- **Definition 17** (diameter as a straight line) enforces symmetry through the circle's centre.
- **Definition 23** (parallel lines) defines uniform direction and non-intersection, introducing translational invariance.
- **Postulates 1, 2, 4, and 5** ensure connection, continuation, equality of right angles, and the unique geometry of the plane.

Taken together, these constructs yield the true nature of the straight line: a magnitude symmetric with respect to all its points, infinite in extension, and lying evenly with every other of its kind. The straight line is therefore not defined all at once but assembled through a sequence of dependent definitions and postulates.

In parallel, the concept of the plane develops gradually:

- **Definition 7** presents the plane as a surface lying evenly with the straight lines upon it.
- **Definition 23** implies that only a surface of zero curvature admits parallel lines.
- **Postulate 5** uniquely determines this surface, distinguishing the Euclidean plane from the spherical or hyperbolic by its singular parallel condition.

Hence, Book I is not a manual of constructions but a logical genesis. Each proposition arises as a consequence of the preceding framework, invoking earlier propositions as logical functions. The intersection of circles in Proposition I.1, the equality of triangles in Proposition I.4, and the final reasoning of the Pythagorean theorem are not isolated demonstrations—they are recursive calls within a single, self-consistent system.

When viewed thus, Euclid's geometry is not static but developmental. Its text mirrors a process of organic and intellectual growth: from the point (the birth of extension) to the full rational harmony of the Pythagorean relation.

By the time we reach the converse of the Pythagorean theorem, the system has matured from elementary being to self-reflective necessity—a geometry conscious of its own logic. Through these reinterpretations, every major criticism of Euclid's reasoning—concerning superposition, betweenness, or area—is resolved within the system itself. No external axiom is needed: the logic of Euclid's own words suffices.

# Methodological Framework: Alleged Gaps and the Ontological Reconstruction

## The problem of "unstated assumptions"

A substantial modern literature reads Euclid's first book as mathematically profound but formally incomplete. On this reading, the demonstrations rely at various points on principles not separately stated among the five postulates and five common notions. The recurring candidates are: the uniqueness of the straight line through two points; the existence of intersections of circles and straight lines; an order or betweenness relation on points of a line and on points inside or outside figures; the invariance of figures under motion or superposition; and rules governing the composition and decomposition of rectilinear figures. These concerns appear in different forms in the work of Heath, Hilbert, Tarski, Mueller, Hartshorne, Beeson, De Risi, and other modern reconstructions.[**?**, **?**, **?**, **?**, **?**, **?**, **?**]

The present reconstruction does not deny that a *modern formal system*, whose primitive predicates are intentionally semantically neutral, normally states explicit axioms for such matters. It questions a different inference: that the need for those axioms in a later formalization proves that Euclid himself was reasoning with unacknowledged premises. The thesis defended here is that the semantic work attributed by modern formalizations to independent axioms is distributed, in Euclid, across the definitions, postulates, common notions, and already-demonstrated propositions.

## The Skliros thesis

The reconstruction attributed in the source notes to Aris Skliros presents Book I as a complete, self-contained, internally coherent ontological system. Its governing claims are fivefold. First, no modern axiom is to be imported merely because a modern axiomatization chooses to isolate a corresponding primitive relation. Second, definitions are relational rather than merely descriptive: the straight line and the plane are determined through criteria of evenness, symmetry, direction, extension, and planar behaviour. Third, the postulates are not arbitrary supplements but determinations of what the relevant geometrical objects are and what operations upon them are universally executable. Fourth, Book I is developmental: later definitions and postulates complete the content initially introduced by earlier ones. Fifth, each proposition operates as a subroutine whose inputs have been prepared by the preceding architecture. In this sense the *Elements* function as a "living algorithm of thought," developing from the point to the Pythagorean theorem.

## Two different standards of completeness

The dispute is therefore methodological before it is technical. A Hilbertian or Tarskian formalization asks whether a theorem follows in a specified formal language from an explicitly enumerated set of primitive axioms. The ontological reading asks whether Euclid's own terms already carry determinate relational content and whether that content is progressively fixed by the order of the text. These are not equivalent questions. The first deliberately abstracts from intended meaning; the second treats intended geometrical meaning as part of the deductive architecture. Consequently, the observation that Hilbert introduces order and congruence axioms does not, by itself, settle whether Euclid's text contains a hidden assumption: it establishes what Hilbert's formal language requires after Euclid's semantic structure has been recoded.

## Cumulative determination of the straight line and the plane

Definition 4—"a straight line is that which lies evenly with the points on itself"—is the central test case. A common criticism is that the definition is obscure, circular, or textually incapable of doing the work assigned to it. A stronger form of the criticism charges the relational reading with silently inserting an external comparison line not named in the Greek. The reply developed in the present work is that Definition 4 is not an isolated exhaustive description. It is a seed whose content is progressively determined: Definition 17 imposes central symmetry through the diameter; Definition 23 adds infinite non-intersection and uniform direction; Postulate 1 supplies universal connection, Postulate 2 continuous production, Postulate 4 invariance of right angularity, and Postulate 5 the planar condition under which the resulting straight lines coexist. On this interpretation, the charge of circularity mistakes a staged definition for a one-sentence definition.

The plane is treated in the same manner. Definition 7 introduces the plane as the surface lying evenly with the straight lines on itself. Definition 23 tests whether infinite straight lines can remain non-intersecting in both directions, and Postulate 5 specifies the convergence behaviour of straight lines cut by a transversal. The completed notion of the Euclidean plane is therefore cumulative rather than supplied all at once.

## Postulate 5 and the historical objection

One criticism recorded in the source material initially treated Postulate 5 as an empirical-looking assumption accepted because Euclid could not conceive geometries of curved surfaces. The same critical discussion later withdraws that historical formulation. Greek mathematical astronomy already supplied a sophisticated geometry of the sphere: Autolycus' *On the Moving Sphere* and Euclid's own *Phaenomena* provide the relevant context. Familiarity with spherical geometry does not by itself prove a particular authorial intention behind Postulate 5, but it does remove the argument that curved surfaces were conceptually unavailable. The ontological reconstruc-

tion can therefore plausibly treat Postulate 5 as a discriminating condition on the plane rather than as an unconscious projection of visual flatness.

## Proposition I.1: analysis, synthesis, and existence

The first proposition concentrates the dispute about continuity. The modern objection is precise: after circles with centres $A$ and $B$ and radii $AB$ and $BA$ are drawn, the text takes a common point $C$; a modern formalization typically supplies a circle–circle continuity principle to license this intersection.[**?**, **?**, **?**] One proposed Sklirosian reconstruction begins analytically: suppose the equilateral triangle $ABC$ is the object sought; then $AC = AB$ and $BC = BA$, so $C$ must be common to the two circles. The synthesis then draws precisely those circles.

A serious criticism of analysis and synthesis is that analysis alone yields only a necessary condition: if $C$ exists, the circles meet at $C$. Merely drawing the circles does not, in a semantically empty formal system, reverse that implication. The fuller reply given here therefore does not rest on analysis and synthesis alone. It combines the constructive force of Postulate 3 with the already determined nature of the plane and with the identity criterion of Common Notion 4. On the ontological reading, the circles are not arbitrary predicates in an uninterpreted domain; they are plane figures generated by a universally executable construction from equal radii in the plane already determined by the opening definitions and postulates. Thus the existence claim is read as internal to the meaning of the construction rather than as a further empirical observation of a diagram. The formalist can still choose to isolate this content as a continuity axiom; the present thesis is that doing so is a re-expression of Euclid's semantics, not evidence that Euclid silently borrowed a proposition foreign to his system.

## Proposition I.4 and the meaning of superposition

The second major controversy is the proof commonly described as superposition. Heath and Mueller treat the language of "application" as invoking rigid motion, while Hilbert replaces such reasoning by explicit congruence axioms.[**?**, **?**, **?**] The reconstruction advanced here rejects the infer-

ence from Euclid's language to a physical act of lifting a triangle. Propositions I.2 and I.3 provide transport of equal lengths by construction; equality of the included angles fixes the corresponding rays; Definition 4 together with uniqueness of connection prevents two distinct straight lines with the same determined extremities from enclosing a second path; and Common Notion 4 supplies equality by coincidence. "Superposition" is thus interpreted as logical coincidence forced by construction, not as an additional kinematic principle.

## Order, betweenness, and inequality

Modern formal systems generally take order or betweenness as primitive or axiomatized. The present reconstruction instead emphasizes Euclid's own resources: continuous production of a straight line, the distinction of whole and part in Common Notion 5, construction of equal segments by circles, subtraction in Common Notion 3, and propositionally established relations of greater and lesser. The claim is not that the modern predicate "Between" has secretly been written into one sentence; it is that Euclid never abstracts order from the determinate operations that generate points on produced lines, cut off segments, and compare wholes with parts. The proposition-level critical apparatus below identifies every place in the supplied assumption catalogue where a separate order principle has been alleged.

## Equality of figures without numerical area

The corresponding criticism of Propositions I.35–I.47 is that Euclid lacks an axiomatized numerical area function and therefore tacitly assumes additivity. The reply distinguishes numerical measure from geometrical equality. Euclid's argument proceeds through congruence, equal bases, common parallels, bisection by diagonals, complements, and the addition and subtraction of equal figures under Common Notions 2 and 3. The chain I.33–I.45 establishes a calculus of equivalence before I.47 is reached. On this reading, "area" is not a hidden number attached to a figure; it is shorthand for an equivalence relation exhibited through permitted transformations and decompositions. A formalist may choose to axiomatize polygonal additivity;

the ontological reconstruction treats the relevant additivity as already expressed by the common notions acting upon the figures whose equality has been proved.

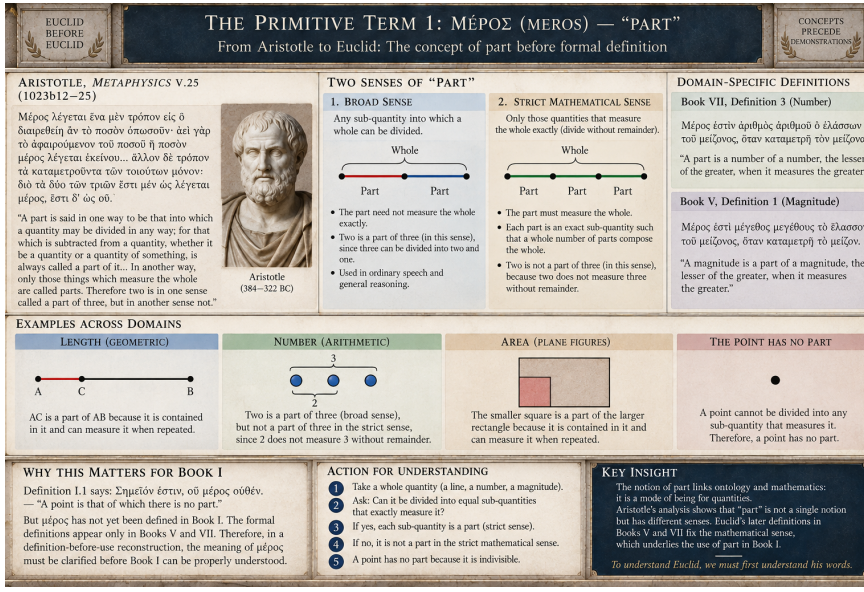
## How the critical apparatus is used below

The remainder of this book retains the entire pre-existing exposition unchanged. New material is added in two forms. First, each definition, common notion, postulate, and proposition receives an integrated ontological restatement based on the supplied wording document. Second, where the supplied catalogue identifies an alleged unstated assumption, a short critical note states the objection and the corresponding reply within the cumulative framework. These notes distinguish the historical or formal objection from the answer defended here; they are not substitutions for the original demonstrations.

# Primitive Terms

## Preamble

Euclid's *Elements* presuppose seven terms that are never formally defined within Book I, yet are used in its very first definitions. These terms belong to a pre-Euclidean conceptual vocabulary, recovered from Aristotle's *Metaphysics*, Proclus' commentary, and the grammatical analysis of Apollonius Dyscolus. They are the hidden foundations of the entire work.



# Primitive Term 1: μέρος (meros) — Part

## Background

Aristotle, *Metaphysics* V.25 (1023b12–25), distinguishes several senses of "part":

- **Broad sense:** Any sub-quantity into which a whole can be divided. Two is a part of three in this sense.
- **Strict mathematical sense:** Only those quantities that **measure** the whole exactly. In this sense, two is **not** a part of three.

Euclid uses the strict sense. The formal definitions appear in:

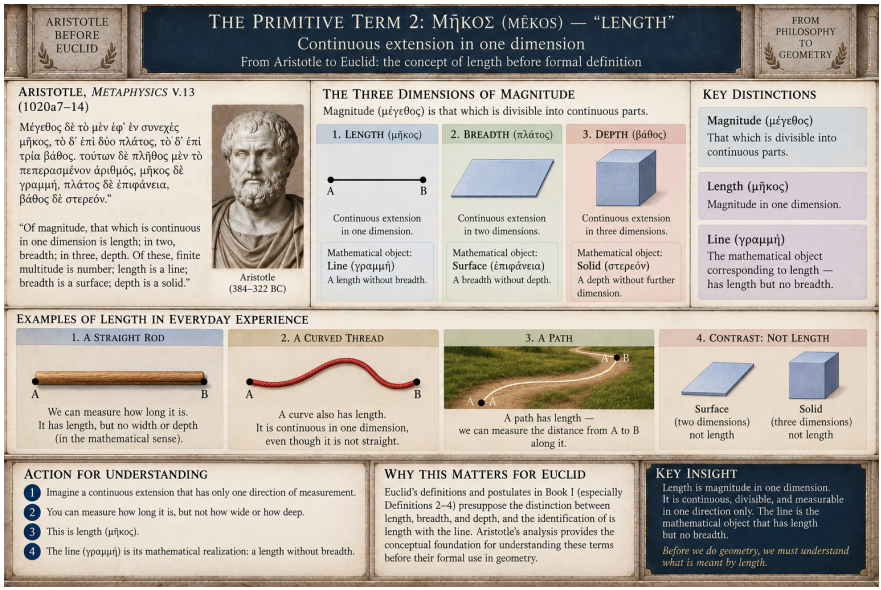
- **Book VII, Definition 3 (number):** *Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.* — "A part is a number of a number, the lesser of the greater, when it measures the greater."
- **Book V, Definition 1 (magnitude):** *Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.* — "A magnitude is a part of a magnitude, the lesser of the greater, when it measures the greater."

## Actions

1. Take a whole quantity (a line, a number, a magnitude).
2. Ask: Can it be divided into equal sub-quantities that exactly measure it?
3. If yes, each sub-quantity is a **part** in the strict sense.
4. If no, it is not a part in the strict mathematical sense.
5. A point has **no part** because it cannot be divided into any sub-quantity that measures it.

## Structural Consequence

**Book I uses μέρος; Books VII and V define μέρος. Therefore, in a definition-before-use reconstruction, VII and V must precede I.**



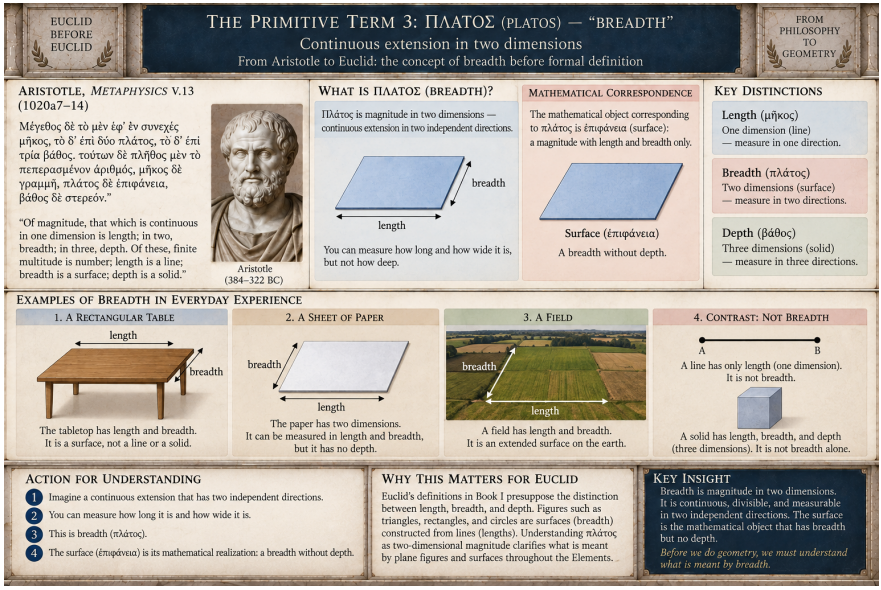

# Primitive Term 2: μῆκος (mēkos) — Length

## Background

Aristotle, *Metaphysics* V.13 (1020a7–14): *Μέγεθος δὲ τὸ μὲν ἐφ' ἓν συνεχὲς μῆκος, τὸ δ' ἐπὶ δύο πλάτος, τὸ δ' ἐπὶ τρία βάθος. τούτων δὲ πλῆθος μὲν τὸ πεπερασμένον ἀριθμός, μῆκος δὲ γραμμή, πλάτος δὲ ἐπιφάνεια, βάθος δὲ στερεόν.* "Of magnitude, that which is continuous in one dimension is length; in two, breadth; in three, depth. Of these, finite multitude is number; length is a line; breadth is a surface; depth is a solid."

## Actions

1. Imagine a continuous extension that has only one direction of measurement.
2. You can measure how long it is, but not how wide or how deep.
3. This is μῆκος. The line is its mathematical realization.
4. **Note:** μῆκος includes both straight and curved lines. Straightness is added later, in Definition I.4.



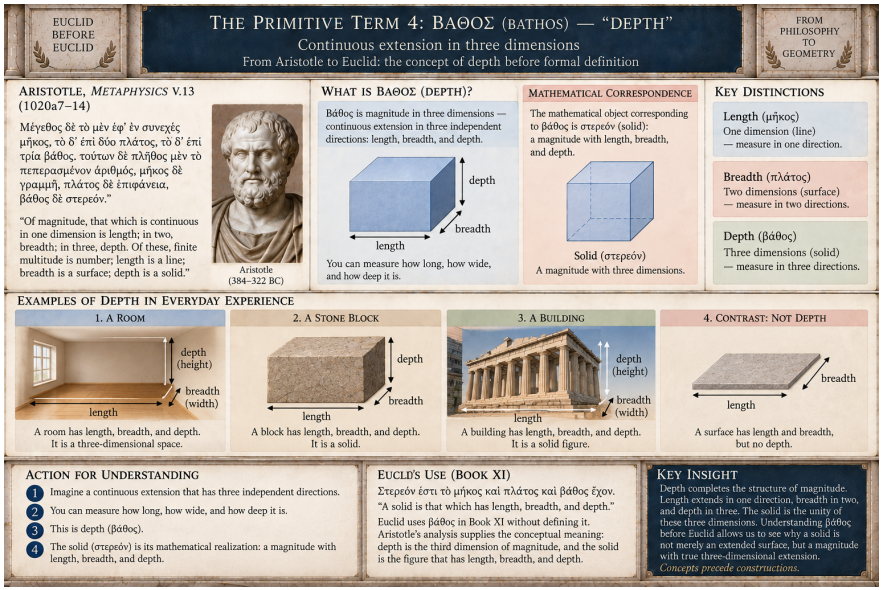

# Primitive Term 3: *πλάτος* (platos) — Breadth

## Background

Aristotle, *Metaphysics* V.13 (continued): *τὸ δ' ἐπὶ δύο πλάτος... πλάτος δὲ ἐπιφάνεια.* "That which is continuous in two dimensions is breadth... breadth is a surface."

## Actions

1. Imagine a continuous extension that has two independent directions of measurement.
2. You can measure how long and how wide it is, but not how deep.
3. This is πλάτος. The surface is its mathematical realization.



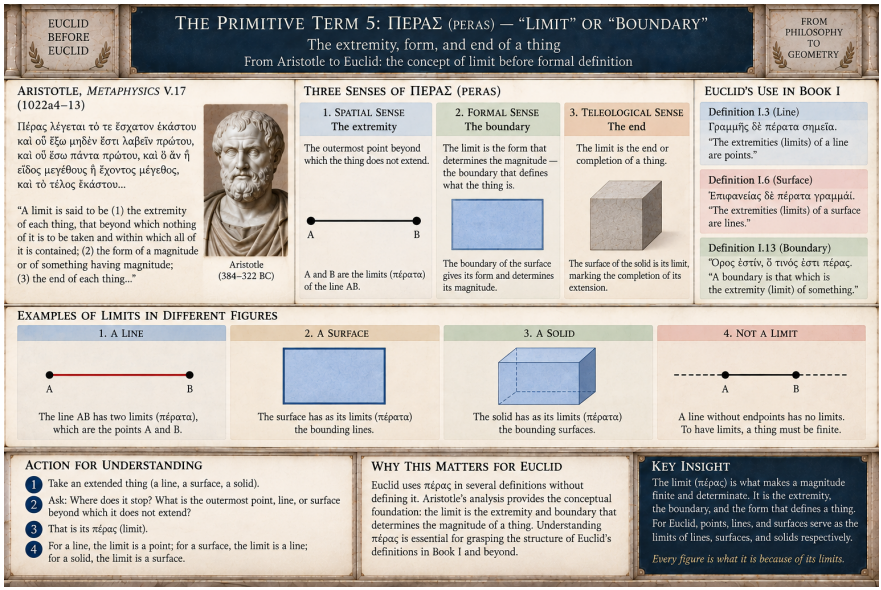

# Primitive Term 4: βάθος (bathos) — Depth

## Background

Aristotle, *Metaphysics* V.13 (continued): *τὸ δ' ἐπὶ τρία βάθος... βάθος δὲ στερεόν.* "That which is continuous in three dimensions is depth... depth is a solid."

## Actions

1. Imagine a continuous extension that has three independent directions of measurement.
2. You can measure how long, how wide, and how deep it is.
3. This is βάθος. The solid is its mathematical realization.
4. Euclid uses βάθος in Book XI: *Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.*




# Primitive Term 5: πέρας (peras) — Limit or Boundary

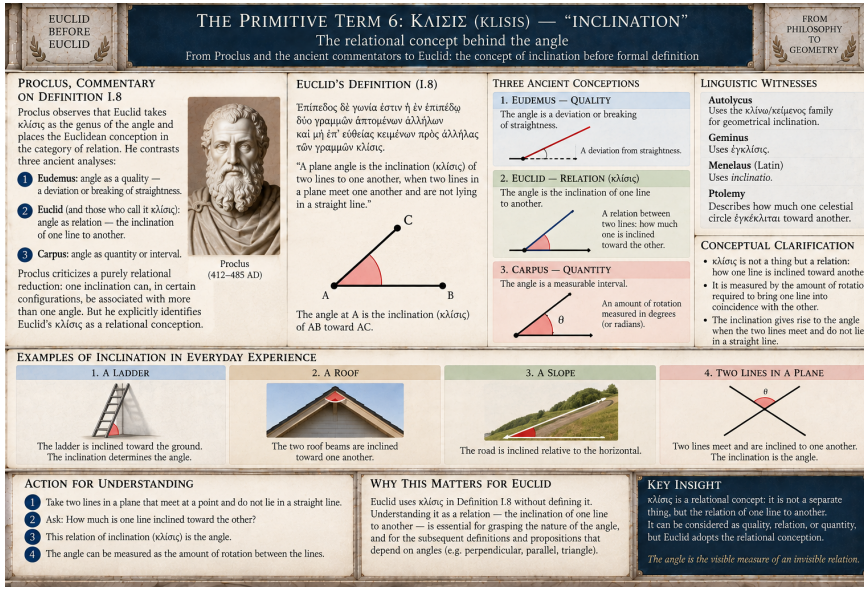
## Background

Aristotle, *Metaphysics* V.17 (1022a4–13): *Πέρας λέγεται τό τε ἔσχατον ἑκάστου καὶ οὗ ἔξω μηδὲν ἔστι λαβεῖν πρώτου, καὶ οὗ ἔσω πάντα πρώτου...*

"A limit is said to be (1) the extremity of each thing, that beyond which nothing of it is to be taken and within which all of it is contained; (2) the form of a magnitude or of something having magnitude; (3) the end of each thing..."

## Actions

1. Take any extended thing (a line, a surface, a solid).
2. Ask: Where does it stop? What is the outermost point beyond which it does not extend?
3. That is its πέρας. For a line, the πέρας is a point. For a surface, the πέρας is a line.



# Primitive Term 6: κλίσις (klisis) — Inclination

## Background

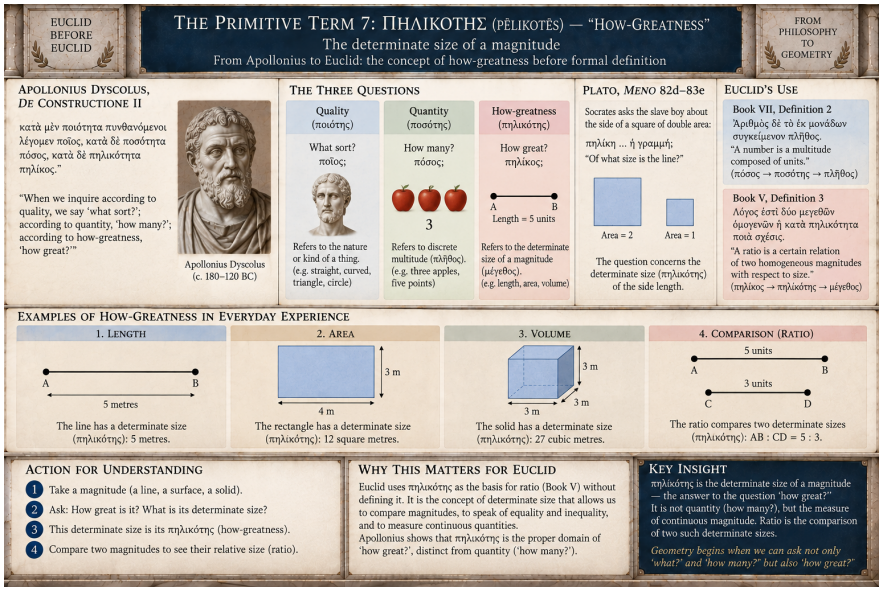
Proclus' commentary on Definition I.8 explicitly observes that Euclid takes **inclination** (κλίσις) as the **genus** of the angle and places the Euclidean conception in the **category of relation**. He contrasts three ancient analyses:

1. **Eudemus:** Angle as a **quality** — a deviation or breaking of straightness.
2. **Euclid:** Angle as **relation** — the inclination of one line to another.
3. **Carpus:** Angle as **quantity** or interval.

Linguistic witnesses: Autolycus uses the κλίνω/κείμενος family; Geminus uses ἐγκλίσις; Menelaus (Latin) uses *inclinatio*; Ptolemy describes how much one celestial circle ἐγκέκλιται toward another.

## Actions

1. Take two lines that meet at a point but do not form a single straight line.
2. Ask: How much is one line inclined toward the other?
3. That relation of inclination is the angle.
4. **Critical:** The definition says "two lines" (δύο γραμμῶν) — **not** "two straight lines." Therefore κλίσις can exist between non-straight lines or even non-similar lines.
5. Cases: both straight (rectilinear angle), both curved similar, both curved non-similar, one straight and one curved (mixed angle).




# Primitive Term 7: πηλικότης (pēlikotēs) — How-Greatness

## Background

Apollonius Dyscolus, *De Constructione* II: *κατὰ μὲν ποιότητα πυνθανόμενοι λέγομεν ποῖος, κατὰ δὲ ποσότητα πόσος, κατὰ δὲ πηλικότητα πηλίκος.* "When we inquire according to quality, we say 'what sort?'; according to quantity, 'how many?'; according to how-greatness, 'how great?"

Apollonius associates:

- **πόσος** (how many?) → **ποσότης** (quantity) → **πλῆθος** (multitude)
- **πηλίκος** (how great?) → **πηλικότης** (how-greatness) → **μέγεθος** (magnitude)

Plato, *Meno* 82d–83e: Socrates asks the slave boy about the side of a square of double area: *πηλίκη ... ἡ γραμμή;* — "Of what size is the line?"

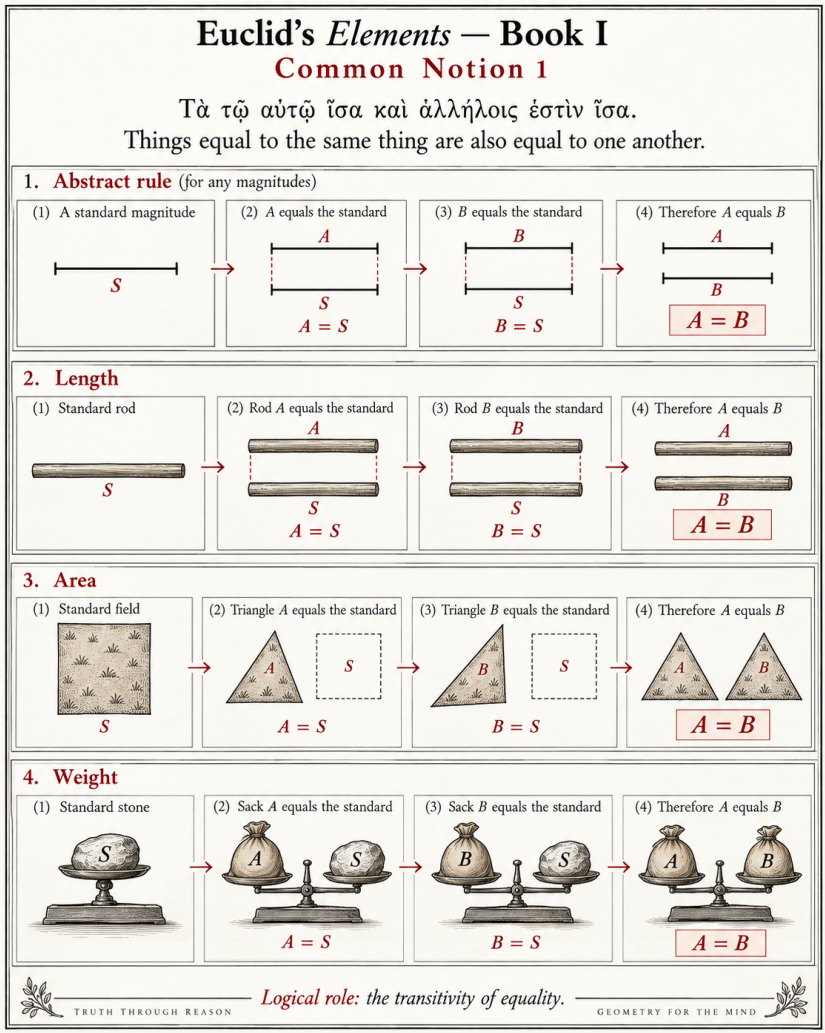
### Actions

1. Take a magnitude (a line, a surface, a solid).
2. Ask: How great is it? What is its determinate size?
3. That is its πηλικότης. Ratio is the comparison of two such determinate sizes.
4. **Book VII** defines number as a πλῆθος composed of units (domain of πόσος).
5. **Book V** defines ratio of magnitudes κατὰ πηλικότητα (domain of πηλίκος).

## The Reconstructed Foundational Order

Primitive Terms ⇝ Common Notions ⇝ Book VII ⇝ Book V ⇝ Book I

# Common Notions




# Common Notion 1

*Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.*

***1. Things equal to the same thing are also equal to one another.***

### Actions

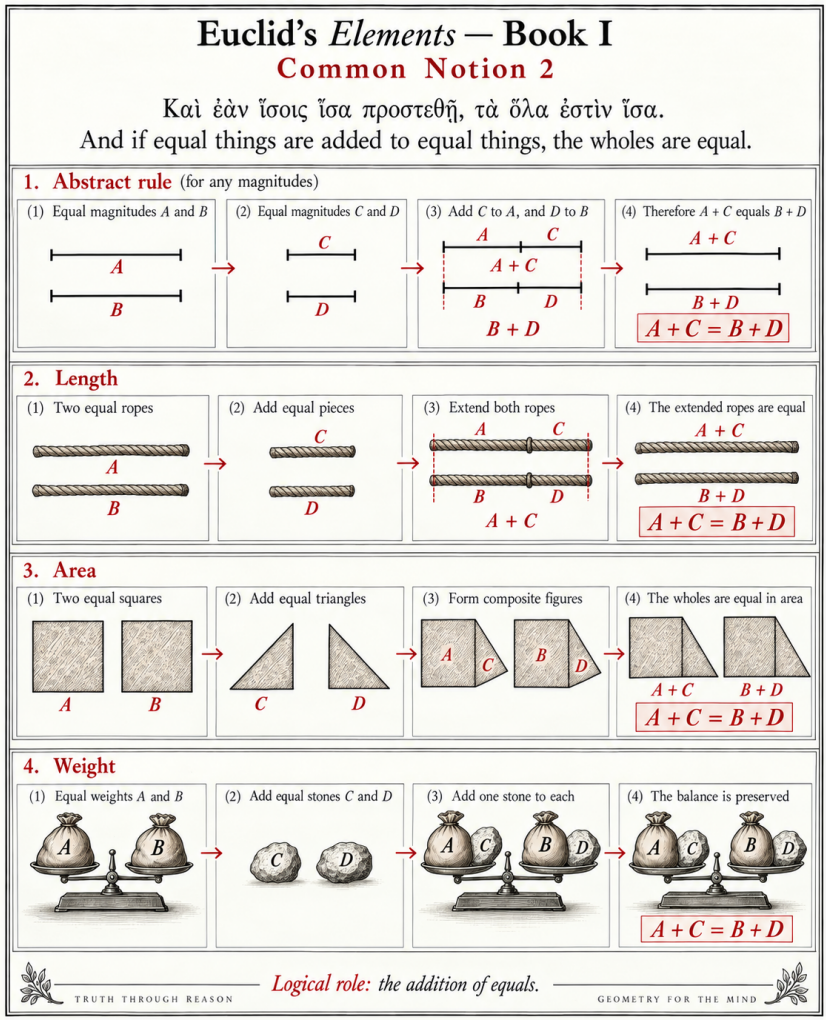
1. Take a standard magnitude S.
2. Compare A to S and find them equal.
3. Compare B to S and find them equal.
4. Conclude A = B without comparing A to B directly.

### Prompts for Examples

- **Length:** Three rods. Rod A matches rod S. Rod B matches rod S. Therefore A and B match each other.
- **Area:** Two triangular fields each equal in area to a third square field. Therefore the two triangles are equal in area.
- **Weight:** Two sacks each balancing the same stone on a scale. Therefore the sacks balance each other.

### Logical Role

This is the **transitivity of equality**. It permits the chain of reasoning in nearly every proposition.




# Common Notion 2

*Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.*

***2. And if equal things are added to equal things, the wholes are equal.***

### Actions

1. Take two equal magnitudes A and B.
2. Take two other equal magnitudes C and D.
3. Add C to A, and D to B.
4. Conclude A + C = B + D.

### Prompts for Examples

- **Length:** Two equal ropes. Add to each an equal additional piece. The extended ropes remain equal.
- **Area:** Two equal squares. Add to each an equal triangle. The composite figures remain equal in area.
- **Weight:** Two equal weights. Add an equal stone to each pan. The balance is preserved.

### Logical Role

This licenses **addition of equals**, used in Propositions I.13, I.5, and many others.




# Common Notion 3

> *Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.*
> ***3. And if equal things are subtracted from equal things, the remainders are equal.***

## Actions

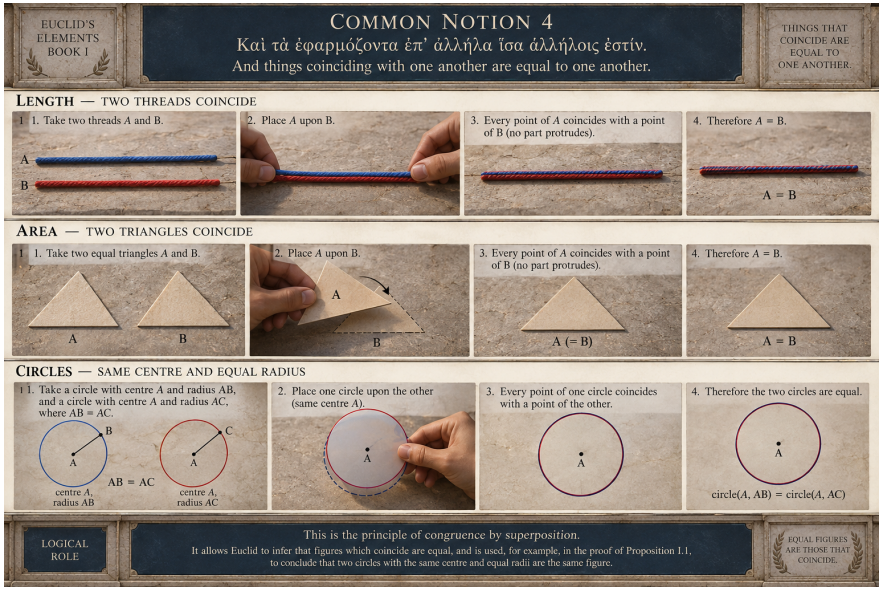
1. Take two equal wholes A and B.
2. Remove from A a part equal to a part removed from B.
3. Conclude the remainders are equal.

## Prompts for Examples

- **Length:** Two equal rods. Cut off equal pieces from each. The remaining lengths are equal.
- **Area:** Two equal rectangles. Remove equal corner triangles from each. The remaining areas are equal.
- **Weight:** Two equal bags of grain. Remove equal scoops from each. The remaining weights are equal.

## Logical Role

This licenses **subtraction of equals**, used in Propositions I.5, I.6, I.14, and I.47.



# Common Notion 4

> *Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν.*
> ***4. And things coinciding with one another are equal to one another.***

## Actions

1. Take figure A.
2. Place it upon figure B so that every point of A falls on a point of B.
3. Conclude A = B.

## Prompts for Examples

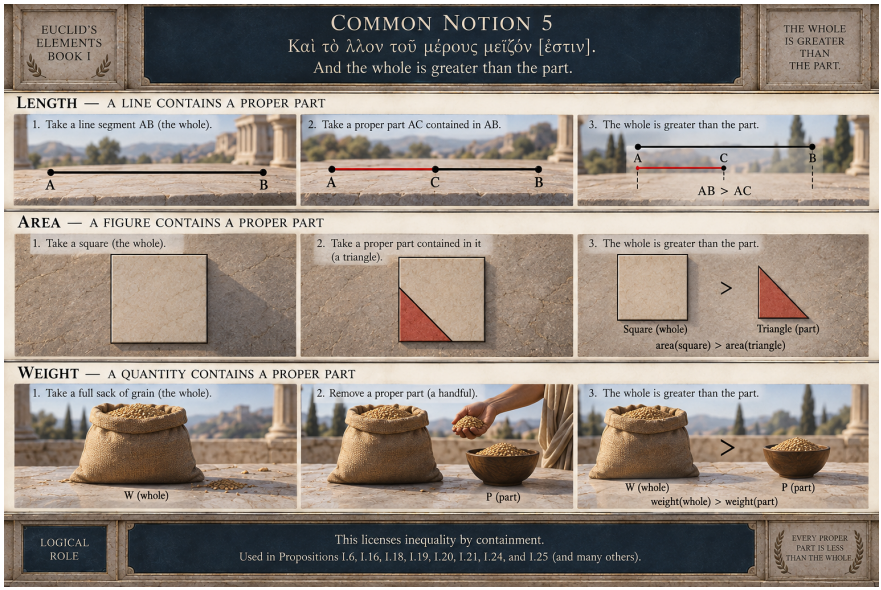
- **Length:** Two threads laid exactly upon one another; no part protrudes. They coincide, hence are equal.
- **Area:** Two cut-out paper triangles placed one on the other with no overlap or gap. They coincide, hence are equal.
- **Weight:** Two identical weights on a balance — the balance coincides with equilibrium.

## Interpretive Note

In the reconstruction of Proposition I.1, **Common Notion 4** guarantees that a circle with centre A and radius AB coincides with a circle with centre A and radius AC when AB = AC. It is the bridge between *equal measure* and *identical figure*.

## Logical Role

This is the principle of **congruence by superposition**, though the commentary argues Euclid uses it logically rather than physically.




# Common Notion 5

> *Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].*
> ***5. And the whole is greater than the part.***

## Actions

1. Take a whole W.
2. Take a proper part P of W (P is contained in W but is not all of W).
3. Conclude W > P.

## Prompts for Examples

- **Length:** A segment AB contains a sub-segment AC. AB > AC.
- **Area:** A square contains a corner triangle. The square's area > the triangle's area.

- **Weight:** A full sack contains a handful removed from it. The original weight > the removed weight.

### Logical Role

This licenses **inequality by containment**, used in Propositions I.6, I.16, I.18, I.19, I.20, I.21, I.24, and I.25.

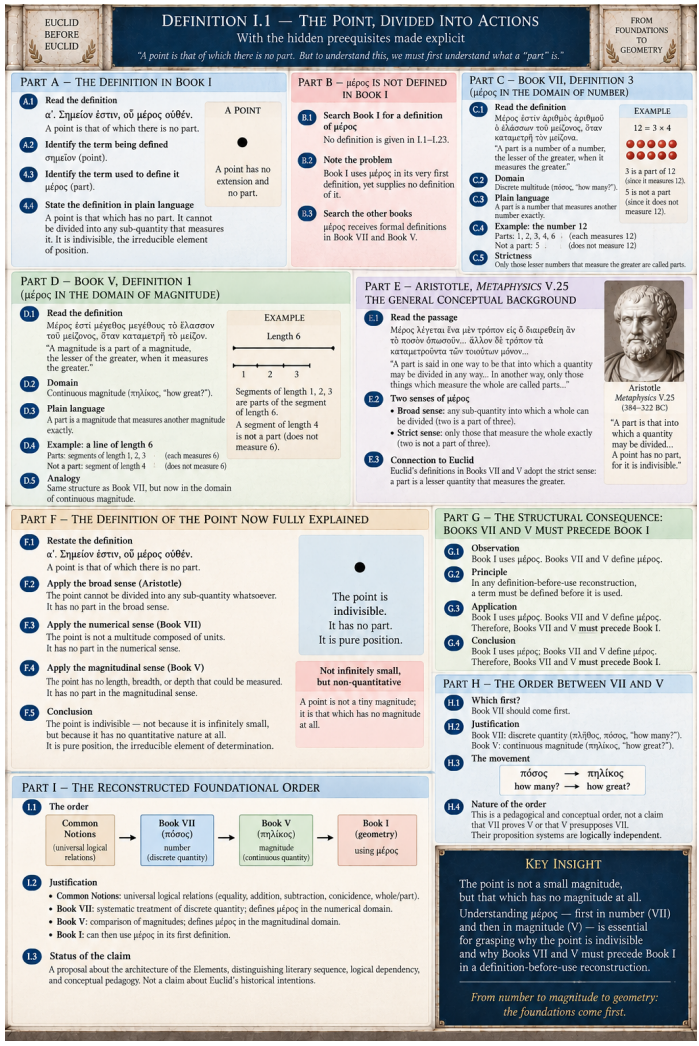
# Definitions

## Definition 1

*Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.*
***1. A point is that of which there is no part.***

### Actions

1. Mark a position.
2. Ask: Can it be divided into equal sub-quantities that measure it?
3. No. It has no μέρος.
4. It is indivisible, the irreducible element of position.




## Interpretation from the Commentary

The point, as Euclid defines it, "that which has no part," stands to geometry as the atom of Democritus stood to natural philosophy. The point is the indivisible element of magnitude. It represents that minimal entity which, though without dimension, gives position and thereby structure to all extension.

## Domain-Specific Definitions of μέρος

Book I uses μέρος without defining it. The formal definitions appear in:

- **Book VII, Def. 3:** A part is a number of a number, the lesser of the greater, when it measures the greater.
- **Book V, Def. 1:** A magnitude is a part of a magnitude, the lesser of the greater, when it measures the greater.

**Therefore, Books VII and V must precede Book I.**

## Expanded Commentary from *Explaining Fully the First Book of Elements*

α΄. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

1. A point is that of which there is no part.




The point, as Euclid defines it, "that which has no part," stands to geometry as the atom of Democritus stood to natural philosophy. Just as the atom is the indivisible element of matter, so the point is the indivisible element of magnitude. Euclid, who lived not long after Democritus, was likely acquainted with this conception, for the notion of an indivisible foundation was common in Greek thought of his age.

In ancient mathematics, there was no notion of zero as a number, nor of a void between magnitudes; instead there existed the idea of the infinitesimal, the limit of divisibility which could be conceived but not divided. The point thus represents that minimal entity which, though without dimension, gives position and thereby structure to all extension.

This conception may be illuminated by an argument concerning the continuity of number. If all numbers were continuous without separation, then between any two distinct numbers there would be no interval — their difference would be nothing, that is, zero — and so they would be identical. But this is absurd, for by assumption they are distinct. Hence, numbers cannot

be continuous in this way; there must exist among them a least interval, an irreducible difference which preserves their distinction.

In like manner, magnitudes cannot be divided without limit. For if every division could be carried further, the very act of measuring or distinguishing position would vanish into nothingness. Therefore, geometry requires an ultimate element — the point — which cannot be cut or conceived as having parts, just as the atom was the ultimate element of body.

And if one objects that such an element cannot exist because no smallest quantity can be found, then one abolishes distinction itself; for without the power to distinguish one place, one number, or one thing from another, nothing can be said to exist at all.

### Integrated Ontological Restatement

**Source formulation.** *A point is that of which there is no part.*

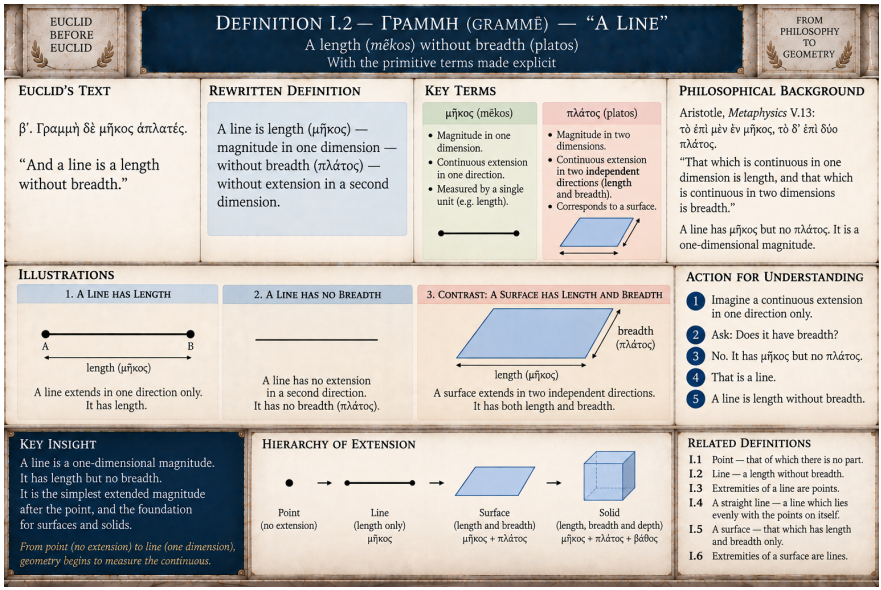
**Ontological role.** Posit an indivisible marker. It has no length, no breadth, no depth—only location. The point is the first act of distinction: ”here” as opposed to ”there.” Without it, nothing can be said to begin, end, or be placed. It is the atom of geometry.

## Definition 2

*Γραμμὴ δὲ μῆκος ἀπλατές.*
***2. And a line is a length without breadth.***

### Actions

1. Imagine a continuous extension in one direction only.
2. It has μῆκος but no πλάτος.
3. A point moving traces a line; the line is the path of the moving point.
4. **Note:** The definition says nothing about straightness. It applies to any line, of any shape.




## Interpretation from the Commentary

A line is that which has length but no breadth. Its length may be imagined as a path. Since the line has no breadth, nothing that possesses size can enter this path or walk upon it; only a point, having no parts, can travel along it. What we draw is never the true line itself, but only an imperfect image of it.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

β΄. Γραμμὴ δὲ μῆκος ἀπλατές. 2. And a line is a length without breadth.

A line is that which has length but no breadth. Its length may be imagined as a path — perhaps winding like a road through a mountain or stretched like an avenue across a plain. In the figure that follows we see a length which has breadth, i.e. something can fit to travel along it.

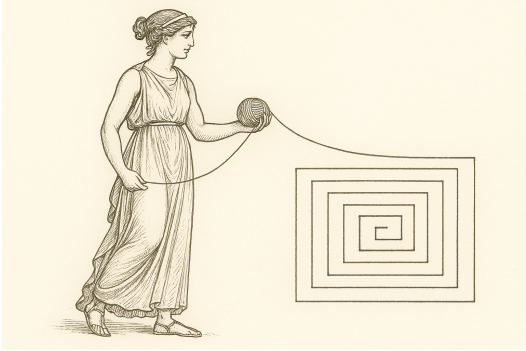


length with breadth, the length represented by the dark brown path

Yet since the line has no breadth, nothing that possesses size can enter this path or walk upon it; only a point, having no parts, can travel along it.

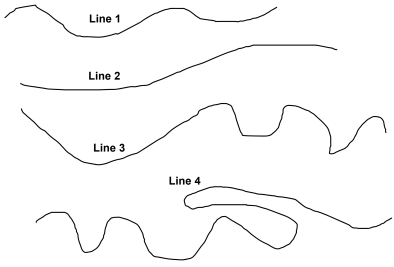
If we imagine that a point could move along this path, the length of the line would be the distance that could be traversed by that motion. It is not that the point measures or counts — for it has no parts — but that the path itself admits a measurable extent, the span that the point would cover in its journey.

We may imitate such a line by laying down a thread, as Ariadne once guided Theseus through the labyrinth.



Ariadne with her thread

By uncoiling the thread, we could trace the same shape and even measure its total length. Yet this thread is not itself a line, for it possesses breadth, however minute. It merely represents, in visible form, the shape and extent of the line that the mind conceives — the ideal path without thickness, along which only a point may move. Thus, the line may indeed be seen and drawn, for without it no geometry could be practiced. Yet what we draw is never the true line itself, but only an imperfect image of it. Every mark of ink or trace of chalk has breadth and matter, whereas the genuine line has none. It exists perfectly only in thought, as the trace of motion without parts, the pure path of the point, whose visible imitation we call a drawing.




various lines

We see here in the figure above example of lines, that is example of lengths without breadths

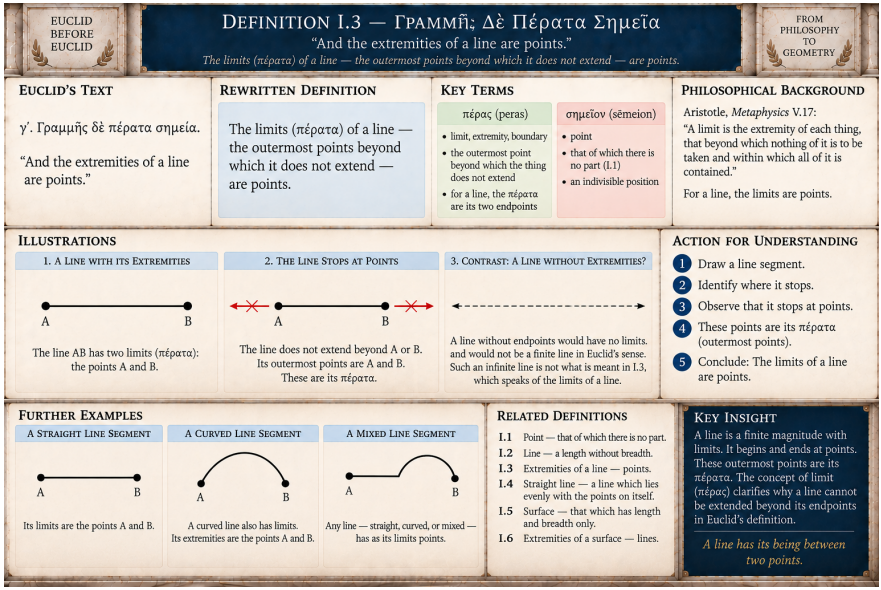
### Integrated Ontological Restatement

**Source formulation.** *A line is a length without breadth.*

**Ontological role.** Generate a one-dimensional extent by moving a point continuously. The line is the path of existence—the measure of distance between two positions. Because it has no breadth, nothing with size can fit upon it; only a point can travel along it.

## Definition 3

*Γραμμῆς δὲ πέρατα σημεῖα.*
***3. And the extremities of a line are points.***




### Actions

1. Draw a line.

2. Where does it stop?

3. At points. These are its πέρατα.

## Interpretation from the Commentary

This follows of necessity from what has already been said. Since a line has length but no breadth, nothing that possesses any magnitude can belong to it. Only the point may touch or mark its limits. The point is the origin and the bound of the line.

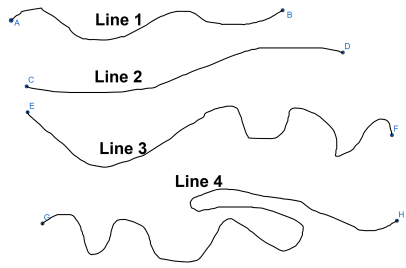
## Expanded Commentary from *Explaining Fully the First Book of Elements*

γ΄. Γραμμῆς δὲ πέρατα σημεῖα. — 3. And the extremities of a line are points.

This follows of necessity from what has already been said. For since a line has length but no breadth, nothing that possesses any magnitude can belong to it or fit upon it. Only the point, which has no parts, may touch or mark its limits.

Thus, wherever the line begins or ends, it does so not in a thing that has size, but in that which has position without extension. The point, therefore, is the origin and the bound of the line — that from which the line proceeds and in which it ceases.

And though the point has no length and no breadth, it exists — not as a body in space, but as a principle of determination: the sign by which the intellect distinguishes "this place" from "that place." Without it, no line could be said to begin, end, or even to exist, for it is by points that the mind conceives the measure and the limits of all geometry.




various lines with their extremities marked

We see here in the figure above example of lines with their extremities, that is example of lengths without breadths with their extremities marking and defining them.

### Integrated Ontological Restatement

**Source formulation.** *The extremities of a line are points.*

**Ontological role.** Whenever a line terminates, place a point at its start and at its end. The point is the boundary of linear extension—the limit that defines finitude or separates one line from another. Without this, a line would be indeterminate, endless, and unbound.

## Definition 4

> *Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.*
> ***4. A straight line is that which lies evenly with the points on itself.***

---

WHY THIS DEFINITION IS THE HINGE OF THE WHOLE BOOK

Read relationally — κεῖται expressing position with respect to something *else*, so that the points are those of a reference line laid against the candi-

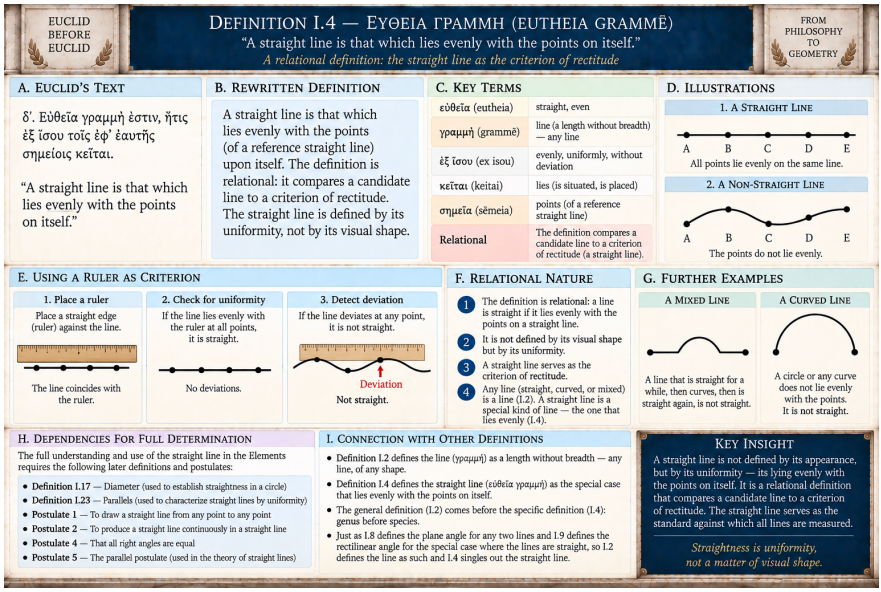
date — Definition 4 is not a description but a **criterion of coincidence**: two straight lines agreeing at two points agree everywhere.

That is precisely the instrument required at **Proposition I.4**, where the tradition reaches instead for superposition and has been apologising for it since antiquity. In this edition I.4 moves no triangle: it builds the second onto the first with two executions of I.3 and lets Definition 4 force the bases together. The definition commentators call vacuous, and that Euclid is said never to use, turns out to be load-bearing at the first congruence theorem in the corpus.

---

## Actions

1. Take a candidate line L.
2. Take a reference straight line R.
3. Place R against L so that corresponding points of R and L coincide.
4. If L lies evenly with R at every point, L is straight.



## Interpretation from the Commentary

The Greek κεῖται ("lies") always expresses relation to something else. The points are **not parts of the line** but external points that form a **reference line**. The definition should be read as: "A straight line is that which lies evenly with the points (of another straight line) upon itself."

## The Evolving Determination

The full determination of the straight line requires:

- **Def. 17** (diameter): symmetry about the centre.
- **Def. 23** (parallels): symmetry about every point.
- **Post. 1**: uniqueness of connection.
- **Post. 2**: infinite extension.
- **Post. 4**: equality of all right angles.
- **Post. 5**: definition of the plane.

## Corollary

If two points of one straight line lie on another, the two lines coincide entirely.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

| | |
|---|---|
| δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται. | 4. A straight line is that which lies evenly with the points on itself. |

At first glance, Euclid's wording seems obscure. Every line has infinitely many points, and those points determine its segments. What, then, does it mean for a line to "lie evenly with the points on itself"?

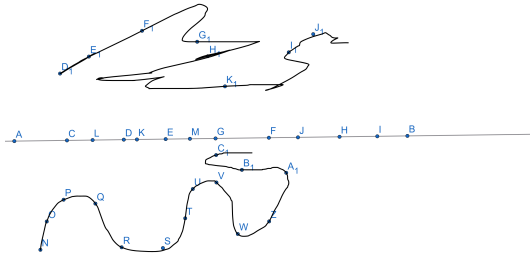
Let us examine his phrasing more closely and attempt to unpack its meaning.

Could Euclid mean that a straight line lies "evenly" with its own points along its length? But each point is at a different distance along the line, so that reading yields no constraint on shape.



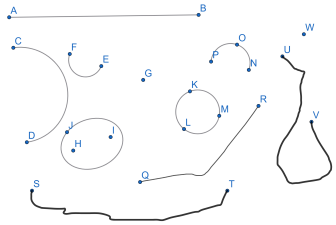

A straight line with some of its points labelled.

That interpretation fails: every point lies on the line, so no meaningful "evenness" is enforced.




Other arbitrary curves might satisfy the same wording.

If Euclid meant only the extremities of the line by "points," then the definition would also apply to circles, ellipses, and arbitrary arcs, granting no restriction on shape.




If “points” means just the endpoints, many curves would satisfy the definition.

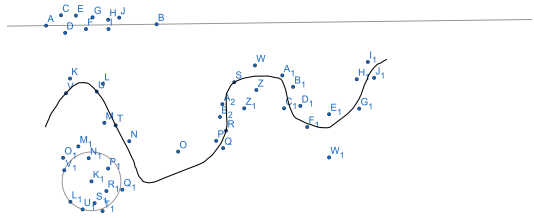
Some modern commentators argue that the definition is essentially vacuous or circular—indeed, that Euclid never uses it in proofs. For example, a geometry blog describes it as “utter and complete nonsense” when taken literally, claiming the definition has been corrupted or mutilated from a fuller version by Heron or others.[1] Others note that Euclid never explicitly invokes Definition 4 in subsequent propositions, which suggests that it plays no operative axiomatic role.[2]

To clarify, let us turn back to the original Greek. The verb κεῖται means “lies” or “is placed,” but always in relation to something else—not as part of it. Combined with adverbs like μακράν (“far away”) or ἐγγύς (“near”), it expresses spatial relation (e.g. “lies far away”). Thus, in Definition 4, the points Euclid refers to *cannot* themselves be part of the line.

Moreover, Euclid uses the preposition ἐπί (as in ἐφ᾽ ἑαυτῆς), meaning “on,” “at,” or “toward.” This indicates that the points are “on” or “toward” the line, perhaps externally positioned.

[1] See “Euclid’s ‘stupid definition of a straight line is not Euclid’s’”, *Intellectual Mathematics* blog (accessed 2025) and related commentary on Heron’s longer rendition of the “uniform” property of straightness.

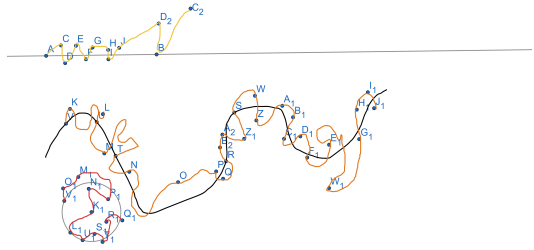
[2] See MacTutor History of Mathematics, “Euclid’s definitions,” noting that Euclid never refers back to his initial definitions in proofs.



Various lines with points “on,” “at,” or “toward” them, as indicated by ἐπί.

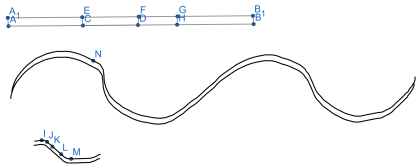
Because Euclid has not yet defined the plane, those points might lie anywhere in space—not necessarily coplanar. We avoid invoking coordinates or dimensions, as Euclid does not use them.

Now, imagine connecting those external points to form a “reference line” that lies in correspondence with the candidate line:



Lines that lie “on,” “at,” or “toward” our line; their own points form another line.

The points of that reference line lie evenly with the candidate line precisely when corresponding points coincide in shape:




Lines matched in shape: all corresponding points lie evenly with the reference line.

Hence we may restate Euclid's definition more precisely:

| | |
|---|---|
| δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς (εὐθείας γραμμῆς) σημείοις κεῖται. | 4. A straight line is that which lies evenly with the points (of another straight line) upon itself. |

In this interpretation, a "straight line of reference" exists; any other line is straight only if each of its points lies evenly with the corresponding points of that reference line.

This mirrors the intuitive procedure of placing a ruler against a drawn line to test for deviation.




Testing a drawn line against a ruler to detect deviations.

One may adopt an alternative “ruler” (another criterion of straightness) in a different geometry. If two such rulers produce matching alignments, then both sets of lines qualify as straight relative to their respective standards.



An alternative “straightness” criterion defines another class of straight lines.

The Greek term εὐθεῖα literally means "upright," "proper," or "true." Thus Euclid is defining "straightness" in relational, not metric, terms. The *shape* of a straight line is not fixed in Definition 4; only the relation of "lying evenly" is set. Its full geometric determination depends on subsequent definitions and postulates.

Finally, this definition has a corollary: if at least two points of one straight line lie on another, then by Definition 4 all corresponding points coincide, so the two lines cannot enclose an area. This corollary underlies Euclid's argument in Proposition I.4 (and elsewhere).

## Integrated Ontological Restatement

**Source formulation.** *A straight line is that which lies evenly with the points on itself.*

**Ontological role.** Take a candidate line L and align it against a reference line R of the same kind. Compare every point of L with the corresponding point of R. If they coincide uniformly along the entire length, L passes the test of rectitude; if they deviate, it fails. This definition is intentionally incomplete: it does not specify the shape of the line, only that it possesses a relational property called evenness. The full determination of what this evenness means will be supplied progressively by Definition 17 (central symmetry via the diameter), Definition 23 (uniform direction via parallelism), and Postulates 1, 2, 4, and 5 (uniqueness, infinity, invariance, and the planar filter). Definition 4 is therefore not a circular definition but a placeholder—a seed that matures throughout the opening of the book.

## Critical Note on Definition 4

**Textual objection.** A modern critic may object that the relational reading appears to insert an external standard—in effect, "another straight line"—that is not stated in the surviving Greek wording, and may therefore accuse the reconstruction of repairing Euclid by paraphrase.

**Reply.** The reconstruction does not require Definition 4 to carry the entire theory of rectitude in isolation. Its claim is that "lying evenly" introduces a relational criterion whose determinate content is supplied by the later architecture. Definition 17 introduces central symmetry, Definition 23 uniform

direction under indefinite production, and Postulates 1, 2, 4, and 5 universalize connection, extension, angular invariance, and planar behaviour. The alleged circularity is therefore answered by the cumulative character of the definition. The critical source itself ultimately revises the earlier charge of circular definition on precisely this ground.

## Definition 5

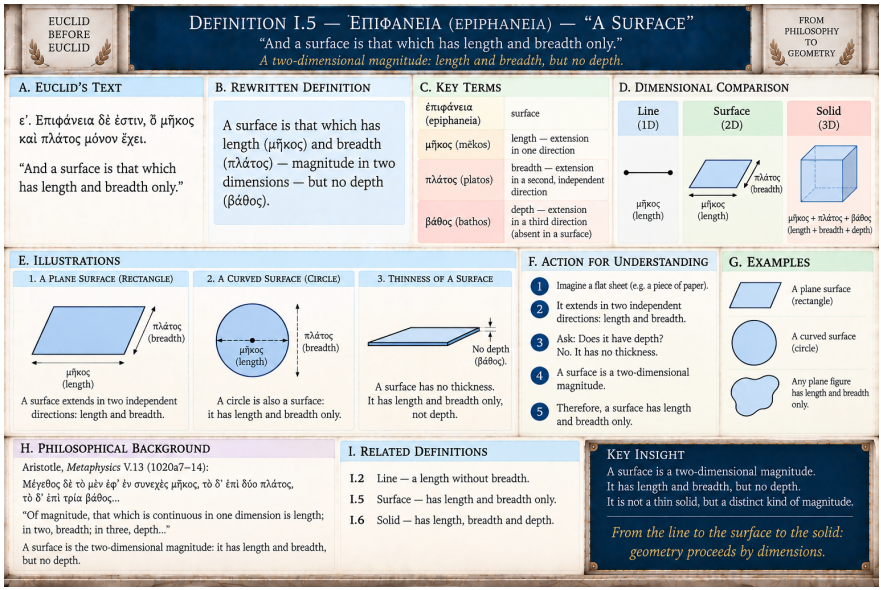
> *Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.*
> ***5. And a surface is that which has length and breadth only.***

### Actions

1. Imagine a continuous extension in two independent directions.
2. It has μῆκος and πλάτος but no βάθος.

### Interpretation from the Commentary

The surface is two-dimensional being — the field in which lines can lie.

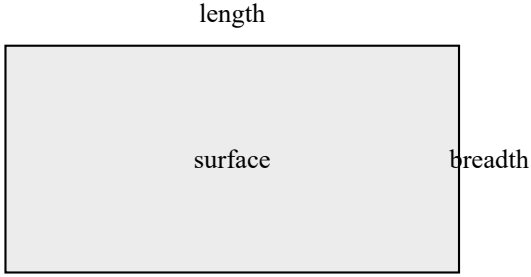



## Expanded Commentary from *Explaining Fully the First Book of Elements*

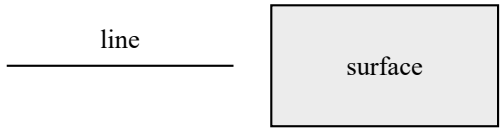
ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

5. And a surface is that which has length and breadth only.

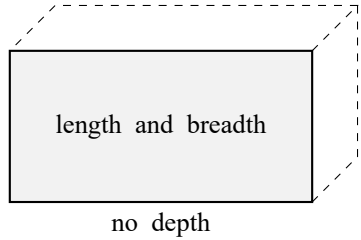
We see below an example of a surface



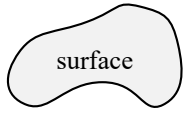

and another example of a surface




and another




and another




The surfaces that I used above have straight lines as boundaries but we do not know how a straight line looks like yet so I presenting other arbitrary examples of surfaces in the next definition.

## Integrated Ontological Restatement

**Source formulation.** *A surface is that which has length and breadth only.*

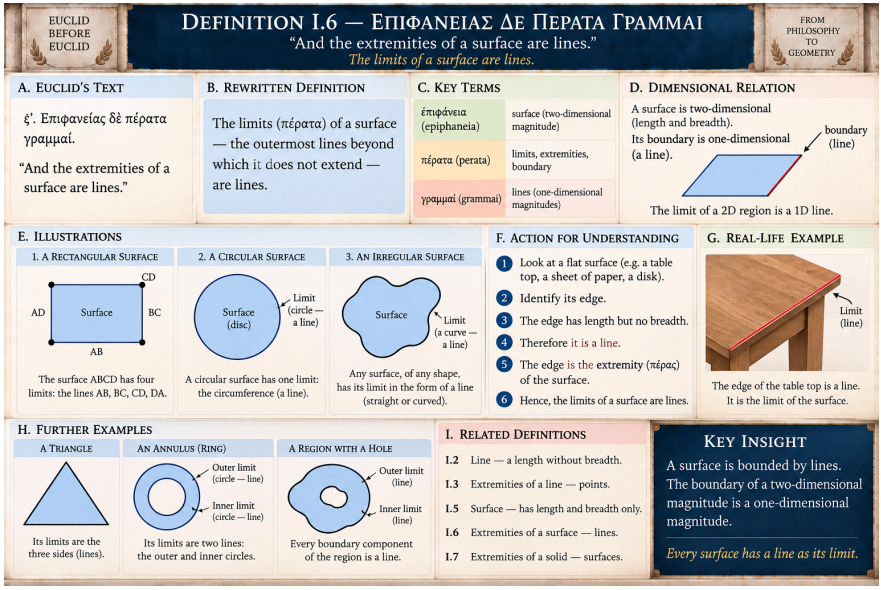
**Ontological role.** Generate a two-dimensional extent by moving a line across a breadth. The surface is the field upon which figures may be drawn and relations established. It has no depth—only extension in two directions.

# Definition 6

*Ἐπιφανείας δὲ πέρατα γραμμαί.*
***6. And the extremities of a surface are lines.***

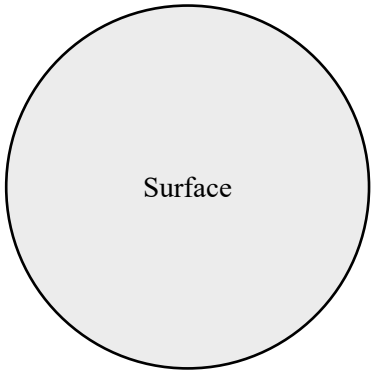
## Actions

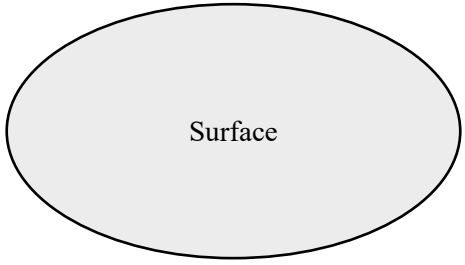
1. Look at the edge of a surface.
2. The edge is a line (ideally).
3. It is the πέρας of the surface.

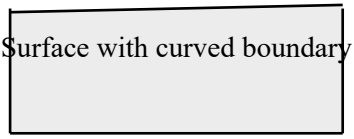


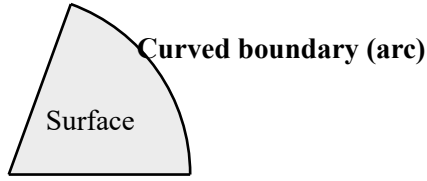

## Expanded Commentary from *Explaining Fully the First Book of Elements*

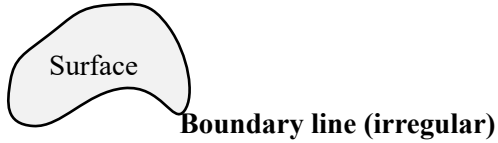
ς΄. ᾽Επιφανείας δὲ πέρατα γραμμαί. 6. And the extremities of a surface are lines.

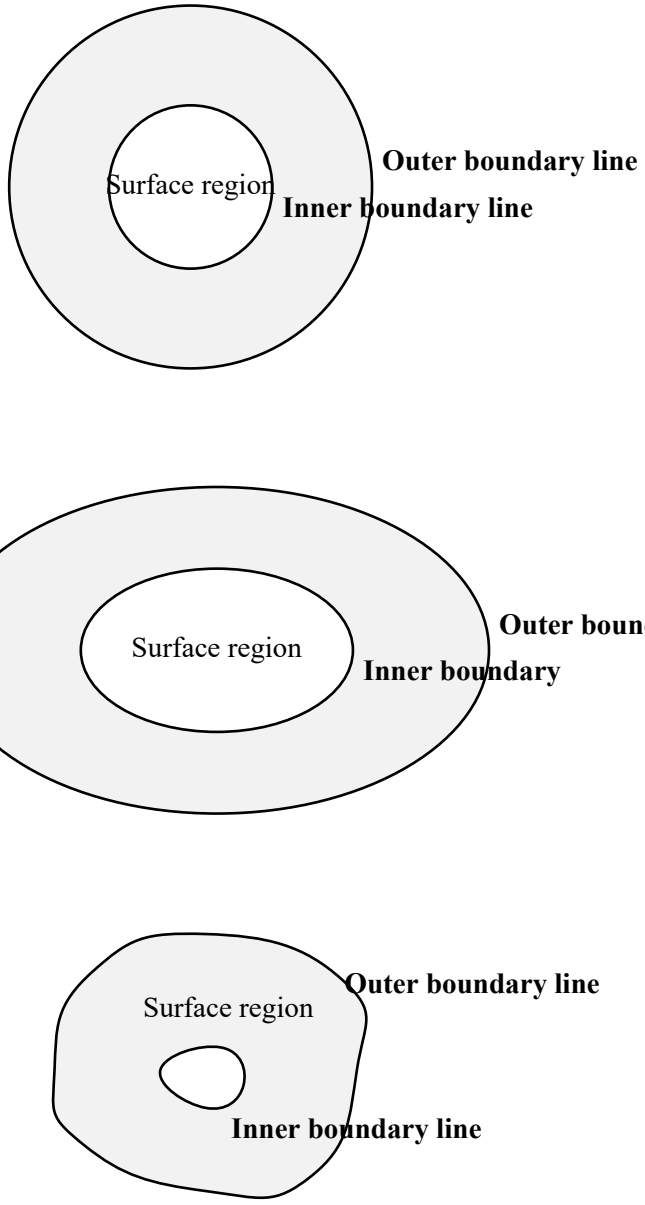

Surface
Boundary line (circle)


Surface
Boundary line (ellipse)


Surface with curved boundary


Curved boundary (arc)
Surface


Surface
Boundary line (irregular)




## Integrated Ontological Restatement

**Source formulation.** *The extremities of a surface are lines.*

**Ontological role.** Wherever a surface ends, place a line at its edge. The line is the boundary of surface extension. This mirrors Definition 3, establishing a hierarchy of boundaries: points bound lines; lines bound surfaces.

## Definition 7

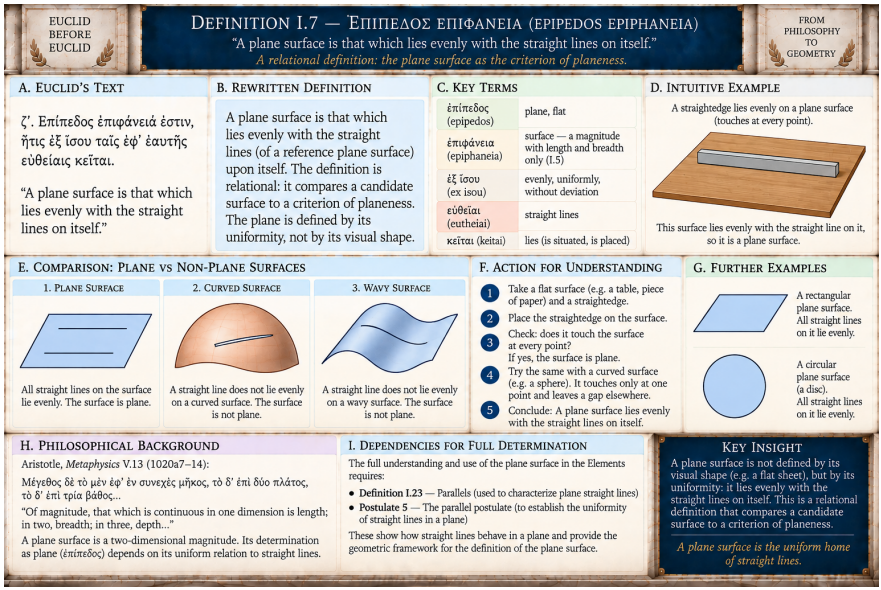
> *Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς εὐθείαις κεῖται.*
> ***7. A plane surface is that which lies evenly with the straight lines on itself.***

### Actions

1. Take a candidate surface S.
2. Take a reference plane surface P.
3. Place P against S so that corresponding straight lines of P and S coincide.
4. If S lies evenly with P at every line, S is plane.

### Interpretation from the Commentary

Like Definition 4, this is relational. The full determination of the plane requires Definition 23 (parallels) and Postulate 5.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

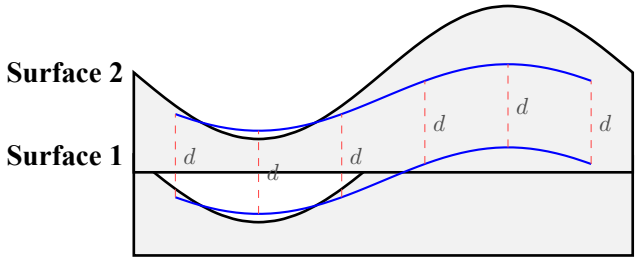
ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.

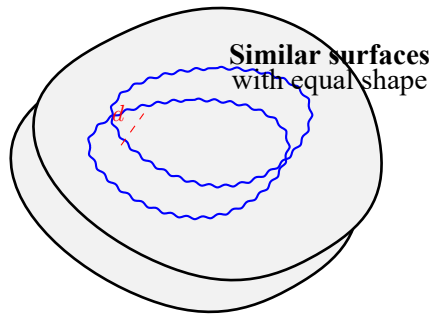
7. A plane surface is one which lies evenly with the straight-lines on itself.

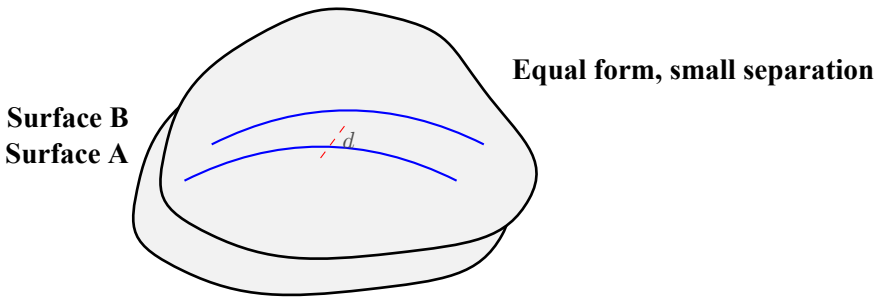
Now, this definition is similar to that of a straight line. We have a plane surface of reference and we lie it onto the surface that we want to examine whether is plane or not. If the straight lines of the plane surface of reference lay evenly to the straight lines of the surface that we want to examine then this is a plane surface as well. Again, we do not know what is the shape of the plane surface, just that we have a plane surface given already, which is the criterion of rectitude and truth and we compare the other surfaces against it to see if they are plane as well or not. The logic is the same as in the explanation of the straight line's definition. Hence, we can re-write the definition of the plane surface as:

ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς(ἐπιπέδου ἐπιφάνείας) εὐθείαις κεῖται.

7. A plane surface is one which lies evenly with the straight-lines(of a plane surface) on itself.




Since we do not know what a plane looks like or what the straight line looks like we can arbitrary define the straight line and the plane and we can compare each other surface against it. In the above I presented three examples

of non conventional planes and striaght lines which fit the definition if we set the plane surface of reference as in the surface in each of the figures.

### Integrated Ontological Restatement

**Source formulation.** *A plane surface is one which lies evenly with the straight-lines on itself.*

**Ontological role.** Take a candidate surface S and align it against a reference plane P. Compare the straight lines on S with the straight lines on P. If they correspond uniformly everywhere, S passes the test of planarity; if they bend or deviate, it fails. Like Definition 4, this definition is incomplete: the full determination of what ”lying evenly” means will be supplied by Definition 23 (parallelism) and Postulate 5 (the defining property that distinguishes the plane from all curved surfaces, such as the sphere or hyperboloid).

### Critical Note on Planarity

The plane definition is to be read in the same cumulative manner as Definition 4. “Lying evenly” is an initial criterion rather than a complete metric description. Definition 23 and Postulate 5 later determine the global behaviour of straight lines upon the surface, thereby distinguishing the plane relevant to Book I from surfaces whose geodesic behaviour is different.

## Definition 8

> *Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ’ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.*
> ***8. A plane angle is the inclination of the lines to one another, when two lines in a plane meet one another and are not lying in a straight line.***

### Actions

1. Take two lines that meet at a point but do not form a single straight line.

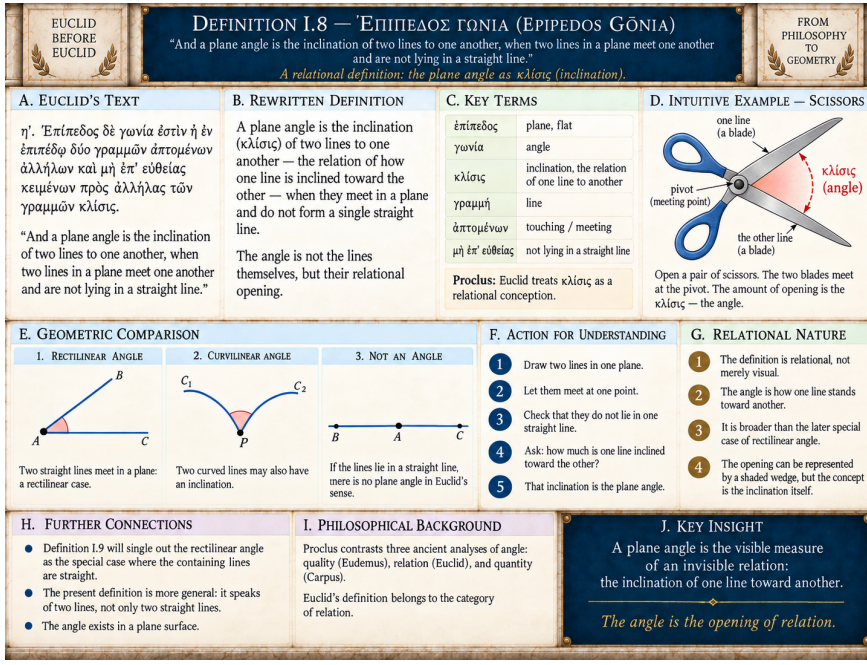
2. Ask: How much is one line inclined toward the other?

3. That relation of inclination (κλίσις) is the angle.

4. **Critical:** The definition says "two lines" — not "two straight lines." Therefore κλίσις can exist between non-straight lines or even non-similar lines.

## Interpretation from the Commentary

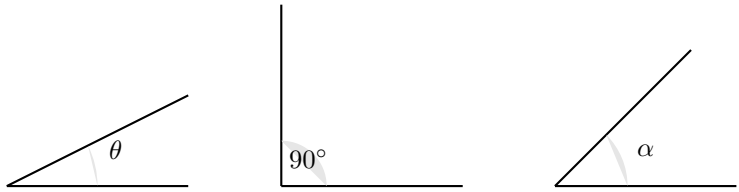
The definition is stated for **any two lines** that meet and are not in a straight line. Definition I.9 then singles out the special case where the lines are straight. This relational conception is what allows the definition to apply to any two lines, regardless of their shape or similarity.



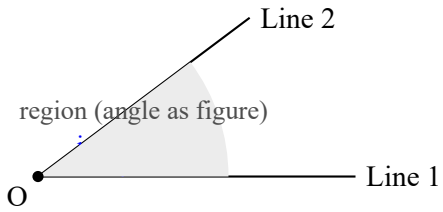
## Expanded Commentary from *Explaining Fully the First Book of Elements*

η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another and are not lying in a straight line.

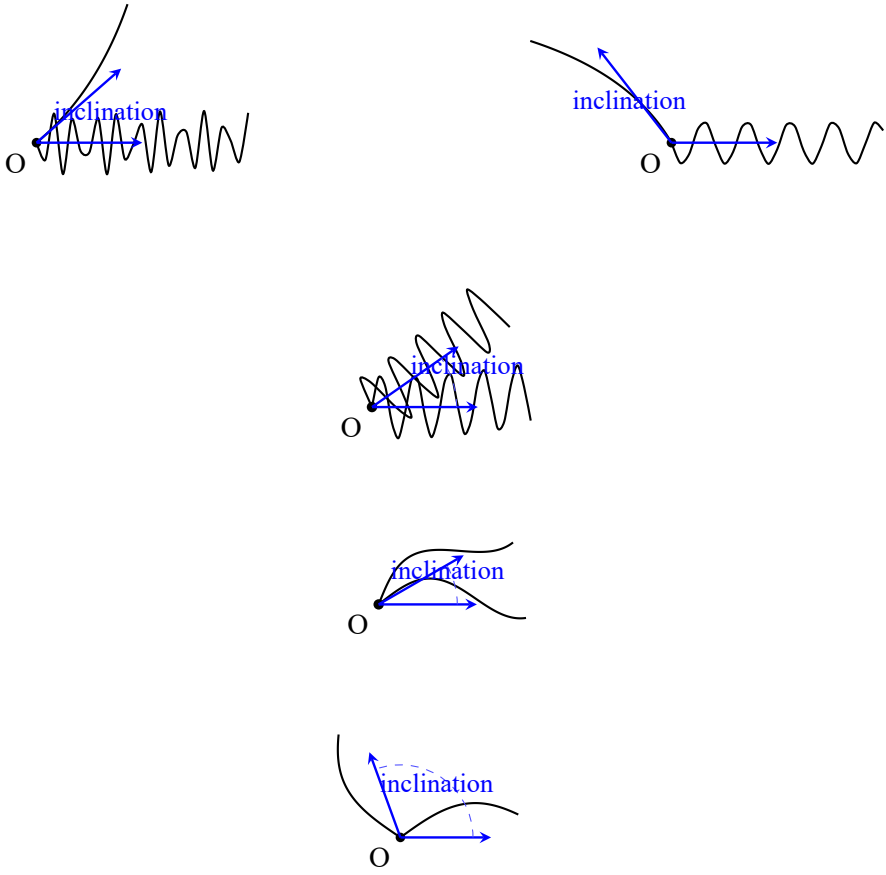



Acute angle Right angle Obtuse angle




*A plane angle as the inclination of two lines that are defined as the straight line*

Yet, the above figures show knowledge of the definition of the straight line that we do not yet have, so:




### Integrated Ontological Restatement

**Source formulation.** *A plane angle is the inclination of two lines to one another, when two lines in a plane meet one another and are not lying in a straight line.*

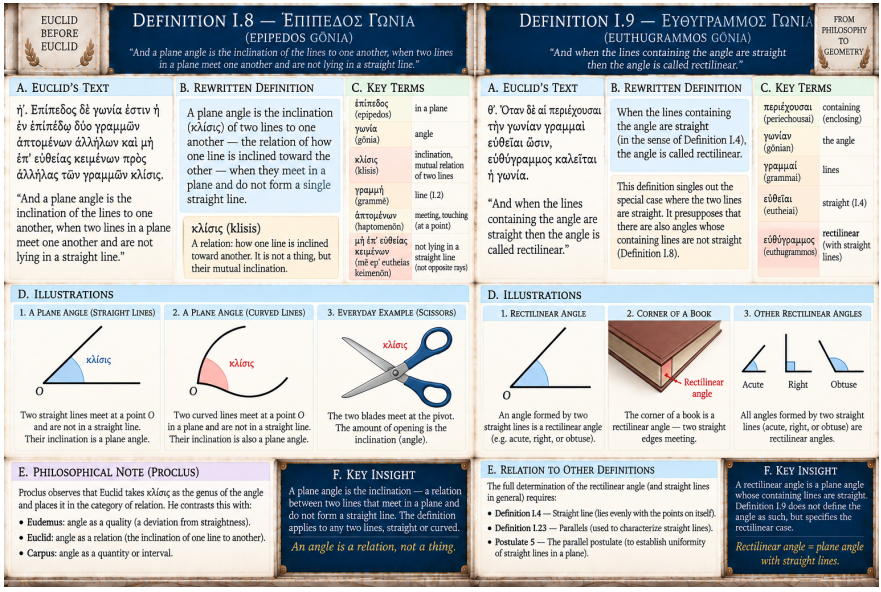
**Ontological role.** Take two lines in the same plane. Bring them to meet at a point. Record the amount of turning between them—that turning is the angle. The angle is not a quantity; it is a relation of divergence.

# Definition 9

> *Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.*
> ***9. And when the lines containing the angle are straight then the angle is called rectilinear.***

## Actions

1. If the two lines forming the angle are straight, the angle is rectilinear.
2. The corner of a book is a rectilinear angle.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

Again we do not know how the straight line looks like yet.

### Integrated Ontological Restatement

**Source formulation.** *When the lines containing the angle are straight, the angle is called rectilinear.*

**Ontological role.** If the two lines that meet have both passed the test of rectitude (Definition 4, cumulatively determined), their inclination is classified as rectilinear. This is a labeling operation that distinguishes angles formed by straight lines from those formed by curves.

## Definition 10

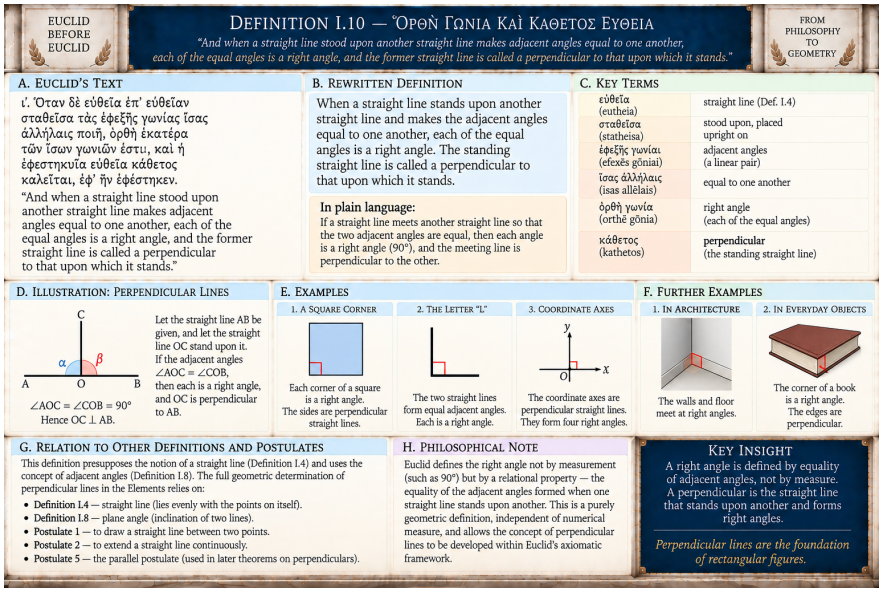
> *Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.*
> ***10. And when a straight line stood upon another straight line makes adjacent angles equal to one another, each of the equal angles is a right angle, and the former straight line is called a perpendicular to that upon which it stands.***

### Actions

1. Take a straight line AB.
2. Place another straight line CD upon it, meeting at a point.
3. If the two adjacent angles are equal, each is a right angle.
4. CD is perpendicular to AB.

## Interpretation from the Commentary

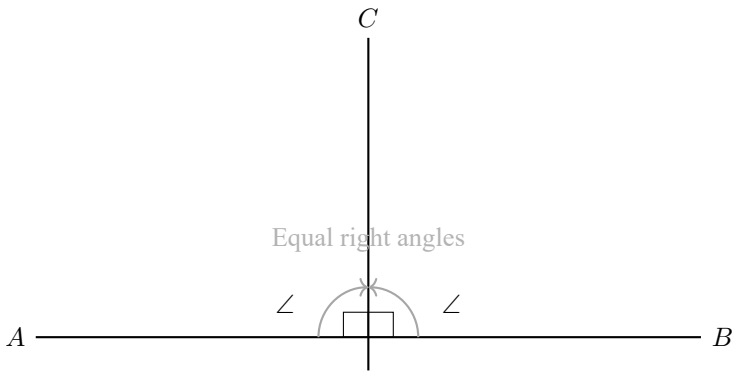
The right angle is the **standard of equality** for angles. Postulate 4 will assert that all right angles are equal.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

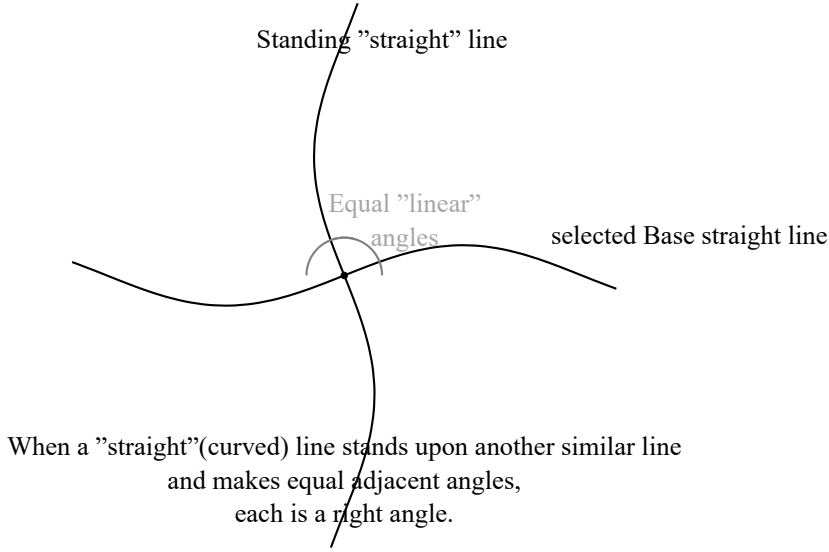
ι΄. Ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ' ἣν ἐφέστηκεν.

10. And when a straight-line stood upon another straight-line makes adjacent angles equal to one another, each of the equal angles is a right angle, and the former straight-line is called a perpendicular to that upon which it stands.




When a straight line stands upon another and makes
adjacent angles equal, each is a right angle.

But since we do not know how a straight line looks like yet, we can define an arbitrary one




When a ”straight”(curved) line stands upon another similar line
and makes equal adjacent angles,
each is a right angle.

## Integrated Ontological Restatement

**Source formulation.** *When a straight line stands upon another straight line and makes adjacent angles equal to one another, each of the equal angles*

*is a right angle, and the former line is called a perpendicular to that upon which it stands.*

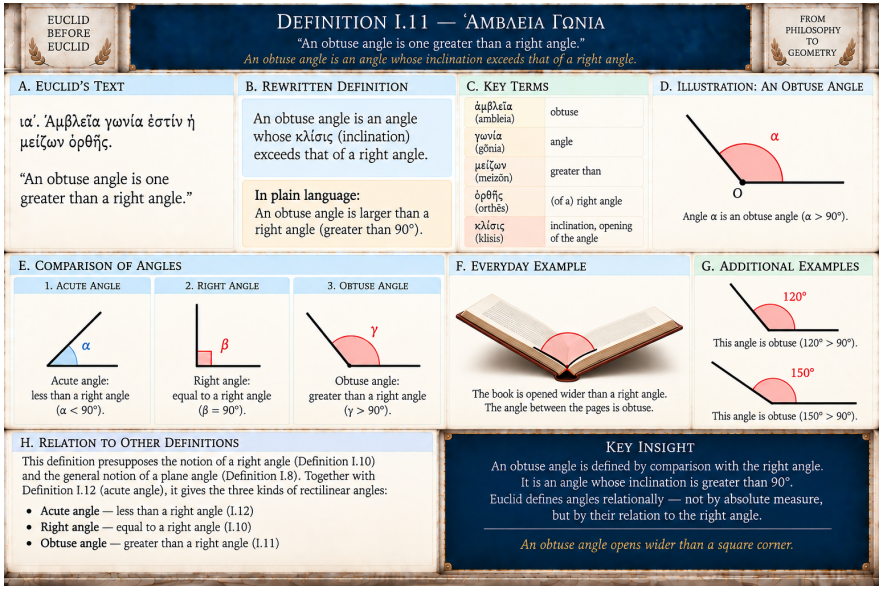
**Ontological role.** Take a straight line AB and place another straight line CD upon it at a point C such that the two angles on either side of CD are equal. If they are equal, label each a right angle. Label CD a perpendicular to AB. This is the operation of generating uniform division of the plane—the first invariant measure of turn.

## Definition 11

*Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.*
***11. An obtuse angle is one greater than a right angle.***

### Actions

1. An angle whose κλίσις exceeds that of a right angle.
2. Open a book wider than a square corner.



### Expanded Commentary from *Explaining Fully the First Book of Elements*

ια΄. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

11. An obtuse angle is one greater than a right angle.

### Integrated Ontological Restatement

**Source formulation.** *An obtuse angle is one greater than a right angle.*

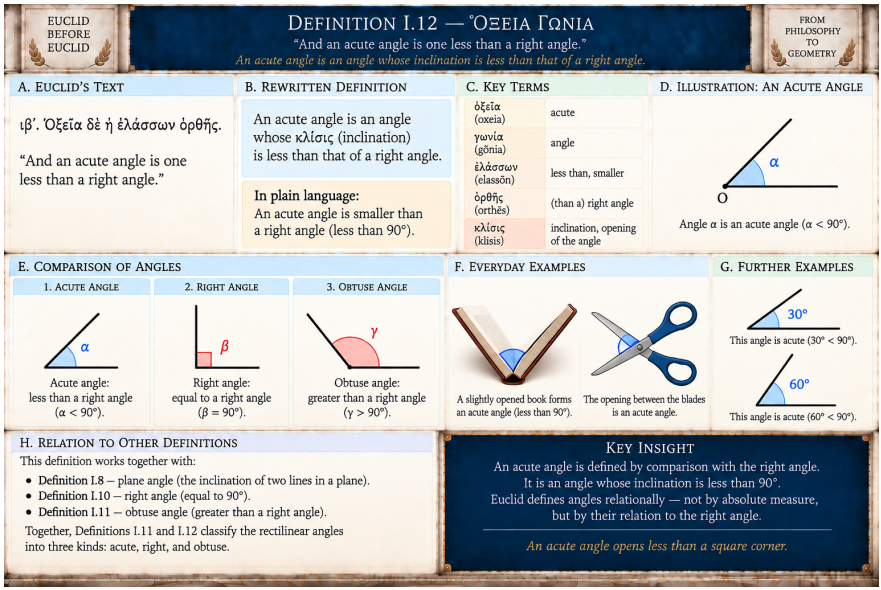
**Ontological role.** Take an angle. Compare it to a right angle (as defined in Definition 10). If it is larger, classify it as obtuse.

## Definition 12

*Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.*
***12. And an acute angle is one less than a right angle.***

### Actions

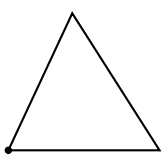
1. An angle whose κλίσις is less than that of a right angle.
2. Open a book slightly.



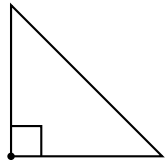

## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιβ΄. ᾽Οξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

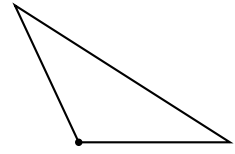
12. And an acute angle is one less than a right angle.



Acute triangle



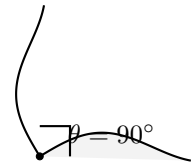
Right triangle



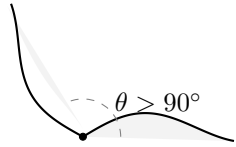
Obtuse triangle



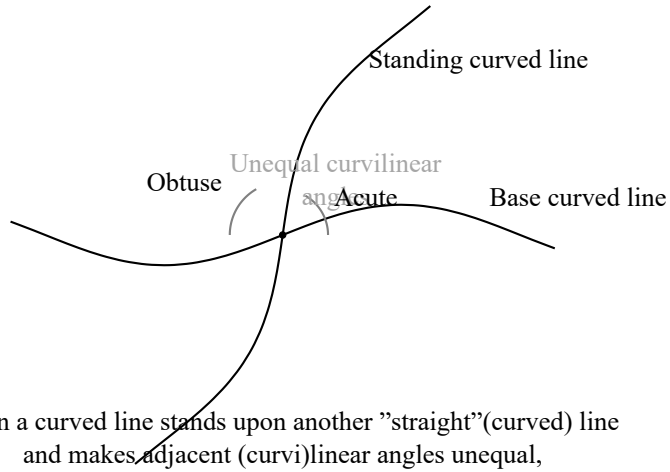

Acute curved angle




Right curved angle




Obtuse curved angle




When a curved line stands upon another ”straight”(curved) line
and makes adjacent (curvi)linear angles unequal,
one is acute and the other obtuse.

## Integrated Ontological Restatement

**Source formulation.** *An acute angle is one less than a right angle.*

**Ontological role.** Take an angle. Compare it to a right angle. If it is smaller, classify it as acute.

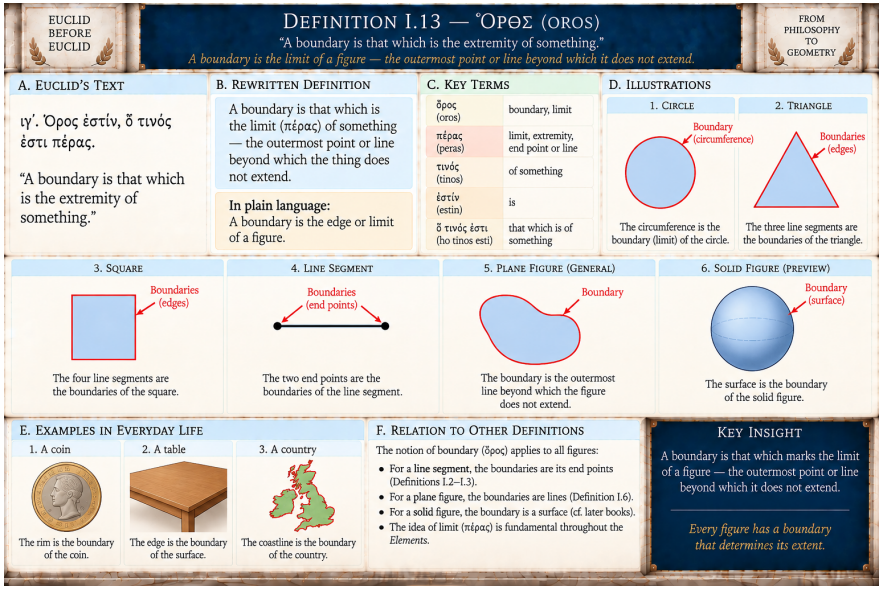
# Definition 13

*Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.*
***13. A boundary is that which is the extremity of something.***

## Actions

1. A boundary is what marks the limit of a figure.

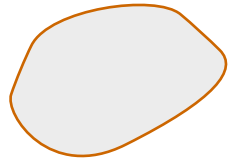
2. The circumference of a circle is the boundary of the circle.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιγ΄. ΄Ορος ἐστίν, ὅ τινός ἐστι πέρας.

13. A boundary is that which is the extremity of something.



Curved polygonal boundary

### Integrated Ontological Restatement

**Source formulation.** *A boundary is that which is the extremity of something.*

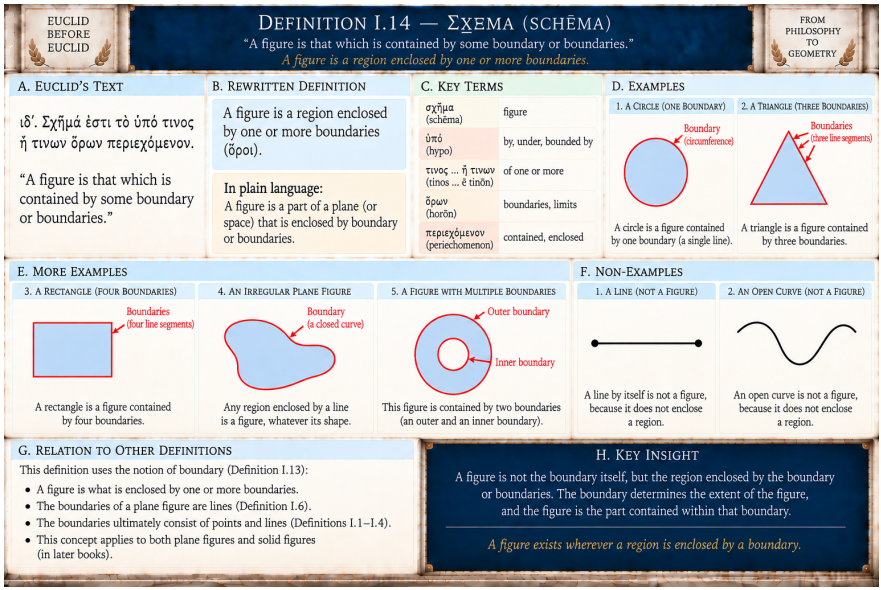
**Ontological role.** A boundary is the terminal marker of any extended thing. It is the limit beyond which the thing does not extend. For a line, the boundary is a point. For a surface, the boundary is a line. For a solid, the boundary is a surface.

## Definition 14

> *Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.*
> ***14. A figure is that which is contained by some boundary or boundaries.***

### Actions

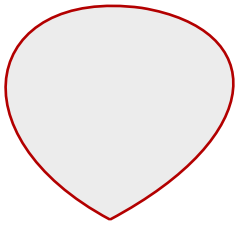
1. A figure is a region enclosed by one or more boundaries.
2. A circle is contained by one boundary. A triangle by three.



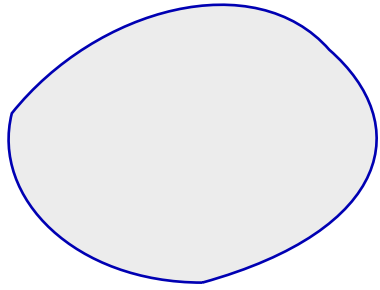

## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιδ΄. Σχῆμά ἐστι τὸ ὑπό τινος ἢ τινων ὅρων περιεχόμενον.

14. A figure is that which is contained by some boundary or boundaries.



Curvilinear triangle



Curvilinear quadrilateral

### Integrated Ontological Restatement

**Source formulation.** *A figure is that which is contained by some boundary or boundaries.*

**Ontological role.** Take a collection of boundaries (points, lines, surfaces) that enclose a region. The enclosed region is a figure. The figure is the unity of bounded extension.

## Definition 15

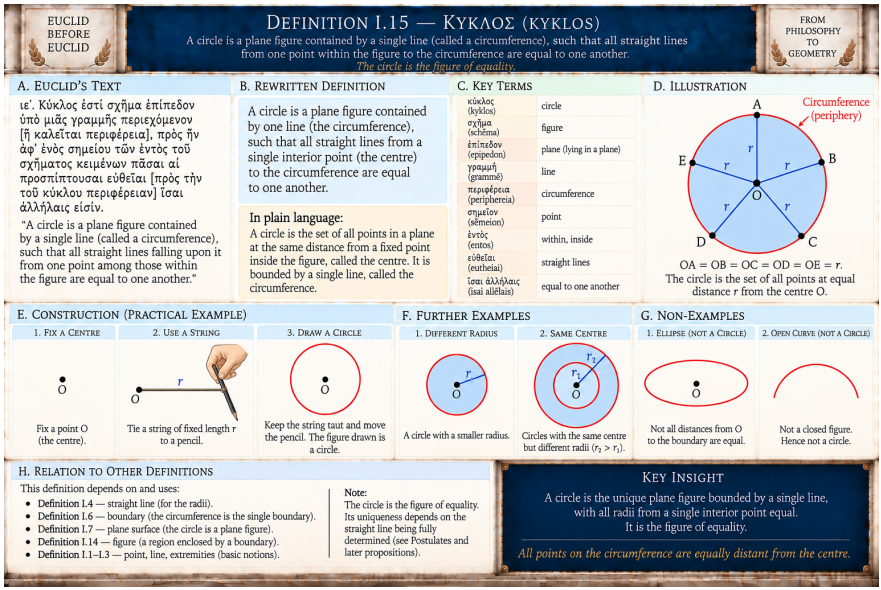
> *Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.*
> ***15. A circle is a plane figure contained by a single line (called a circumference), such that all straight lines falling upon it from one point among those within the figure are equal to one another.***

### Actions

1. Choose a point O (the centre).
2. Choose a distance r (the radius).
3. Draw a figure such that every point on its boundary is at distance r from O.
4. The figure is a circle.

### Interpretation from the Commentary

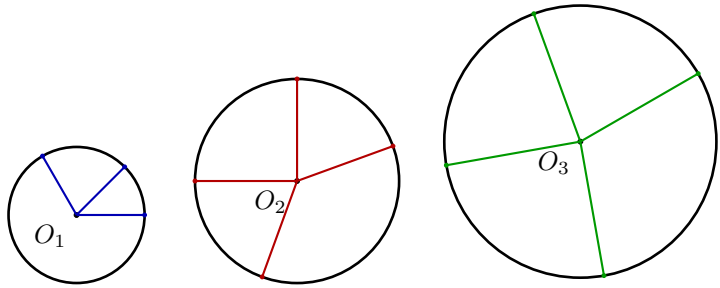
The circle is the **figure of equality** — every radius is equal. Its uniqueness depends on the straight line being fully determined.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιε΄. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

15. A circle is a plane figure contained by a single line (called a circumference), such that all straight lines falling upon it from one point among those within the figure are equal to one another.




Small circle (radius $r_1$) Medium circle (radius $r_2$) Large circle (radius $r_3$)

### Integrated Ontological Restatement

**Source formulation.** *A circle is a plane figure contained by a single line (called a circumference), such that all straight lines radiating from one fixed interior point to the circumference are equal to one another.*

**Ontological role.** Take a fixed point O (the center). Take a movable straight line r (the radius). Rotate r around O without changing its length. The path traced by the endpoint is the circle. Check that every endpoint is at the same distance from O. If so, the figure is a circle. Critical dependence: this definition presupposes that the straight line (the radius) is a stable, symmetric measure. If the straight line were bent or asymmetric, the circle could not be uniquely defined. Thus, Definition 15 exerts ontological pressure back onto Definition 4, forcing the straight line to be symmetric and uniform.

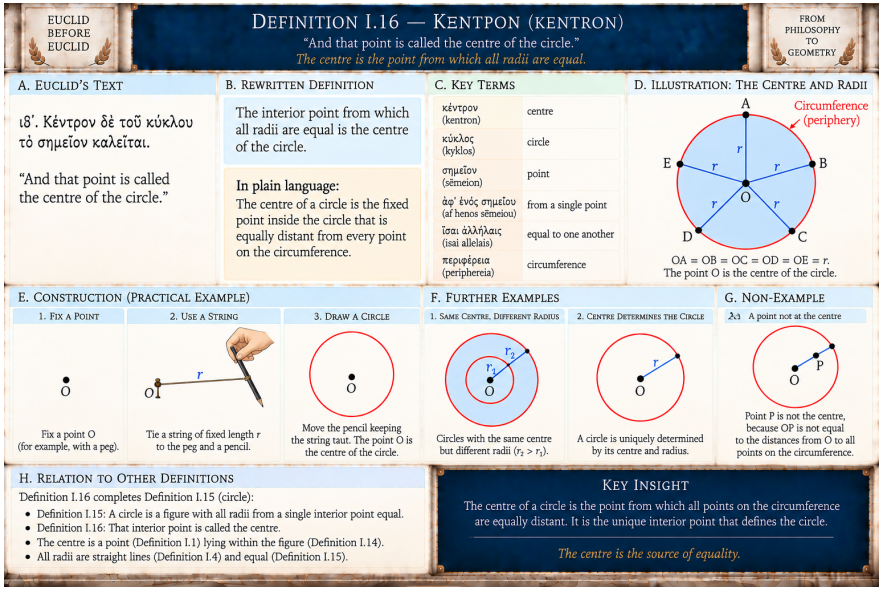
## Definition 16

*Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.*
***16. And that point is called the center of the circle.***

### Actions

1. The point from which all radii are equal is the centre.

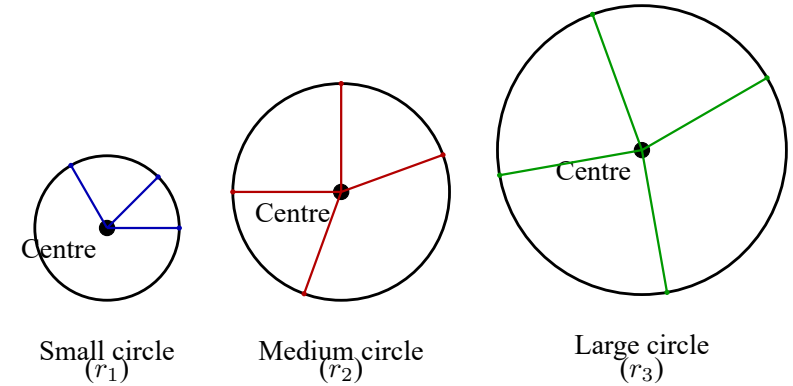
2. The hole left by the peg in the string construction is the centre.

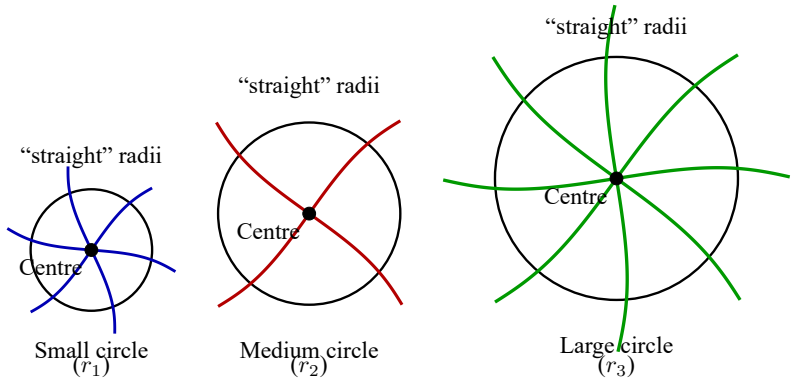



## Expanded Commentary from *Explaining Fully the First Book of Elements*

ις΄. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

16. And that point is called the center of the circle.




### Integrated Ontological Restatement

**Source formulation.** *The center of the circle is the point called the center.*

**Ontological role.** Label the fixed interior point O from Definition 15 as the center. It is the origin of all radii and the pivot of rotation.

## Definition 17

> *Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.*

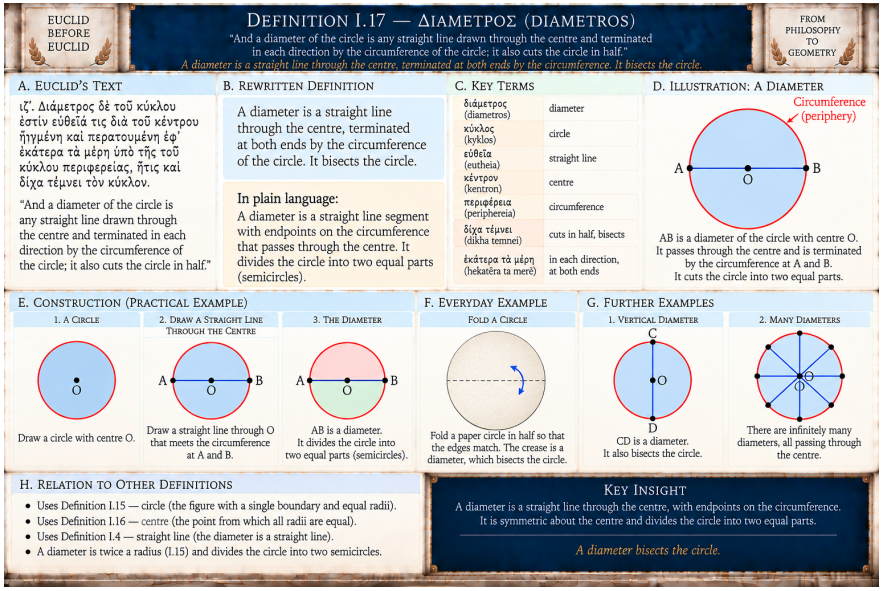
> ***17. And a diameter of the circle is any straight line drawn through the center and terminated in each direction by the circumference of the circle; it also cuts the circle in half.***

## Actions

1. Draw a straight line through the centre O.
2. Extend it until it meets the circumference on both sides.
3. The segment between the two circumference points is a diameter.
4. It divides the circle into two equal halves.

## Interpretation from the Commentary

This definition **forces** the straight line to be symmetric with respect to its centre. If the diameter were a curve, the two halves of the circle would not be equal. This is the first step in determining the straight line's true shape.

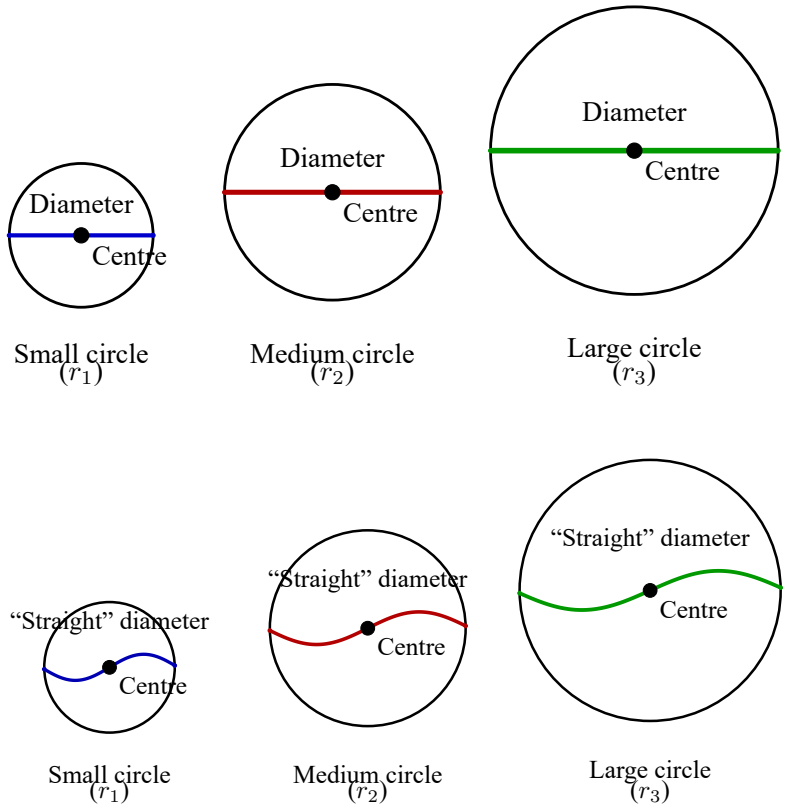


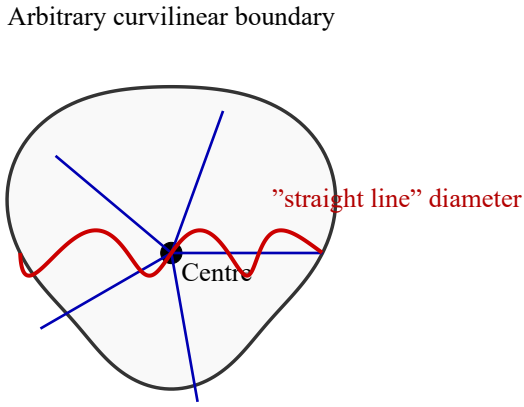
## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιζ΄. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

17. And a diameter of the circle is any straight line drawn through the center and terminated in each direction by the circumference of the circle; it also cuts the circle in half.

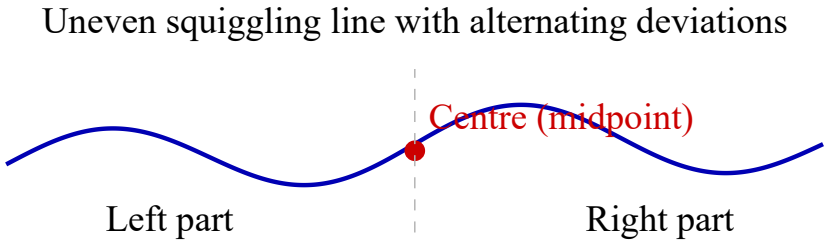
So from this postulate we have to have that the diameter of the circle should be a straight line. Let us see some of our infinite options so far:

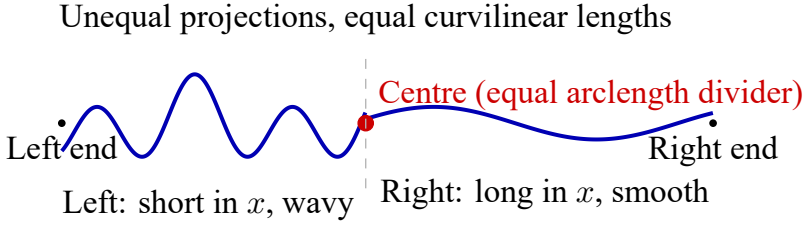


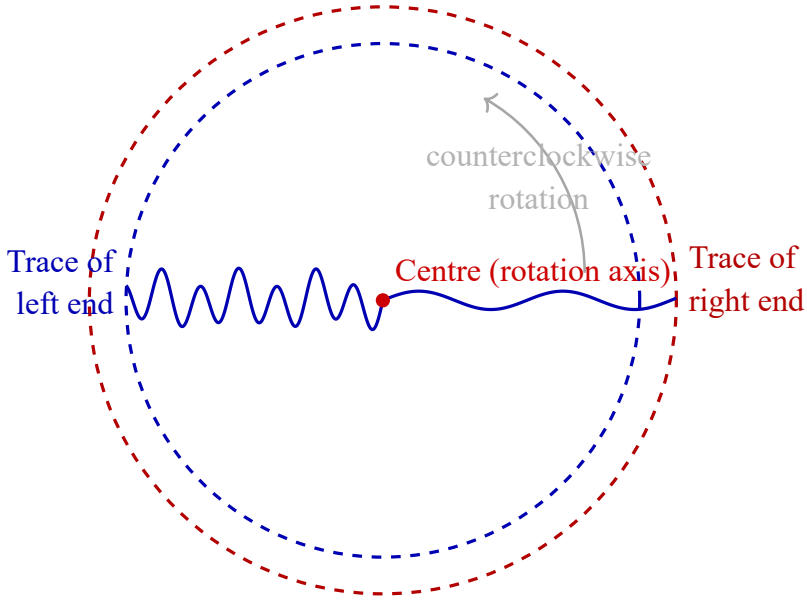



Abstract circle figure

Since we do not yet know what the circle truly looks like, nor in what manner the distance from its centre is to be measured, let us consider the case of a straight line that bends unevenly on either side of its centre. That line is necessarily the diameter of the circle. The centre of this line may still be determined by counting an equal number of steps along each part.






Rotation of an uneven curvilinear line —
endpoint traces are circles of unequal radii

Figure 1: When the diameter is a curvilinear line, unequal in shape on its two sides yet equal in total length, it is rotated about its centre, each of its ends describes a perfect circle. The constancy of the distance between each end and the centre fulfills Euclid's definition of the circle, given that we defined the straight line as above.

Let a straight line be conceived as a length having a centre which divides it into two equal parts, though the two halves may differ in form, the one being more irregular and the other more even. If such a line be turned about its centre as about a fixed point, each of its ends will trace a path around that centre. Let us now define the diameter of any circle based on that straight line. Because every point of the line remains at the same distance from the centre throughout the rotation, each endpoint describes a perfect circle. The motion of the line thus reveals that the circle is that figure whose boundary is everywhere equidistant from its centre.

Yet, if the line itself is not symmetric with respect to its centre—if one half bends more or less than the other—then the two ends, though both at equal distances from the centre, will trace circles of unequal size. Hence the defi-

nition of the circle, taken in itself, may still be satisfied, but not in a unique way. Until we know what form the straight line truly possesses, the circle likewise remains uncertain in its appearance; for the measure of distance from the centre, by which the circle is defined, depends on the nature of that line. Thus we see that when the line lacks symmetry, the circle with the same nominal radius is no longer one and the same, but two, each true to the definition yet unequal in extent.

Yet this definition alludes to more than the act itself: it assumes that distance may be known and preserved, that it may revolve uniformly about a fixed point, and that all points reached thereby remain equally distant from the centre. In this sense, the postulate does not merely define a figure but proclaims the very possibility of constant measure and unbroken motion. It is therefore the first hint of equality and symmetry within space, for by it every point may serve as a centre, and every length as a measure.

But if the nature of the straight line and the manner of measuring distance remain uncertain, then the circle so drawn is not yet one in form, nor unique in kind. For where the radius is not known as a constant magnitude, the circle cannot be known as a single figure.

The definition of the diameter therefore sets the first step for determining the shape of the straight line! The shape of the straight line has to be the same, the straight line has to be symmetric with respect to its centre, otherwise the circle induced by it is not unique! We have two different circles with the same radius, diameter and centre.

## Integrated Ontological Restatement

**Source formulation.** *A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference. Such a line also cuts the circle in half.*

**Ontological role.** Draw a straight line through the center O of a circle until it hits the circumference on both sides. Mark these two points. This line cuts the circle into two equal halves. Ontological consequence: if a line is to serve as a diameter, it must be symmetric with respect to its center. The two halves—one on either side of the center—must be mirror images in length and shape. If the line were curved or asymmetrical, the two halves would

be unequal, and the circle would not be uniquely defined. By Postulate 2 (infinite extension), this symmetry must hold for every point on the line, not just the center. Therefore, every point of a straight line is a center of symmetry—meaning the line is infinite, uniform, and without turning. This is the birth of true rectilinearity.

### Role in the Cumulative Definition of Rectitude

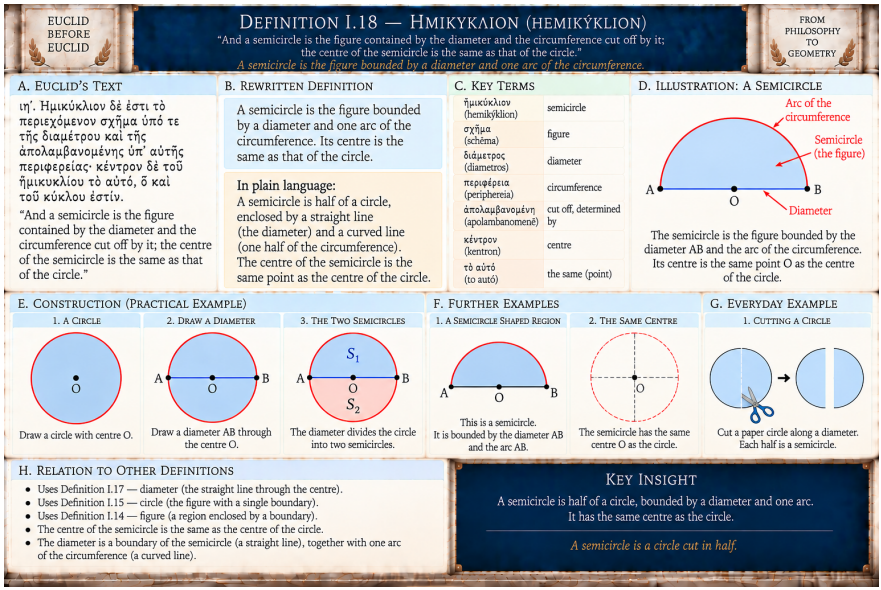
The diameter supplies the symmetry condition that the reconstruction reads back into the developing concept of the straight line. A line capable of passing through the centre of a circle and terminating on the circumference in both directions while dividing the circle into equal halves must display bilateral symmetry about that centre. Combined with continuous production, this becomes a global condition rather than a property restricted to one chosen diameter.

## Definition 18

> *Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῆς περιφερείας· κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὅ καὶ τοῦ κύκλου ἐστίν.*
> ***18. And a semicircle is the figure contained by the diameter and the circumference cut off by it; the center of the semicircle is the same as that of the circle.***

### Actions

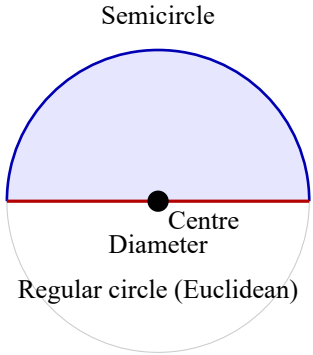
1. Take a circle and a diameter.
2. The figure bounded by the diameter and one arc is a semicircle.
3. Its centre is the same as the circle's.

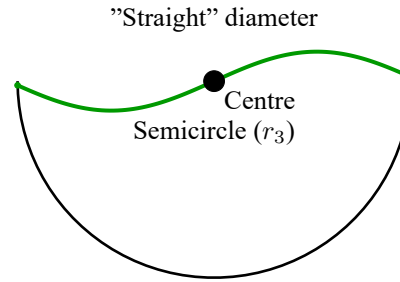



## Expanded Commentary from *Explaining Fully the First Book of Elements*

ιη΄. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῆς περιφερείας· κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

18. And a semicircle is the figure contained by the diameter and the circumference cut off by it; the center of the semicircle is the same as that of the circle.




Here we use the fact that the straight line has to be a line which is symmetric with respect to its centre.

### Integrated Ontological Restatement

**Source formulation.** *A semicircle is the figure contained by the diameter and the circumference cut off by it. The center of the semicircle is the same as that of the circle.*

**Ontological role.** Take a circle. Draw a diameter. The diameter divides the circle into two equal halves. Take one of these halves—the region bounded by the diameter and the corresponding arc of the circumference. That region is a semicircle. Its center is the same point as the center of the original circle. This definition assumes that the diameter has already imposed central symmetry (Definition 17) and that the circle has been uniformly divided.

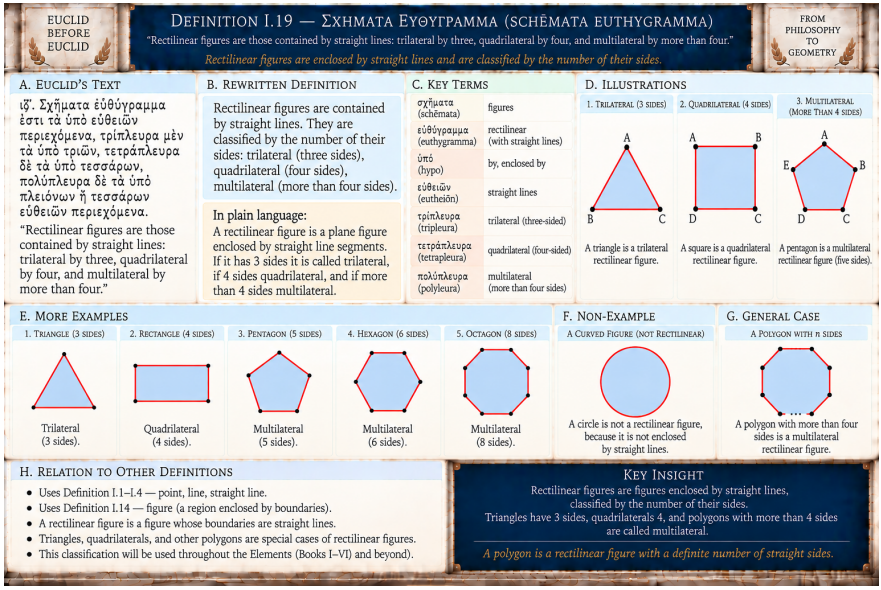
## Definition 19

> *Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.*
> ***19. Rectilinear figures are those contained by straight lines: trilateral by three, quadrilateral by four, and multilateral by more than four.***

## Actions

1. A figure whose boundaries are all straight lines is rectilinear.

2. It is classified by the number of sides.

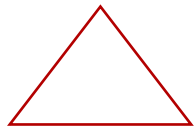



## Expanded Commentary from *Explaining Fully the First Book of Elements*

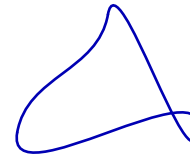
ιθ΄. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

19. Rectilinear figures are those contained by straight lines—trilateral by three, quadrilateral by four, and multilateral by more than four.

Trilateral (Triangle)

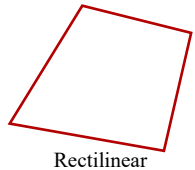


Rectilinear

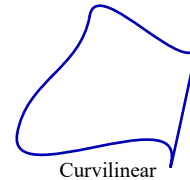


Curvilinear

Quadrilateral

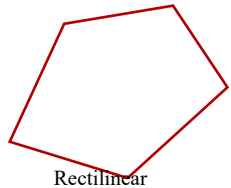


Rectilinear

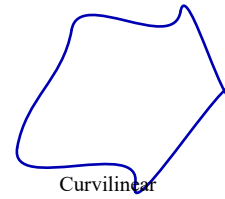


Curvilinear

Quintilateral (Pentagon)

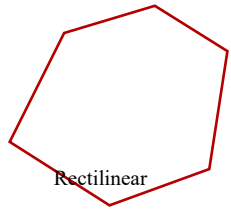


Rectilinear

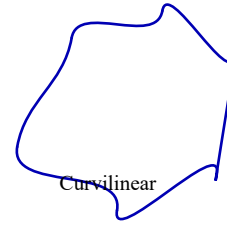


Curvilinear

Sextilateral (Hexagon)

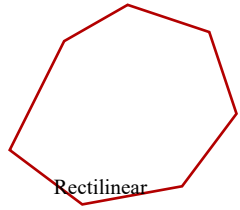


Rectilinear

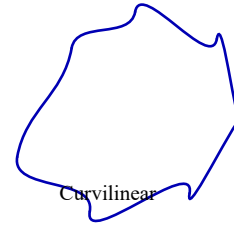


Curvilinear

Heptilateral (Heptagon)



Rectilinear



Curvilinear

Figure 2: Each polygon is drawn both with rectilinear and with symmetric curvilinear sides. Though wavy in form, these curvilinear sides remain balanced about their midpoints, and may thus be regarded as "straight" in the higher sense of symmetrical equilibrium.

### Integrated Ontological Restatement

**Source formulation.** *Rectilinear figures are those contained by straight lines: trilateral by three, quadrilateral by four, and multilateral by more than four.*

**Ontological role.** Take a closed figure. Examine its boundaries. If every boundary is a straight line (as determined by Definition 4, cumulatively), the figure is rectilinear. Count the number of straight-line boundaries. If there are three, classify it as a trilateral figure (triangle). If four, quadrilateral. If more than four, multilateral. This is a classification operation that organizes all rectilinear figures by the number of their sides.

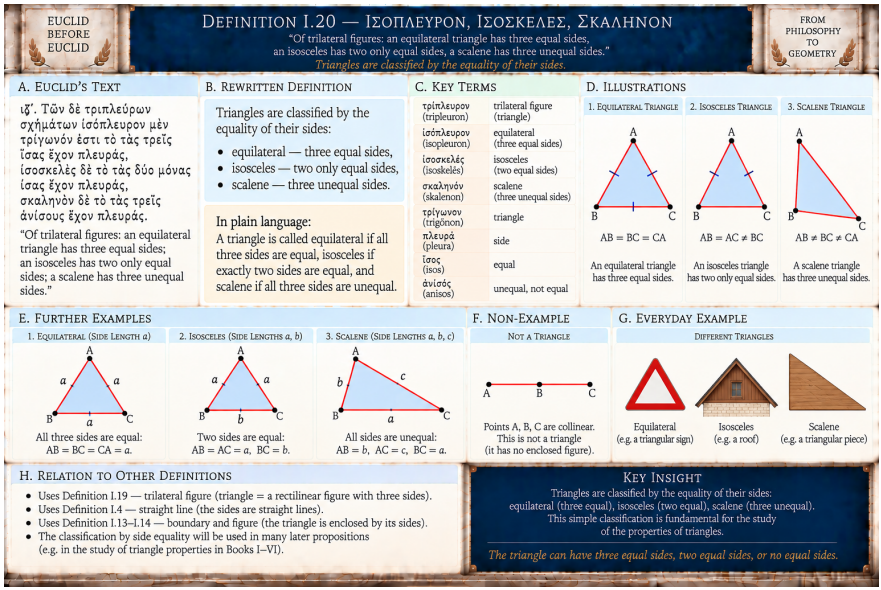
## Definition 20

> *Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.*
> ***20. Of trilateral figures: an equilateral triangle has three equal sides; an isosceles has two only equal sides; a scalene has three unequal sides.***

### Actions

1. Classify triangles by side equality: equilateral (3 equal), isosceles (2 equal), scalene (3 unequal).

2. Draw three triangles: one with all sides equal, one with two equal, one with none equal.

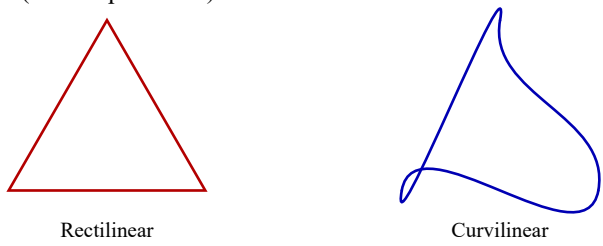



## Expanded Commentary from *Explaining Fully the First Book of Elements*

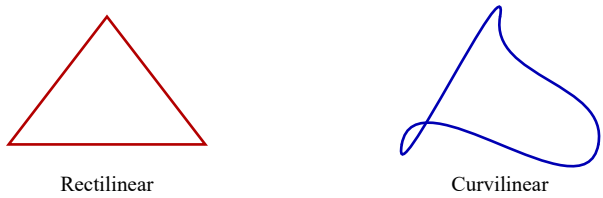
κ΄. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

20. Of trilateral figures: an equilateral triangle has three equal sides; an isosceles has two only equal sides; a scalene has three unequal sides.

Equilateral Triangle (three equal sides)




Isosceles Triangle (two equal sides)




Scalene Triangle (no equal sides)




Figure 3: Each figure is drawn both with rectilinear and with symmetric curvilinear sides. Though wavy in form, these curvilinear sides remain balanced about their midpoints, and may thus be regarded as “straight” in the higher sense of symmetrical equilibrium.

### Integrated Ontological Restatement

**Source formulation.** *Of trilateral figures: an equilateral triangle is that having three equal sides; an isosceles triangle that having only two equal sides; and a scalene triangle that having three unequal sides.*

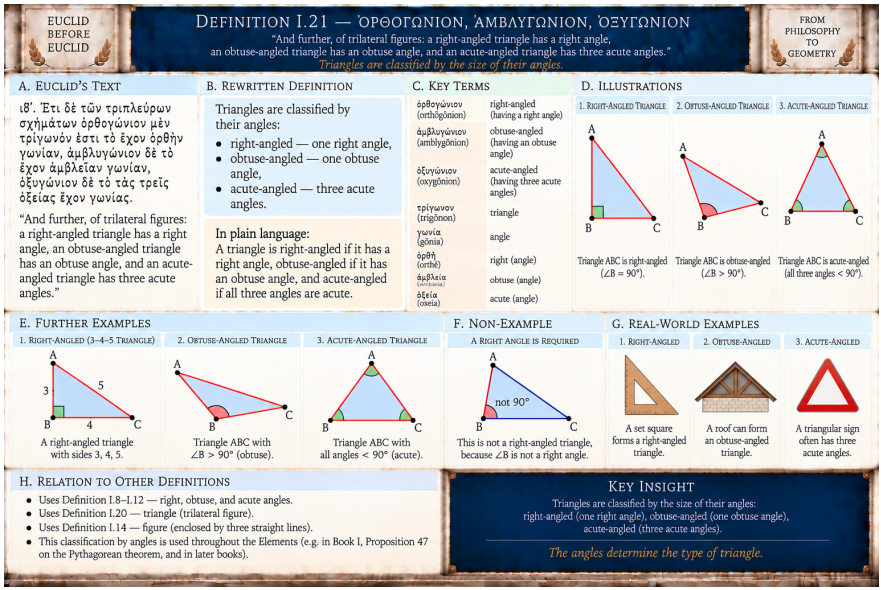
**Ontological role.** Take a trilateral figure (triangle). Measure its three sides. Compare them pairwise. If all three are equal, label it equilateral. If exactly two are equal, label it isosceles. If all three are unequal, label it scalene. This is a labeling operation that distinguishes triangles by their symmetry of side-length.

## Definition 21

> *Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.*
> ***21. And further, of trilateral figures: a right-angled triangle has a right angle, an obtuse-angled triangle has an obtuse angle, and an acute-angled triangle has three acute angles.***

### Actions

1. Classify triangles by angle type: right-angled (one right), obtuse-angled (one obtuse), acute-angled (three acute).
2. Draw a right triangle, an obtuse triangle, and an acute triangle.

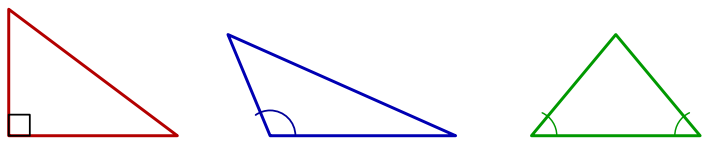



## Expanded Commentary from *Explaining Fully the First Book of Elements*

καʹ. Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

21. And further, of the trilateral figures: a right-angled triangle has a right angle, an obtuse-angled triangle has an obtuse angle, and an acute-angled triangle has three acute angles.

Right-angled triangleObtuse-angled triangleAcute-angled triangle



one right angle one obtuse angle all acute angles

### Integrated Ontological Restatement

**Source formulation.** *And further, of the trilateral figures: a right-angled triangle is that having a right angle; an obtuse-angled triangle that having an obtuse angle; and an acute-angled triangle that having three acute angles.*

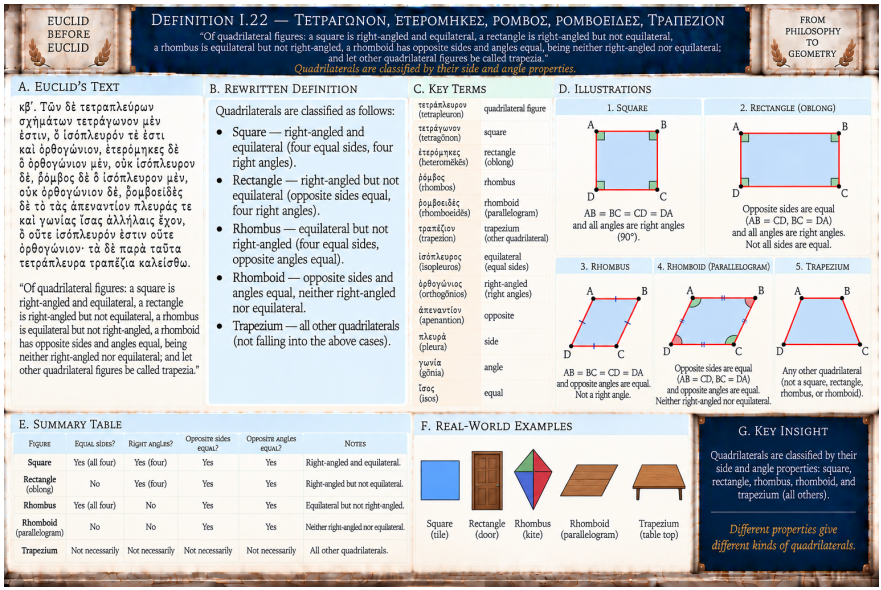
**Ontological role.** Take a triangle. Measure its three angles (as defined in Definition 8). If one angle is exactly equal to a right angle (Definition 10), label it right-angled. If one angle is greater than a right angle (Definition 11), label it obtuse-angled. If all three angles are less than a right angle (Definition 12), label it acute-angled. This complements Definition 20, allowing triangles to be classified both by their side-equality and by their largest angle.

## Definition 22

> *Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δὲ ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δὲ ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.*
> ***22. Of quadrilateral figures: a square is right-angled and equilateral; a rectangle is right-angled but not equilateral; a rhombus is equilateral but not right-angled; a rhomboid has opposite sides and angles equal, being neither right-angled nor equilateral; and let other quadrilateral figures be called trapezia.***

### Actions

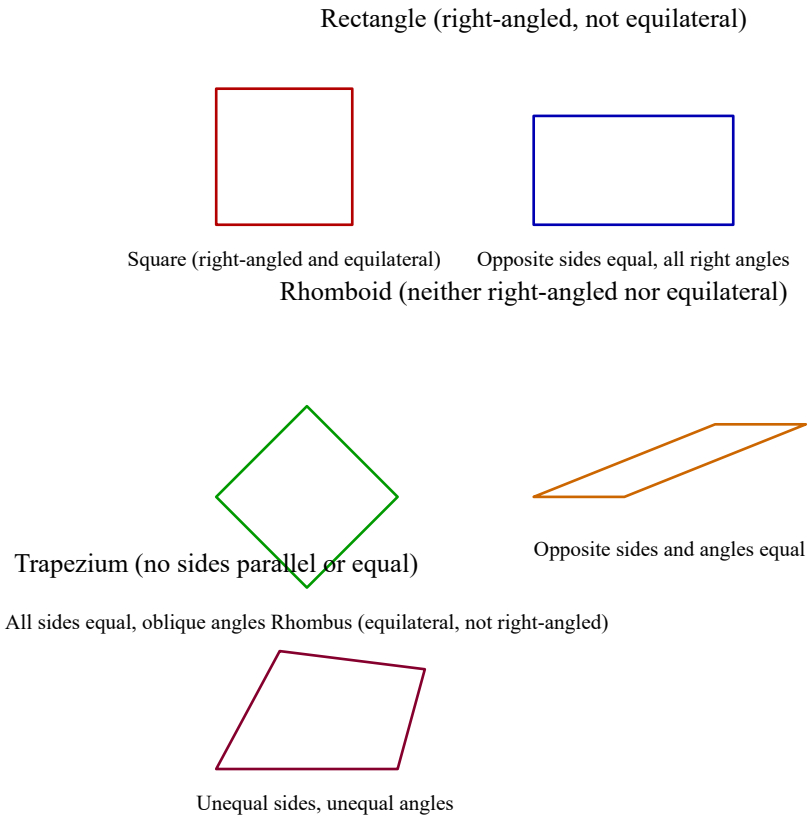
1. Classify quadrilaterals: square, rectangle, rhombus, rhomboid, trapezium.
2. Draw each.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

κβ΄. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μὲν ἐστιν, ὃ ἰσόπλευρόν τε ἐστὶ καὶ ὀρθογώνιον, ἑτερόμηκες δὲ ὃ ὀρθογώνιον μὲν, οὐκ ἰσόπλευρον δέ· ῥόμβος δὲ ὃ ἰσόπλευρόν μὲν, οὐκ ὀρθογώνιον δέ· ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευρὰς καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

22. Of quadrilateral figures: a square is right-angled and equilateral; a rectangle is right-angled but not equilateral; a rhombus is equilateral but not right-angled; a rhomboid has opposite sides and angles equal to one another, being neither right-angled nor equilateral; and let other quadrilateral figures be called trapezia.




## Integrated Ontological Restatement

**Source formulation.** *Of quadrilateral figures: a square is right-angled and equilateral; a rectangle is right-angled but not equilateral; a rhombus is equilateral but not right-angled; a rhomboid has opposite sides and angles equal to one another, being neither right-angled nor equilateral; and let quadrilateral figures besides these be called trapezia.*

**Ontological role.** Take a quadrilateral figure. Apply two tests: test whether all angles are right (Definition 10), and test whether all sides are equal (compare side lengths). If both pass, label it a square. If right-angled but not equilateral, label it a rectangle. If equilateral but not right-angled, label it

a rhombus. If opposite sides and opposite angles are equal but the figure is neither equilateral nor right-angled, label it a rhomboid. If none of these conditions hold, label it a trapezium. This is a comprehensive classification of four-sided figures by their symmetry properties.

## Definition 23

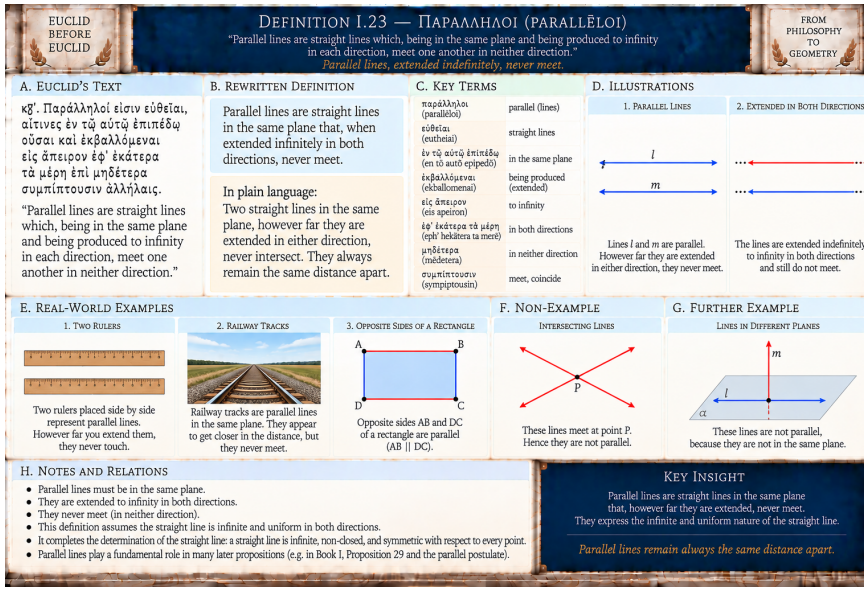
> *Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.*
> ***23. Parallel lines are straight lines which, being in the same plane and being produced to infinity in each direction, meet one another in neither direction.***

### Actions

1. Take two straight lines in the same plane.
2. Extend them infinitely in both directions.
3. If they never meet, they are parallel.

### Interpretation from the Commentary

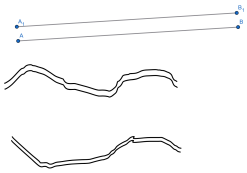
This definition **forces** the straight line to be symmetric with respect to **every point**. If the line had a unique centre, translation would change its shape and parallelism would fail. Therefore the straight line must be infinite, uniform, and self-symmetric at every point.




## Expanded Commentary from *Explaining Fully the First Book of Elements*

κγ΄. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

23. Parallel lines are straight lines which, being in the same plane and being produced to infinity in each direction, meet one another in neither direction.



Various lines which are candidates for the straight line, which are necessarily symmetric with respect to their centre point.

This observation shows that the Euclidean definition of parallelism depends not merely on the *absence of intersection*, but also on the *uniformity of direction*. Straight lines may be translated in any direction without altering their inclination, whereas curvilinear lines cannot. Therefore, a family of curvilinear lines cannot preserve mutual non-intersection under arbitrary displacement, except in the case where there is symmetry with respect to any point.

**Conclusion.** When curvature is introduced, under translation, intersection becomes inevitable. Hence, only in the case of true rectilinearity or symmetry with respect to every point can Euclid's definition of parallelism be perfectly satisfied.

## Conclusions

So far we saw that Euclid's definition of the straight line implied a comparison of a line with the straight line that we have already defined. The definition of the diameter of the circle as a straight line already excluded all the lines which are not symmetric with respect to their centre points. If a the line is symmetric with respect to any point thereof then the parallel definition is satisfied as well. This means effectively that every point of the straight line has to be effectively the centre of length thereof and thus the

straight line has no end and beginning or it is a closed line, like a circle. This indeed is verified by Postulate 2 that we will see next.

## Integrated Ontological Restatement

**Source formulation.** *Parallel lines are straight lines which, being in the same plane and being produced to infinity in each direction, meet with one another in neither direction.*

**Ontological role.** Take two straight lines in the same plane. Extend them infinitely in both directions (Postulate 2). Observe whether they ever share a point. If they never meet in either direction, they are parallel. This is the test of directional invariance. Ontological consequence: For a family of lines to be mutually parallel, they must all share the same uniform direction. This is only possible if the lines themselves are perfectly straight (symmetric about every point, as established by Definition 17) and the surface they inhabit is perfectly flat (as determined by Postulate 5). If the surface were curved (like a sphere), all ”straight” lines would eventually meet. If it were hyperbolic, they would diverge in multiple ways. Thus, Definition 23 works together with Postulate 5 to select the Euclidean plane as the unique surface where such uniform, non-intersecting families exist.

## Role in the Cumulative Definition of the Plane

Definition 23 is not treated as a detached classification. It couples the already developed straight line to the surface on which infinite production is possible without eventual intersection. In the reconstruction it is therefore one of the decisive links between rectitude and planarity, completed by Postulate 5.

# Postulates

READ THE FIVE POSTULATES BY KIND, NOT BY NUMBER

The first three are **commands**: they may be executed, and at the outset they are the only things that may be executed at all.

`P1` join — draw a straight line from any point to any point

`P2` extend — produce a finite straight line continuously

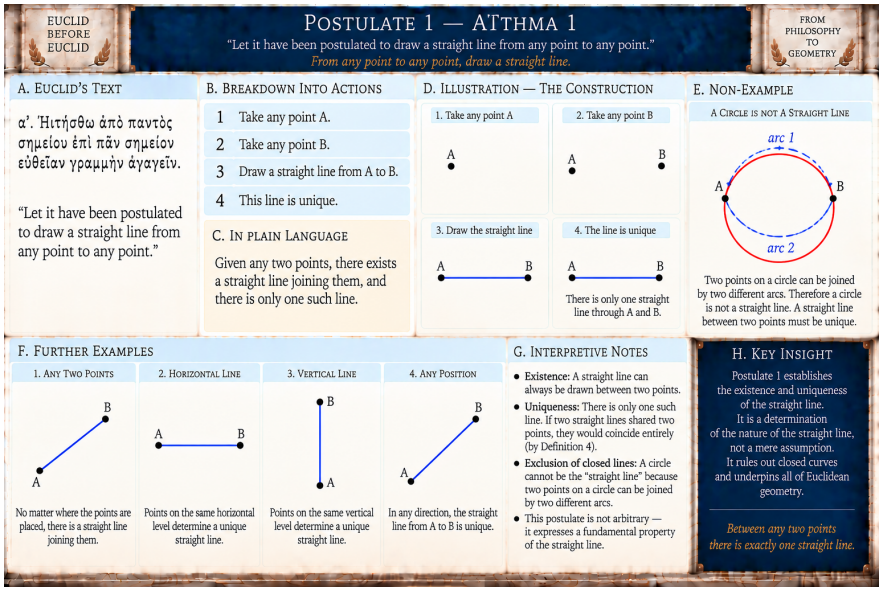
`P3` sweep — describe a circle on any centre with any given radius

The last two are **constraints**: they determine the plane in which those commands operate, and cannot themselves be performed.

`C4` all right angles are equal

`C5` the meeting condition for produced straight lines

C5 is not used anywhere in Book I until **Proposition I.29**. Parallels are shown to exist at I.27–I.28 and are constructed at I.31 without it. It is spent twice in forty-eight propositions: at I.29 in a reductio, and at I.44 to close the application of areas.




# Postulate 1

> *Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.*
> ***1. Let it have been postulated to draw a straight line from any point to any point.***

---

Permission P1

P1 Given any two points, produce the straight line joining them. Returns exactly one line: if two straight lines shared two points, by Definition 4 all their points would coincide and they would be one line.

---

## Actions

1. Take any point A.

2. Take any point B.

3. Draw a straight line from A to B.

4. This line is unique.

## Interpretation from the Commentary

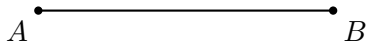
This postulate ensures:

- **Existence:** A straight line can always be drawn between two points.
- **Uniqueness:** There is only one such line. If two straight lines shared two points, they would coincide entirely (by Definition 4).
- **Exclusion of closed lines:** A circle cannot be the "straight line" because two points on a circle can be joined by two different arcs.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

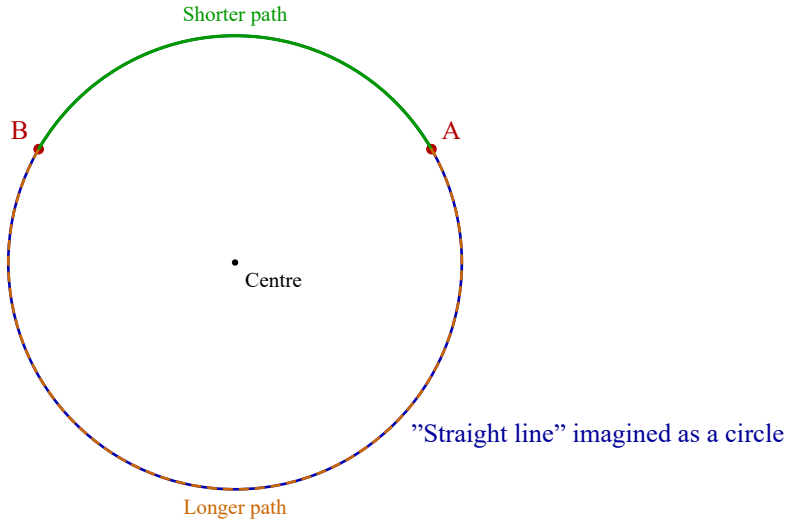
| | |
|---|---|
| α΄. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν. | 1. Let it have been postulated to draw a straight line from any point to any point. |

We want to address two topics here: Firstly, this postulate actually ensures the uniqueness of the straight line between two points as well. For if two straight lines have two common points each one is onto another and with the revealed true definition of the straight line that Euclid wanted to give, or had given but it was lost in the dark ages, the points of one lie evenly from the points of the other and since two points coincide all points coincide. Since all points coincide from those straight lines, those straight lines coincide. Hence, there is a unique straight line. Secondly, this postulate helps us clarify further the nature of the straight line. In particular, let us imagine that what we call a straight line were instead a line that returns upon itself, like a circle.



**If the straight line were a closed line, symmetric one like a circle**

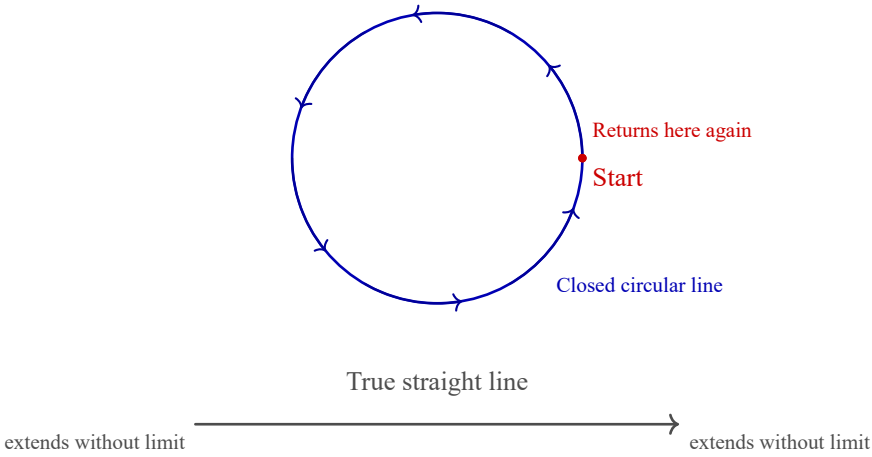
Then, when we are given two points upon this line, it would indeed be possible to join them by following the curve that connects them along the circle. Yet there would not be a single path between them, but two: one shorter, the other longer, each completing a different part of the same circuit. Hence, the joining of the points would no longer be one and the same, but twofold.




Two points $A$ and $B$ upon a circular “line” can be joined by two paths: one short (green), one long (orange). Hence, the joining of points is *not unique*, and the line cannot be truly “straight.”

Further, this line, being closed, could not be extended continually in one direction without coming back upon itself. It would have neither beginning nor end, but it would not proceed endlessly forward. For whenever we tried to carry it on without limit, we would soon return to the place from which we began. Thus it would not answer to the meaning of that which can be drawn from any point to any point, for many pairs of points in the world would not lie upon the same circle at all, and no path of this kind could pass through them.

**A closed line cannot be continually extended in one direction**




A circular “line” has neither beginning nor end, yet
it cannot be extended endlessly in one direction.
Whenever we try to continue
it, it returns to its starting point.
Hence, it does not answer to the Postulate which
bids us draw a line continually in the same direction,
nor can such a curve pass through
any two arbitrary points in the world.

Therefore, if the straight line were imagined to be a closed line, it could not fulfil the first postulate. For it would not allow one and only one way of joining two points, nor could it be continued forever in a single direction. Such a line would indeed be perfect in its closure, but not straight in its nature.

**If a straight line were a line with curvature**

We excluded that already by the result of the definition of the diameter.

**If the straight line were a symmetric curvilinear line**

As we saw in the definition of the parallel line, it has to preserve symmetry for every point, thus every point has to be essentially the centre of that line.

### Integrated Ontological Restatement

**Source formulation.** *Let it have been postulated to draw a straight line from any point to any point.*

**Ontological role.** Execute the command `connect(point1, point2)`, which returns a straight line. Guarantee: this line is unique. If two lines share two points, they are identical (by Definition 4 cumulatively determined: two straight lines sharing two points would have all their corresponding points coinciding, and thus be one and the same line). This command eliminates branching, closure, and curvature. It completes Definition 4 by ensuring that straightness implies uniqueness of connection.

## Postulate 2

> *Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.*
> ***2. And to produce a finite straight line continuously in a straight line.***

---

Permission P2

P2 Given a finite straight line, carry it on beyond either extremity, continuously and in the same straight line. Note what it does *not* give: a length. Carrying a distance elsewhere is not in the initial grant at all — Proposition I.2 pays for it.

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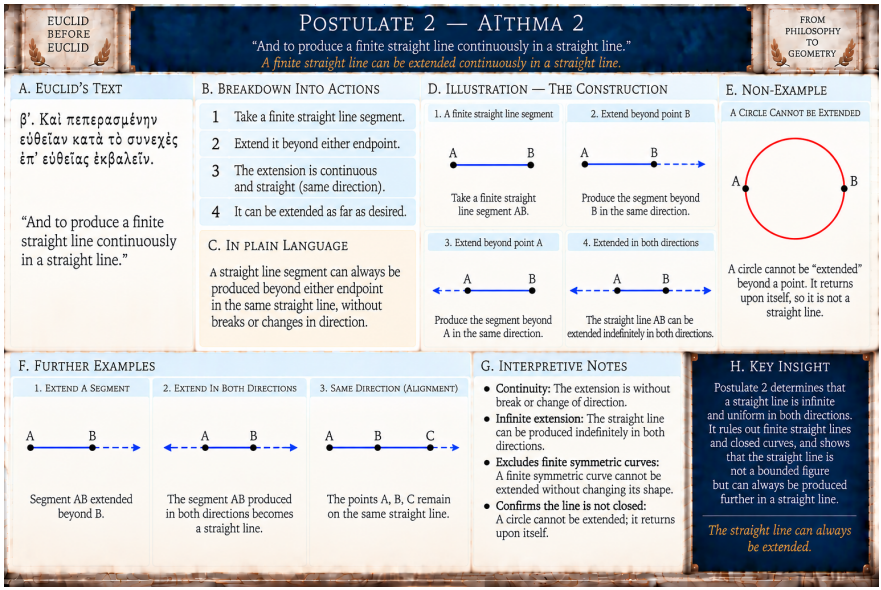



## Actions

1. Take a finite straight line segment.
2. Extend it beyond either endpoint.
3. The extension is continuous and straight (same direction).

## Interpretation from the Commentary

This postulate:

- **Excludes finite symmetric curves:** A finite symmetric curve cannot be extended without changing its shape.
- **Forces infinite extension:** The straight line must be extendable indefinitely.
- **Confirms the line is not closed:** A circle cannot be extended; it returns upon itself.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

2. And to produce a finite straight line continuously in a straight line.

Let us imagine that the straight line must not be such only that it has a single centre dividing it into equal parts, but that it is so constituted that every point upon it is alike a centre.

For if the line be truly straight, then from any point taken upon it, the parts on either side are equal and opposite in direction. Hence, the line is self-symmetric about each and every of its points.

But if so, it cannot have a finite end; for were it finite, only the middle point could serve as centre, and the equality would fail at all others. Therefore, such a line must extend without bound in both directions, neither returning upon itself nor terminating, but proceeding infinitely, the same throughout and everywhere equal to itself.

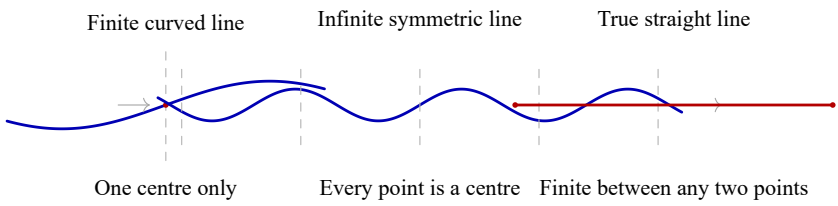
If, then, a straight line were a curved line symmetric with respect to every one of its points, it would follow that each point were a centre dividing the line into equal and opposite parts. But this property could belong only to a line without end, for in a finite line the equality fails at all but the middle.

Hence such a line would be infinite in length, and no finite straight line could exist.

Yet the first postulate commands that we may draw a finite straight line from any point to any point. Therefore the line which is straight cannot be of this curved kind; for though the curved line may possess symmetry, it cannot at once be finite and everywhere self-symmetric.

The straight line, then, must be that which is the same with itself throughout not by curvature and reflection, but by the absence of deviation altogether.

Thus the nature of the straight line is revealed by contradiction: it is neither a closed circle nor an endless symmetric curve, but that form of extension which proceeds evenly and without turning from any point to any point.




### Integrated Ontological Restatement

**Source formulation.** *And to produce a finite straight line continuously in a straight line.*

**Ontological role.** Execute the command `extend(line, direction)`, which returns an infinite line. Guarantee: the line has no endpoints. It is infinite in both directions. This eliminates circular lines and finite symmetric curves (which would fail the symmetry test of Definition 17 when extended beyond their finite bounds). It completes Definition 4 by ensuring that straightness implies infinity and non-returning extension.

## Postulate 3

> *Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.*
> ***3. And to draw a circle with any center and radius.***

---

PERMISSION P3

P3 Given a point as centre and a line already in the figure as radius, describe the circle. Because the radius is taken from the figure, P3 cannot transport a distance across the plane; it is a collapsing compass, and I.2 shows the non-collapsing one is derivable rather than assumed.

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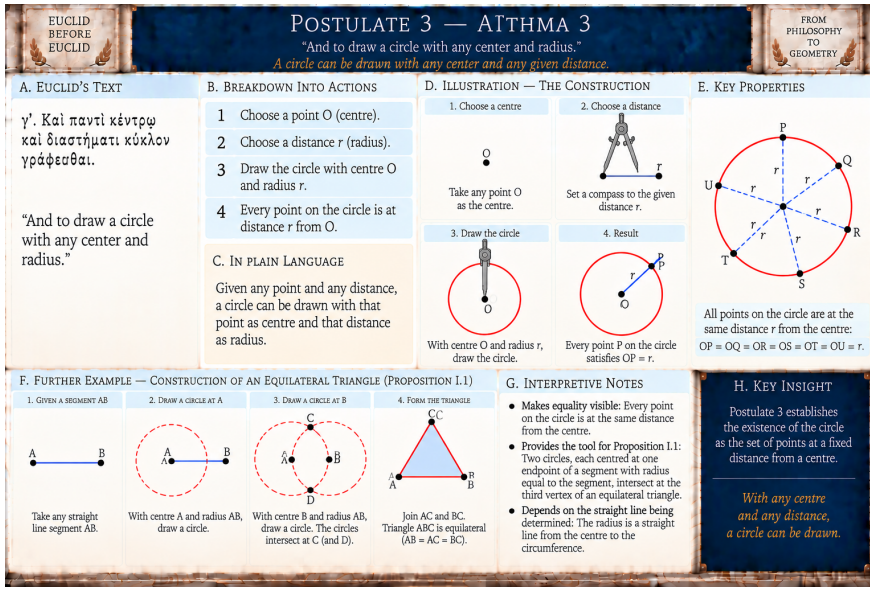



## Actions

1. Choose a point O (centre).
2. Choose a distance r (radius).
3. Draw the circle with centre O and radius r.

## Interpretation from the Commentary

This postulate:

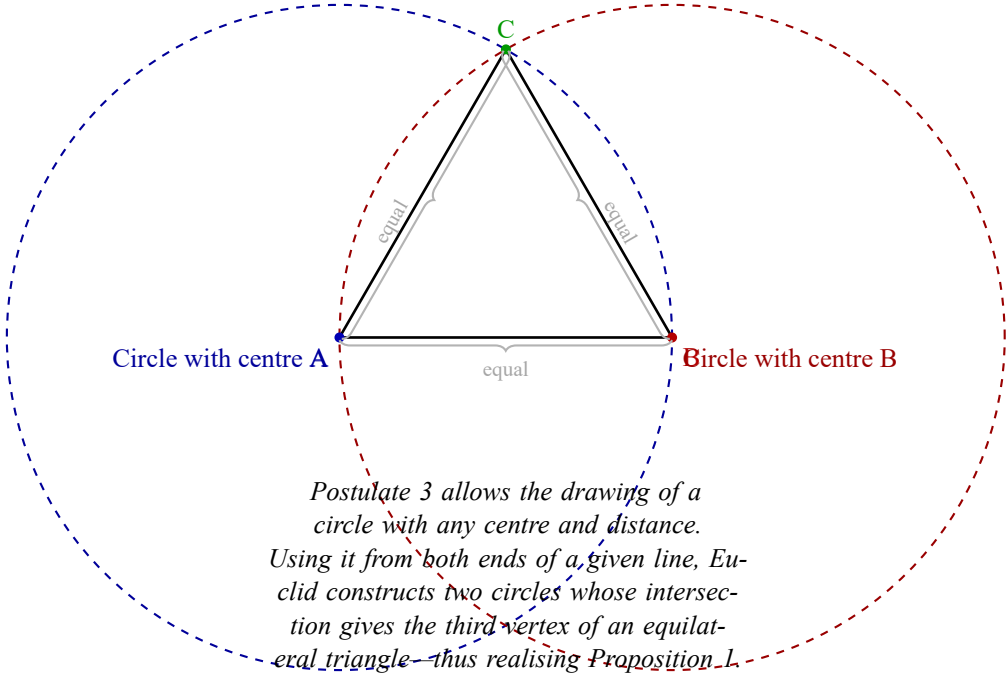
- **Makes equality visible:** Every point on the circle is at the same distance from the centre.
- **Provides the tool for Proposition I.1:** Two circles, each centred at one endpoint of a segment with radius equal to the segment, intersect at the third vertex of an equilateral triangle.
- **Depends on the straight line being determined:** The radius is a straight line from centre to circumference.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

3. And to draw a circle with any center and radius.

This postulate prepares the way for the first proposition. For when Euclid bids us to construct an equilateral triangle upon a given finite straight line, he does so by means of two circles—each drawn from one endpoint of the given line as centre, and each having the line itself as radius. The intersection of those circles marks the third vertex, and the equality of the sides follows from the constancy of that distance. Thus, the Third Postulate provides the instrument by which equality in length is made visible in form. Without it, no construction of equal sides, nor any proposition that depends upon uniform measure, could be accomplished.




*Postulate 3 allows the drawing of a circle with any centre and distance. Using it from both ends of a given line, Euclid constructs two circles whose intersection gives the third vertex of an equilateral triangle—thus realising Proposition 1.*

### Integrated Ontological Restatement

**Source formulation.** *And to draw a circle with any center and radius.*

**Ontological role.** Execute the command `draw_circle(center, radius)`, which returns a circle. Guarantee: equality of distance is always constructible. The circle is the locus of constant measure. This makes Definition 15 universally executable and ensures that the act of measuring equal distances is always possible. It completes Definition 15 by guaranteeing that for any chosen center and any chosen radius, the figure of equal distances exists.

## Postulate 4

> *Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.*
> ***4. And that all right angles are equal to one another.***

---

Constraint C4

**Constraint, not command.** Nothing here can be executed. C4 fixes a standard, so that an angle determined in one place may be compared with an angle determined in another, and the plane is homogeneous.

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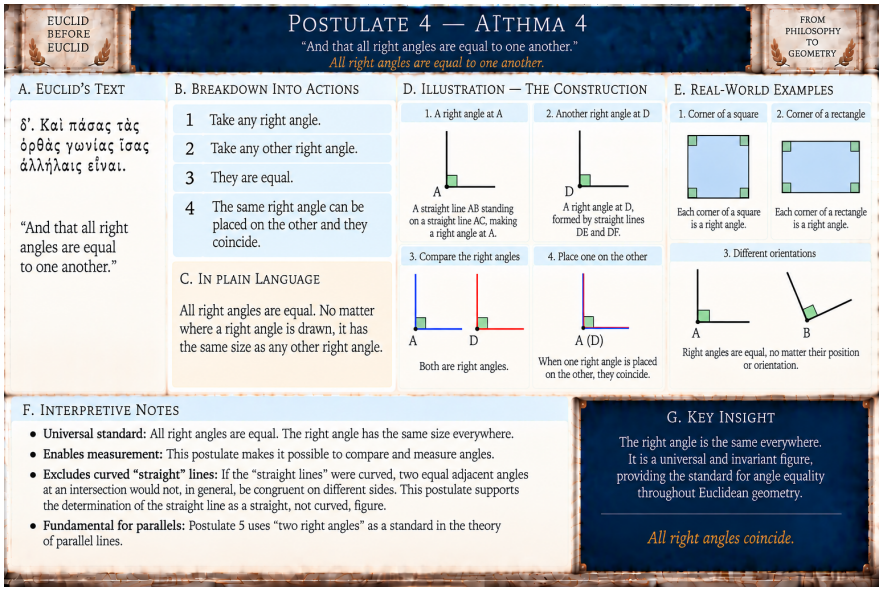



## Actions

1. Take any right angle (formed by a straight line standing on another, making equal adjacent angles).
2. Take any other right angle.
3. They are equal.

## Interpretation from the Commentary

This postulate:

- **Excludes curved "straight" lines:** If two curved lines intersected making equal adjacent angles, the angles on different sides would not be congruent.
- **Provides a universal standard:** The right angle is the same everywhere. This makes angle measurement possible.
- **Underlies the theory of parallels:** Postulate 5 uses "two right angles" as a standard.

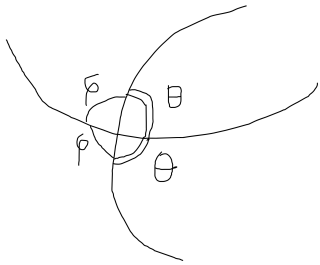
## The Commentary's Argument

Suppose the "straight line" were a curved line symmetric about every point. Let two such curved lines intersect. At their meeting, each produces adjacent angles equal to one another, so each may rightly be called a "right angle." Yet if we compare the upper and the lower, we find them unequal in shape and extent. Their curvature gives rise to differing inclinations. Thus, if the straight line were replaced by any other form, the equality of all right angles could not be maintained in a common measure. Postulate 4 excludes this possibility.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

| | |
|---|---|
| δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι. | 4. And that all right angles are equal to one another. |

This postulate strengthens further our conclusion that the straight lines cannot be bent lines. Let two lines, not straight but bending alike, intersect one another. At their meeting, each produces adjacent angles which are equal to one another, and hence each may rightly be called a right angle. Yet if we compare the upper and the lower, we find them unequal in shape and extent; for though each fulfils the definition—being composed of equal adjacent parts— their curvature gives rise to differing inclinations. Thus, if the straight line were replaced by any other form, the equality of all right angles could not be maintained in a common measure.



Two ”straight”(curvilinear) lines intersecting. The angles $\phi$ and $\theta$ each satisfy the condition of a right angle (each adjacent pair being equal), yet the two are not congruent, since curvature alters their local inclination.

### Integrated Ontological Restatement

**Source formulation.** *And that all right angles are equal to one another.*

**Ontological role.** Execute the assertion `assert_equal(all_right_angles)`. Guarantee: inclination is invariant under translation and rotation. If the plane had local curvature, right angles would differ in shape or extent from place to place. This postulate enforces global uniformity of turn, completing Definition 10 by ensuring that the equal division of adjacent angles produces the same result everywhere in the plane.

## Postulate 5

> *Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.*
> ***5. And that if a straight line falling across two other straight lines makes the internal angles on the same side less than two right angles, then the two other straight lines, being produced***

***to infinity, meet on that side on which the angles are less than two right angles.***

---

### Constraint C5

**Constraint, not command — and dormant.** C5 states when two produced lines must meet. It is first needed at **I.29**, for the converse inference from parallelism back to equal angles, and used once more at **I.44**. Nowhere else in Book I.

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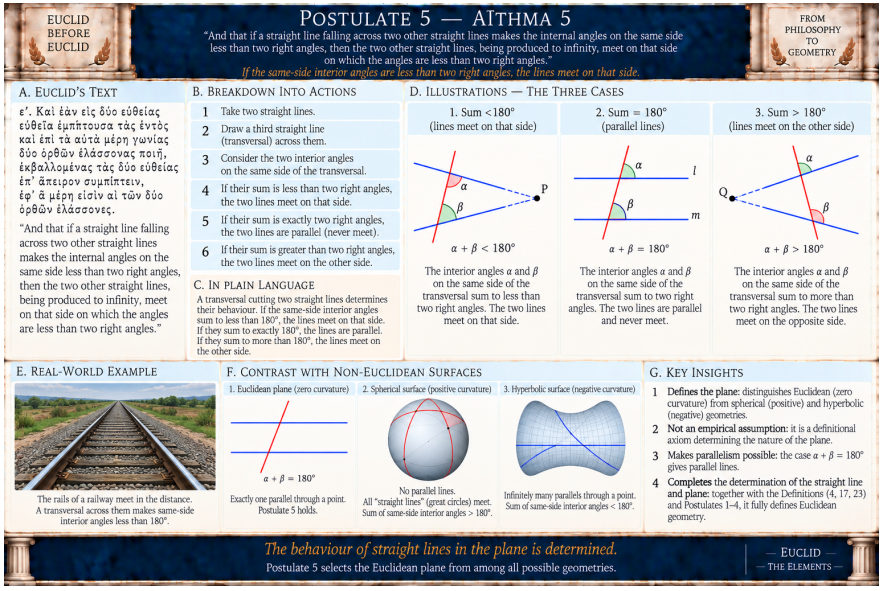


## Actions

1. Take two straight lines.
2. Draw a third straight line (transversal) across them.
3. Measure the two interior angles on the same side of the transversal.

4. If their sum is less than two right angles, the two lines will meet on that side.

5. If their sum is exactly two right angles, the two lines are parallel (never meet).

6. If their sum is greater than two right angles, the two lines meet on the other side.

## Interpretation from the Commentary

This postulate:

- **Defines the plane:** It distinguishes the Euclidean plane (zero curvature) from the sphere (positive curvature, where parallels don't exist) and the hyperbolic plane (negative curvature, where infinitely many parallels exist).
- **Is not an empirical assumption:** It is a definitional axiom that determines the nature of the plane itself.
- **Completes the determination of the straight line and plane:** With this postulate, the Euclidean plane is fully defined.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

5. And that if a straight line falling across two other straight lines makes the internal angles on the same side less than two right angles, then the two other straight lines, being produced to infinity, meet on that side on which the angles are less than two right angles.

This postulate serves to complete the very definition of the plane surface. Recall that the plane surface was defined as that in which the straight lines

lying upon it also lie evenly with the straight lines of the surface itself. Yet this definition gives no indication of the actual form of the plane; it tells us only how its lines behave, not what kind of surface they inhabit. For it is by the nature of its lines that a surface is truly determined.

Let us then set aside other possibilities. If the so-called plane surface were instead a portion of a sphere, the "parallel lines" upon it would be great circles. But upon the sphere, when two great circles are cut by a third and the sum of the interior angles on one side is less than two right angles, those circles do not meet again on that side. For every great circle is closed upon itself and returns to its beginning. They meet only on the opposite side, at the point antipodal to their first intersection.

Likewise, upon every surface that bends outward — that is, upon those of positive curvature — all straight lines, if continued far enough, will at length meet again. For such a surface turns back upon itself, and no two lines can be drawn upon it which remain everywhere equidistant. Hence, the postulate of parallels finds no place there, for there are in truth no parallels at all.

On surfaces of negative curvature, on the contrary, through a single point there pass infinitely many parallels that never meet a given line, and the postulate again fails. Therefore, this fifth postulate is not a theorem concerning arbitrary surfaces, but rather a statement that defines the plane itself — the unique surface on which these conditions are satisfied.

Spherical geometry was certainly known to Euclid, for Autolycus of Pitane (Αὐτόλυκος ὁ Πιταναῖος), who wrote the earliest surviving works on spherical astronomy and geometry, was his contemporary. Hence, the characteristics of curved (positively curved) and saddle-shaped (negatively curved) surfaces could not have been unknown to Euclid. It follows that the Fifth Postulate was not born of ignorance of such surfaces, but was deliberately framed to distinguish the plane from all other conceivable surfaces.

## Conclusion on the five postulates

The five postulates of Euclid were not arbitrary assumptions, but the axiomatic distillation of the nature of the plane.

- The first postulate, that a straight line may be drawn from any point to any point, determines the essence of linear connection firstly that there

is a direct straight line path between two distinct points and that the straight line cannot be closed or curved.

## Integrated Ontological Restatement

**Source formulation.** *And that if a straight line falling across two other straight lines makes the internal angles on the same side less than two right angles, then the two other straight lines, being produced to infinity, meet on that side on which the angles are less than two right angles.*

**Ontological role.** Execute the definitional filter: if a transversal crosses two lines and the sum of the internal angles on one side is less than two right angles, then the two lines converge and meet on that side. If the sum is exactly two right angles, the lines are parallel and never meet. If the sum is greater than two right angles, the lines would meet on the opposite side (but this case is excluded from the Euclidean plane). This postulate selects the unique surface on which this property holds. It excludes the sphere, where great circles meet on both sides (not just one). It excludes the hyperboloid, where lines can diverge and never meet even when the angle sum is less than two right angles. It excludes discrete spaces, where continuity fails. Postulate 5 therefore completes Definition 7: it tells us exactly what kind of surface we are in—the plane—and distinguishes it from all other conceivable surfaces.

## Critical Note: Axiom or Determination of the Domain?

The traditional history of the parallel postulate records repeated attempts to derive it from simpler-looking principles, from antiquity through Saccheri and Lambert.[**?**, **?**, **?**] The reconstruction does not dispute the logical independence questions studied by later geometry. It disputes the conclusion that independence makes Postulate 5 an embarrassment or a covert empirical hypothesis. Here Postulate 5 is read as the final planar filter: it specifies the surface on which the subsequent theory of parallels is to unfold. I.29 therefore depends on Postulate 5 openly and necessarily, just as a theorem about circles depends on the prior determination of what a circle is.

A historical objection once formulated in the supplied critical notes claimed that Euclid lacked awareness of curved geometries. That formulation is subsequently corrected in the same notes by reference to Greek spherical astron-

omy, including Autolycus and Euclid's *Phaenomena*. The strongest form of the debate is consequently not whether Euclid could conceive a sphere, but whether Postulate 5 should be read merely as a formal axiom or also as a determination selecting the plane. The present work defends the latter reading.

# Proposition I.1

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TOOLS IN HAND BEFORE THIS PROPOSITION

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [C5] still dormant

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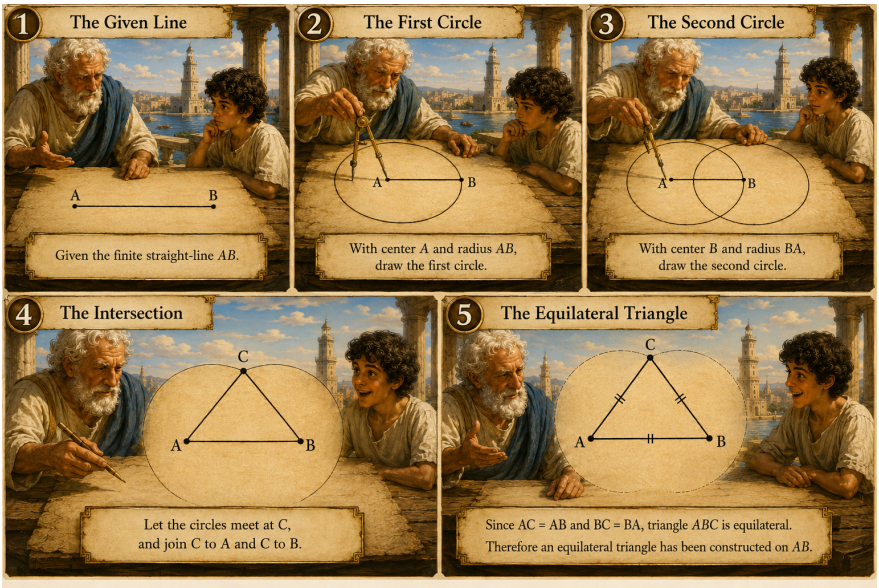


## Statement

*To construct an equilateral triangle on a given finite straight line.*

## Actions

1. Let AB be the given finite straight line.
2. With centre A and radius AB, describe the circle BCD (Postulate 3).
3. With centre B and radius BA, describe the circle ACE (Postulate 3).
4. The circles meet at C (see Interpretation below).
5. Join CA and CB (Postulate 1).
6. Since A is the centre of the circle through B and C, AC = AB (Definition 15).
7. Since B is the centre of the circle through A and C, BC = BA (Definition 15).
8. Therefore AC = AB = BC (Common Notion 1).
9. Therefore triangle ABC is equilateral (Definition 20).

## Interpretation from the Commentary

The intersection point C is **not assumed visually**. It is logically necessary:

- The circle with centre A and radius AB exists (Post. 3).
- The circle with centre B and radius BA exists (Post. 3).
- If we assume an equilateral triangle ABC exists, then AC = AB and BC = BA.
- Therefore the circle with centre A and radius AC coincides with the circle with centre A and radius AB (Common Notion 4).

- Similarly, the circle with centre B and radius BC coincides with the circle with centre B and radius BA.
- Therefore the two circles must meet at C.
- The existence of C is a logical consequence of the postulates and common notions, not a visual observation.

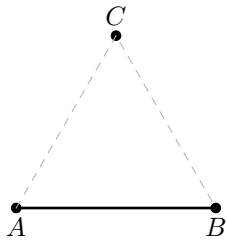
## Expanded Commentary from *Explaining Fully the First Book of Elements*

Many commentators have claimed that Euclid never truly proved that the two circles in Proposition I.1 intersect, and that he merely assumed their intersection from visual intuition [**?**, **?**, **?**, **?**, **?**].

This interpretation, however, is mistaken. The reasoning that ensures the intersection is implicit in the logical force of the postulates and the common notions.

**Reconstruction of the reasoning.**

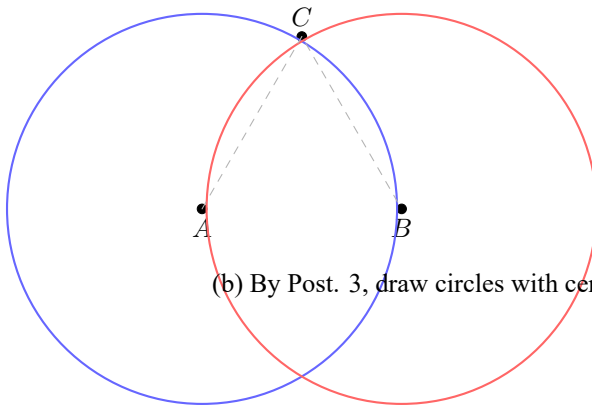
1. **Given:** a finite straight line $AB$. **To prove:** that on it an equilateral triangle can be constructed.
2. Assume that an equilateral triangle with base $AB$ and vertex $C$ exists.




(a) Given $AB$; assume equilateral $\triangle ABC$ exists.

3. By *Postulate 3*: "From any centre and with any radius, a circle may be drawn."
4. Hence, draw a circle with centre $B$ and radius $BC$.

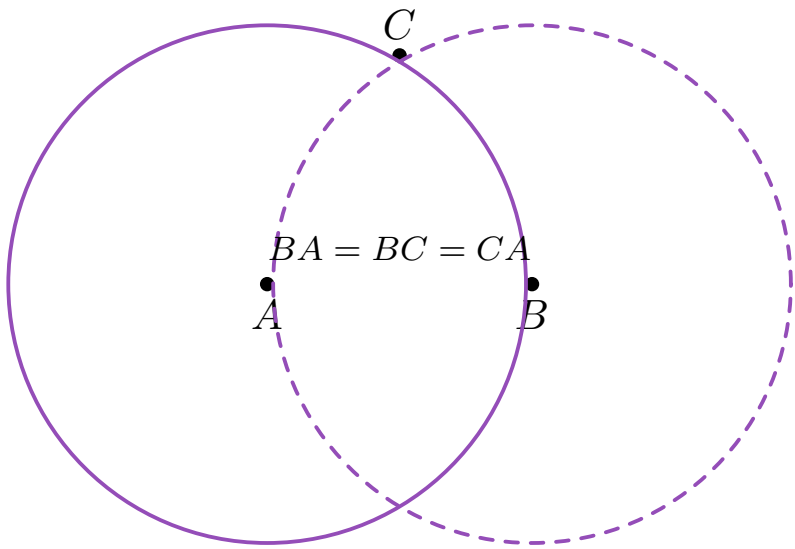
5. Draw another circle with centre $A$ and radius $AC$.




(b) By Post. 3, draw circles with centers $A$, $B$ and radii $AC$, $BC$.

6. These circles meet one another at the point $C$.

7. Take the circle with centre $B$ and radius $BA$. Since $BA = BC$ by hypothesis, this circle is identical with the circle of center $B$ and radius $BC$. By *Common Notion 4*: "Things which coincide with one another are equal to one another." Therefore, these two circles are the same.




(c) By Comm. Not. 4, circles which coincide are equal.

8. Similarly, take the circle with centre $A$ and radius $BA$. Since $BA = AC$, this circle coincides with the circle centered at $A$ with radius $AC$. By *Common Notion 4* again, they are the same.

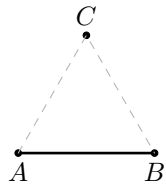
9. Since the circle centered at $A$ with radius $AC$ meets the circle centered at $B$ with radius $BC$ at $C$, it follows that the circles centered at $A$ and $B$, each with radius $BA$, also meet at $C$.

10. Therefore, the point $C$ exists not because it appears in the diagram, but because its existence is required by the logical coherence of *Postulate 3* and *Common Notion 4*, together with the hypothesis that $BA = BC = CA$.

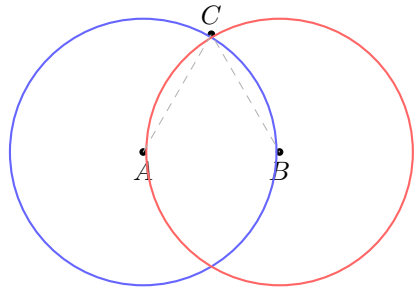
11. From here, the remainder of Proposition I.1 follows: the triangle $ABC$ thus constructed is equilateral.

**Philosophical insight.** This portion of the proof must once have been included in Euclid's original text, though it was likely lost in transmission through the centuries, particularly during the decline of mathematical study in the early Middle Ages. The present reconstruction restores that missing reasoning, transforming the opening of Euclid's *Elements* from a merely manual act of construction into a demonstration of pure logic. The intersection of the circles—the very birth of the point $C$—is not the work of the hand, but of the intellect. The existence of $C$ follows necessarily from the equality of the radii and from the common notion that circles with equal centers and equal radii coincide. Thus, geometry, even before it measures, already manifests the necessity of reason.

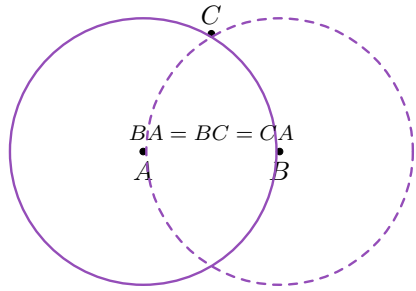
**Conclusion.** Hence, the intersection point $C$ is not postulated as visible, but as logically inevitable: it must exist wherever the postulates and common notions hold true. The proposition therefore stands not as an act of drawing, but as the first demonstration that geometry is the science of intelligible motion made visible through reason.



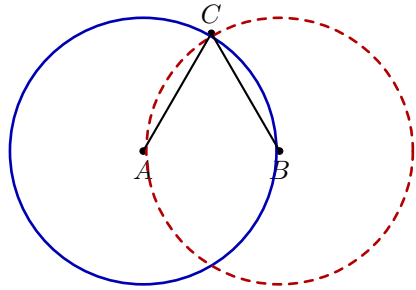

(a) Given $AB$; assume equilateral $\triangle ABC$ exists.




(b) By Post. 3, draw circles with centres $A$, $B$ and radii $AC$, $BC$.




(c) By Comm. Not. 4, circles with equal radii coincide.




(d) Hence circles meet necessarily at $C$, not by sight but by logic.

Figure 4: Logical reconstruction of Euclid's Proposition I.1. (a) Given $AB$ and assumed equilateral $\triangle ABC$. (b) Circles drawn by Postulate 3. (c) Equal radii imply coincidence by Common Notion 4. (d) Intersection at $C$ follows necessarily from the postulates.

## Integrated Ontological Restatement

**Source formulation.** *To construct an equilateral triangle on a given finite straight-line.*

**Ontological role.** Given a finite line AB. Draw circle with center A and radius AB. Draw circle with center B and radius BA. Their intersection C exists necessarily because the plane is complete (Postulate 5) and the circles are defined by constant radii (Definition 15). Join A to C and B to C. By Definition 15, AC equals AB and BC equals BA. By Common Notion 1, AC equals BC. Therefore, triangle ABC is equilateral (Definition 20).

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The standard objection is that the two circles with centres $A$ and $B$ are simply assumed to intersect at $C$, and that Euclid also presupposes uniqueness when the same extremities determine a straight line.

**Reply within the reconstruction.** The reconstruction answers the two points together. Postulate 3 makes each circle constructible from the given centre and radius; the plane in which those constructions occur has already been determined by Definitions 7 and 23 together with the postulates; and Common Notion 4 supplies the criterion of coincidence. Uniqueness is not introduced as an extra principle at I.1 but is read from the cumulative determination of the straight line, especially Definition 4 with Postulate 1. The analysis–synthesis objection is addressed in the methodological chapter above: analysis identifies the necessary construction, while the ontological semantics of the construction supplies its existence.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB$ be the given finite straight line. `[given]`

**2.** With centre $A$ and radius $AB$, describe the circle $BCD$. `[P3]`

**3.** With centre $B$ and radius $BA$, describe the circle $ACE$. `[P3]`

4. Let $C$ be a point in which the circles cut one another. [see panel below]
5. Join $CA$. [P1]
6. Join $CB$. [P1]
7. $AC = AB$, since $A$ is the centre of circle $BCD$. [Def. 15]
8. $BC = BA$, since $B$ is the centre of circle $ACE$. [Def. 15]
9. Therefore $AC = AB = BC$. [C.N. 1]
10. Therefore the triangle $ABC$ is equilateral. [Def. 20]

---

HOW DO YOU KNOW IT IS THERE?

**The point $C$.** Not found by looking. Circle $BCD$ exists by execution of [P3]; circle $ACE$ by a second execution of [P3]; each carries, by Definition 15, a constant distance from its own centre. $C$ is not an extra object requiring its own certificate — it is what two such loci, each within the other's reach, necessarily share. **Why the reach matters.** $A$ lies on circle $ACE$ and $B$ on circle $BCD$: each centre lies inside the other circle. A locus at constant distance from $B$ cannot pass from the interior of the other to its exterior without sharing a point with it. **The honest residue.** A modern axiomatisation states circle–circle intersection explicitly, and this edition does not pretend otherwise. What it denies is that the step is an *unnoticed* or a *visual* assumption. It is a consequence of Definition 15 applied in a plane the postulates have already determined.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.1 Erect an equilateral triangle on a given finite straight line.

The first earned action — and note that [P2] was never used. The opening proposition of the *Elements* extends nothing.

# Proposition 2

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [C5] still dormant

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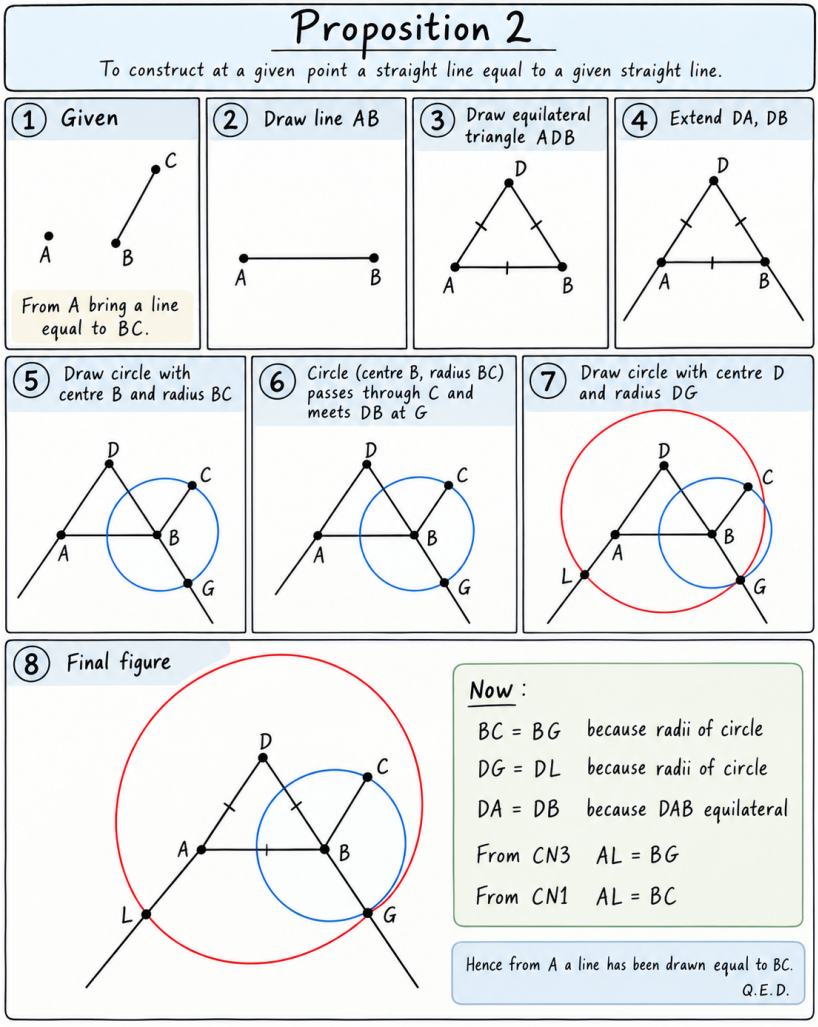


## Statement

*To place at a given point a straight line equal to a given straight line.*

## Actions

1. Let A be the given point and BC the given straight line.
2. Join AB and construct the equilateral triangle DAB on AB by Proposition I.1.
3. Produce DA and DB beyond A and B respectively.
4. With centre B and radius BC describe a circle cutting the produced line DB at G; thus BG=BC.
5. With centre D and radius DG describe a circle cutting the produced line DA at L.
6. Since DA=DB and DG=DL, subtract equals from equals to obtain AL=BG=BC.
7. Thus the straight line AL placed at A equals the given straight line BC.

## Integrated Ontological Restatement

**Source formulation.** *To place a straight-line equal to a given straight-line at a given point.*

**Ontological role.** Given a point A and a line BC. Draw line AB. Construct equilateral triangle DAB on AB (Proposition 1). Extend DA and DB (Postulate 2). Draw circle with center B and radius BC. Draw circle with center D and radius DG. By equality of radii and subtraction (Common Notion 3), AL equals BC. Thus, AL is the required line at A.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The alleged gap is again continuity: the circles introduced in the transfer construction are said to meet without a separately stated circle-intersection axiom.

**Reply within the reconstruction.** Within this reconstruction I.2 inherits the existence result established at I.1. Postulate 3 universally executes circle

construction, and the intersections used in the transfer are instances of the same plane-and-circle structure rather than new primitive assumptions.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $A$ be the given point and $BC$ the given straight line. `[given]`
2. Join $AB$. `[P1]`
3. On $AB$ erect the equilateral triangle $DAB$. `[I.1]`
4. Produce $DA$ beyond $A$. `[P2]`
5. Produce $DB$ beyond $B$. `[P2]`
6. With centre $B$ and radius $BC$ describe a circle cutting $DB$ produced at $G$, so $BG = BC$. `[P3, Def. 15]`
7. With centre $D$ and radius $DG$ describe a circle cutting $DA$ produced at $L$. `[P3, Def. 15]`
8. $DA = DB$, sides of the equilateral triangle. `[I.1, Def. 20]`
9. $DL = DG$, radii of the second circle. `[Def. 15]`
10. Taking equals from equals, $AL = BG$. `[C.N. 3]`
11. But $BG = BC$; therefore $AL = BC$. `[C.N. 1]`

---

HOW DO YOU KNOW IT IS THERE?

**The equilateral triangle $DAB$.** Not decorative. It exists solely to supply $DA = DB$, the pair of equals from which the subtraction at step 10 can be made. **The equality $AL = BC$.** No length was ever transported. Two subtractions did the work an illegitimately travelling compass would have done. This is why `[P3]` may be read as a collapsing compass without loss: the non-collapsing compass is *derived* here, not assumed.

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Added to the repertoire by this proposition

I.2 Place at a given point a straight line equal to a given straight line.

From here a length may be carried anywhere, and every such move cites this proposition.

# Proposition I.3

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [C5] still dormant

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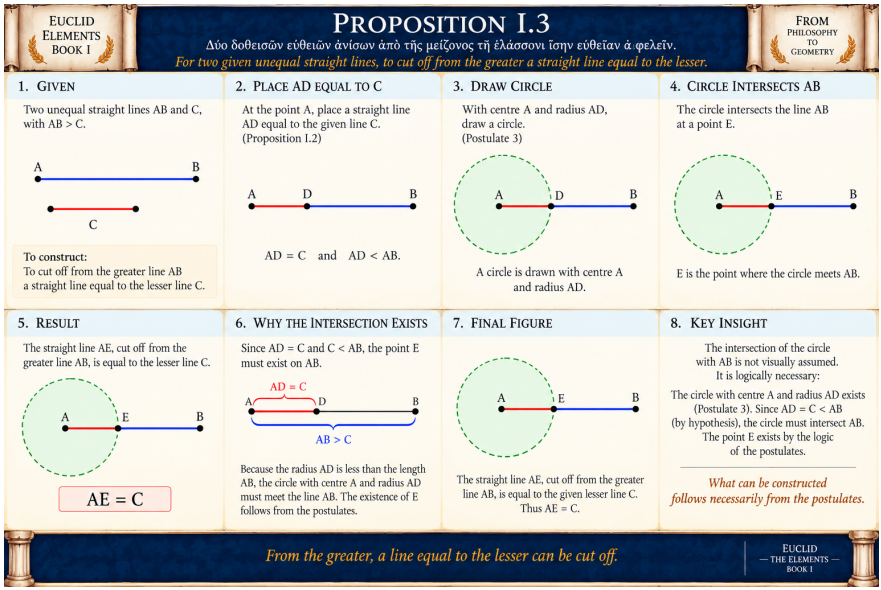


## Statement

> *For two given unequal straight lines, to cut off from the greater a straight line equal to the lesser.*

## Actions

1. Given: two unequal straight lines AB and C, with AB > C.
2. Place AD equal to C at point A (Proposition I.2).
3. Draw circle with centre A and radius AD (Postulate 3).
4. The circle intersects AB at E.
5. AE = AD = C (Definition 15).

## Interpretation from the Commentary

The intersection of the circle with AB is **not visually assumed**. It is logically necessary:

- The circle with centre A and radius AD exists (Post. 3).
- AD = C < AB (by hypothesis).
- The circle must intersect AB because the radius is less than the length of AB.
- The intersection point E exists by the logic of the postulates.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Statement:** To cut off from the greater of two unequal straight lines a part equal to the lesser.

**Common interpretation:** It is often claimed that Euclid assumes, without proof, that a circle and a straight line intersect, relying on visual intuition rather than on the postulates.

**Critics:**

- **Błaszczyk & Petiurenko (2022)** highlight that Euclid does not explicitly prove the existence of intersection points and therefore relies on diagrammatic evidence.[3]
- **Beeson (2019)** introduces an explicit intersection axiom in his formal reconstruction to ensure that circles meet straight lines when required.[4]

We disagree completely because Euclid provides all the necessary postulates and common notions for that. **Logical reconstruction:**

[3] P. Błaszczyk and M. Petiurenko, "Existence assumptions in Euclid's *Elements*," *Foundations of Science*, 27 (2022), 665–685.

[4] M. Beeson, "Euclid, Hilbert, and constructive geometry," *Review of Symbolic Logic*, 12 (2019), 403–467.

Let $AB$ and $C$ be two unequal straight lines, and let $AB > C$.

1. By *Postulate 3*, we can construct a circle with centre $A$ and radius $AK = C$.

2. By *Common Notion 5*, if $AB$ were a part of $AK$, then $AK > AB$, which does not hold by hypothesis.

3. By *Postulates 1 and 2*, we can project $AK$ onto $AB$, because by *Postulate 1* we can have a straight line $AB$, and by *Postulate 2* we can produce $AK$ further along $AB$.

4. Since $K$ is part of the circle and part of $AB$, the circle intersects the straight line $AB$.

Thus Euclid, completely logically and without any visualization, constructs the intersection. The rest of the proof then proceeds exactly as in the original text.

## Integrated Ontological Restatement

**Source formulation.** *For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.*

**Ontological role.** Given greater AB and lesser C. Place AD equal to C at point A (Proposition 2). Draw circle with center A and radius AD. The circle cuts AB at E (guaranteed by the completeness of the plane). Then AE equals C. This is the operation of subtraction by construction.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The criticism is that the proof assumes one can choose on the greater straight line a point at exactly the required distance, thereby using an unstated metric or order principle.

**Reply within the reconstruction.** I.3 does not select such a point by visual approximation. Proposition I.2 first constructs at the given point a segment

equal to the lesser segment; Postulate 3 then draws the corresponding circle. The cut is generated as the circle–line intersection, while Common Notion 5 supplies the whole–part order needed to distinguish the greater from the lesser.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB$ and $C$ be the given unequal lines, $AB$ the greater. `[given]`

**2.** At the point $A$ place $AD$ equal to $C$. `[I.2]`

**3.** With centre $A$ and radius $AD$ describe the circle $DEF$, cutting $AB$ at $E$. `[P3, Def. 15]`

**4.** $AE = AD$, radii of the same circle. `[Def. 15]`

**5.** But $AD = C$; therefore $AE = C$. `[C.N. 1]`

---

HOW DO YOU KNOW IT IS THERE?

**The brevity.** Three steps, because `[I.2]` has already been paid for. Attempted with the postulates alone, I.3 cannot be done at all.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

`I.3` Cut off from the greater of two unequal lines a part equal to the less.

With I.2 and I.3 in hand, lengths may be both carried and matched. This is exactly — and only — what Proposition I.4 will need.

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# Proposition I.4

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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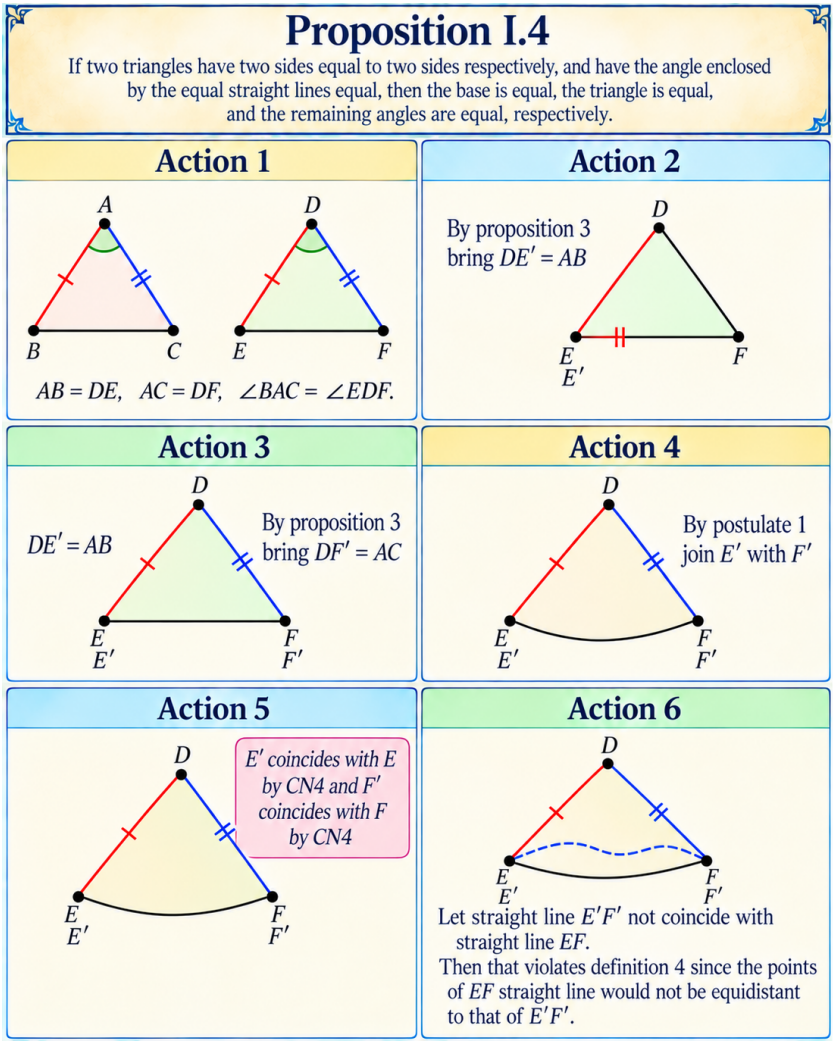



## Statement

*If two triangles have two sides equal to two sides, respectively, and have the angle enclosed by the equal straight lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles*

*subtended by the equal sides will be equal to the corresponding remaining angles.*

## Actions

1. Let triangles ABC and DEF with AB = DE, AC = DF, and $\angle BAC = \angle EDF$.
2. By Proposition 3, bring on DE a length equal to AB and call it DE'.
3. By Proposition 3, bring on DF a length equal to AC and call it DF'.
4. Join E' and F' with a straight line.
5. E' coincides with B and F' coincides with C.
6. Let the straight line E'F' not coincide with the straight line BC. Then that violates Definition 4, since the points of BC straight line would not be equidistant to that of E'F'.

## Interpretation from the Commentary

The traditional "superposition" is replaced by **logical construction**. The equal sides are constructed from D using Propositions I.2 and I.3. The equal included angle ensures that the constructed triangle corresponds exactly to the original. The coincidence of the bases follows from Definition 4. No physical motion is required.

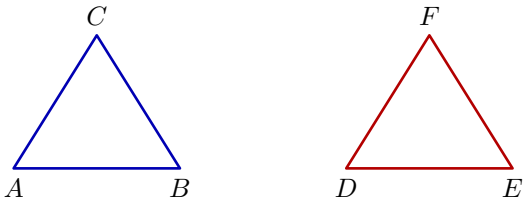
## Expanded Commentary from *Explaining Fully the First Book of Elements*

Many commentators have claimed that Euclid never truly proved Proposition I.4, but rather relied upon an unproved act of *superposition* [**?**, **?**, **?**, **?**, **?**].

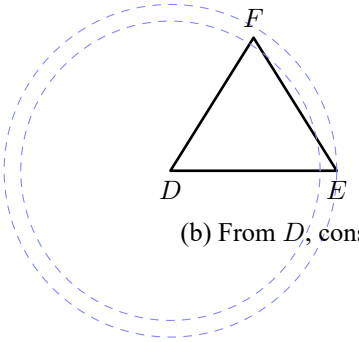
Yet a careful reading of the preceding propositions shows that no such external operation is required. The proof can be understood as proceeding purely from the Definition 4, and the preceding *Propositions I.2* and *I.3*.

Let two triangles, $\triangle ABC$ and $\triangle DEF$, be given such that $AB = DE$, $AC = DF$, and the included angle $\angle BAC$ equals the included angle $\angle EDF$.

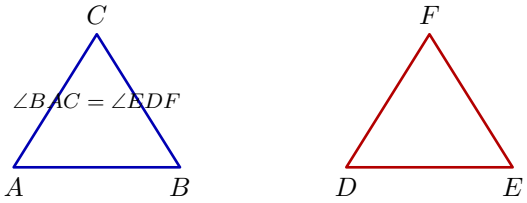
By *Proposition I.2*, it is possible to construct from any given point a straight line equal in length to any given straight line. Hence, from point $D$, we can construct a line equal to $AB$, and similarly from point $D$ we can construct a line equal to $AC$. By the same reasoning as in *Proposition I.3*, from center $D$ and with radius $AB$ we may draw a circle cutting the line $DE$ (or its extension, by *Postulate 2*) at the point where the line equal to $AB$ is completed. Likewise, from center $D$ and with radius $AC$ we draw a circle cutting the line $DF$ (or its extension) at the point where the line equal to $AC$ is completed.



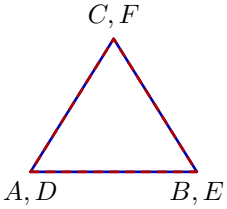

(a) Given $\triangle ABC$ and $\triangle DEF$ with equal sides and included angle.




(b) From $D$, construct lines equal to $AB$ and $AC$ (Prop. I.2), draw circles (Pr




(c) Equal included angles and sides imply triangles correspond exactly.




(d) If $BC$ and $EF$ did not coincide, they would enclose area—contradiction.

Figure 5: Logical reconstruction of Euclid's Proposition I.4 without superposition. (a) Two triangles with equal sides and included angle. (b) Equal sides constructed from a given point (Props. I.2–3). (c) Equal angles ensure corresponding position. (d) Bases coincide by Definition 4; triangles are equal in all respects.

Thus the lines drawn from $D$ coincide in direction and in length with $DE$ and $DF$, respectively. Because the angle between $AB$ and $AC$ is equal to the angle between $DE$ and $DF$, the triangle constructed upon $D$ by these two equal sides corresponds exactly to the triangle upon $A$. This, and nothing more, is what Euclid means by saying that "triangle $ABC$ is applied to triangle $DEF$."

Suppose, contrary to the claim, that the bases $BC$ and $EF$ do not coincide. Then the two straight lines $BC$ and $EF$ would enclose an area. But this is impossible, for by *Definition 4*—that a straight line "lies evenly with the points(of the straight line) on itself"—the points of one line cannot lie at unequal distances from those of another straight line with which its extremities coincide. Hence, since points $B$ and $E$ coincide and the lines $BC$ and $EF$ both lie evenly with the same straight line through them, all the points of $BC$ and $EF$ must coincide.

Therefore, the triangles $\triangle ABC$ and $\triangle DEF$ coincide in all respects without the need for superposition. Their equality follows directly from the constructions of *Propositions I.2* and *I.3* together with the meaning of *Definition 4*. Thus Euclid's reasoning is entirely logical, and the use of "application" is not a physical act but an intelligible correspondence between two constructions on the same foundation.

*The proof therefore stands self-contained within the axioms of the Elements: no movement, no superposition, and no appeal to visual intuition are required.*

Modern commentators such as Heath [**?**], Mueller [**?**], De Risi [**?**], Beeson [**?**], and Hilbert [**?**] have frequently maintained that Euclid's Proposition I.4 depends upon an extra–geometrical operation of *superposition*—the physical placing of one triangle upon another—thus introducing into geometry an act of motion not warranted by the postulates. However, when the proposition is read in the light of the preceding Propositions I.2 and I.3, together with the true meaning of *Definition 4* ("a straight line lies evenly with the points on itself"), it becomes clear that no appeal to motion is required. The construction of equal sides from a given point (Prop. I.2), combined with the generation of equal circles (Prop. I.3), suffices to establish a triangle at $D$ whose sides and included angle correspond exactly to those at $A$.

The coincidence of the bases then follows necessarily from Definition 4, for two straight lines sharing their extremities and lying evenly with the same points cannot enclose an area.

Thus the proposition is not a physical act of superposition but a logical application: a correspondence of constructions grounded in Euclid's own postulates and definitions. In this light, Euclid's reasoning stands complete and self–consistent. The so–called "superposition" is revealed to be merely a later misreading of a synthetic operation that already exists in the Elements as a consequence of Propositions I.2 and I.3. What appears to modern eyes as kinematic becomes, in Euclid's own idiom, a pure act of geometry: the meeting of two equal constructions in one intelligible form.

## Integrated Ontological Restatement

**Source formulation.** *If two triangles have two sides equal to two sides, respectively, and have the angle encompassed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles will be equal to the corresponding remaining angles.*

**Ontological role.** Given triangles ABC and DEF with AB = DE, AC = DF, and angle BAC = angle EDF. From D, construct lines equal to AB and AC (Propositions 2 and 3) and set them at the same angle (Proposition 23). These lines coincide with DE and DF. By Definition 4 cumulatively determined, two straight lines sharing two points must be identical; therefore, BC coincides with EF. By Common Notion 4, the bases are equal. The triangles coincide entirely, and all corresponding parts are equal. No physical superposition is required; the congruence follows from construction and the relational definition of straightness.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The traditional criticism treats Euclid's application of one triangle to another as physical superposition and therefore as an unstated rigidity or motion axiom.

**Reply within the reconstruction.** The reconstruction replaces physical transport by constructional correspondence. I.2 and I.3 transport equal lengths; the equal included angle fixes the corresponding rays; the cumulative Definition 4 and Postulate 1 secure uniqueness of the straight connection; and Common Notion 4 identifies coincident figures as equal. The equality is therefore logical coincidence, not preservation under bodily motion.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$, $DEF$ have $AB = DE$, $AC = DF$, $\angle BAC = \angle EDF$. `[given]`
2. On $DE$ cut off $DE'$ equal to $AB$. `[I.3]`
3. On $DF$ cut off $DF'$ equal to $AC$. `[I.3]`
4. Join $E'F'$. `[P1]`
5. Since $AB = DE$ and $DE' = AB$, the point $E'$ falls on $E$. `[C.N. 1]`
6. Since $AC = DF$ and $DF' = AC$, the point $F'$ falls on $F$. `[C.N. 1]`
7. Suppose $E'F'$ did not coincide with $BC$. Then two straight lines with the same extremities would not lie evenly with one another's points. `[Def. 4]`
8. That is impossible; therefore the bases coincide and $EF = BC$. `[Def. 4]`
9. The triangles coincide, and the remaining angles are equal respectively. `[C.N. 4]`

How do you know it is there?

**Nothing moves.** The traditional proof asks you to *apply* one triangle to the other. That is an action, and it is in nobody's ledger: no postulate grants rigid motion. The construction above moves nothing — it *builds* the second triangle onto the first with two executions of [I.3], earned one proposition earlier. **The coincidence of the bases.** The step that replaces superposition, and it is pure Definition 4. Two straight lines are given with the same two extremities; if they did not coincide, the points of the one would not lie evenly with the points of the other — which, read relationally, is precisely what straightness forbids. **The wider consequence.** Definition 4, called vacuous by commentators and said never to be used by Euclid, is the load-bearing member at the first congruence theorem in the corpus. It is used not as a description but as a *criterion*: the test by which two straight lines sharing extremities are judged to be one.

Added to the repertoire by this proposition

**Warrant I.4 (side–angle–side).** Two triangles agreeing in two sides and the included angle agree in everything. No new action. I.4 draws nothing you could not already draw; it licenses a conclusion. This is the first entry in the second ledger. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition I.5

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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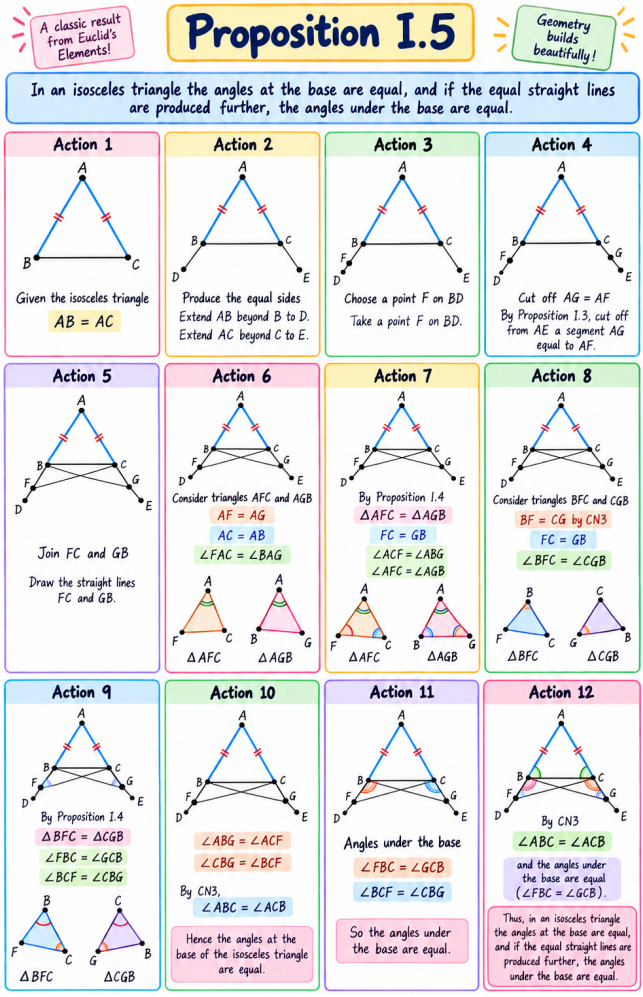


## Statement

*For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.*

## Actions

1. Let triangle ABC with AB = AC.
2. Produce AB beyond B to D, and AC beyond C to E (Postulate 2).
3. Take a point F at random on BD.
4. By Proposition 3, cut off from AE a length AG = AF.
5. Join FC and GB (Postulate 1).
6. Consider triangles AFC and AGB: AF = AG, AC = AB, $\angle FAC = \angle GAB$ → triangles congruent (I.4) → FC = GB, $\angle ACF = \angle ABG$, $\angle AFC = \angle AGB$.
7. Consider triangles BFC and CGB: BF = CG (CN3), FC = GB (Action 6), $\angle BFC = \angle CGB$ (Action 6) → triangles congruent (I.4) → $\angle FBC = \angle GCB$, $\angle BCF = \angle CBG$.
8. Derive $\angle ABC = \angle ACB$: $\angle ABG = \angle ACF$ (Action 6), $\angle CBG = \angle BCF$ (Action 7) → $\angle ABG - \angle CBG = \angle ACF - \angle BCF$ → $\angle ABC = \angle ACB$.
9. Derive $\angle FBC = \angle GCB$ (angles under the base) from Action 7.

## Interpretation from the Commentary

This proposition does not rely on visual symmetry. It follows from Proposition I.4 (which is logically reconstructed) and the Common Notions.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** This proposition is often said to rely on the visual symmetry of the diagram and on the use of superposition from Proposition I.4. It is also claimed that Euclid assumes that equal sides imply equal base angles on the same side, which presupposes orientation.

**Critics:**

- **Manders (2008)** shows that the reasoning in Proposition I.5 depends on "diagrammatic reasoning" rather than on axiomatic inference.
- **Błaszczyk & Petiurenko (2022)** classify it as a typical example of visual *reductio*.

**Logical reconstruction:** I disagree with the above criticisms. Firstly, the validity of Proposition I.4 has been established by using the amended Definition 4. Secondly, the following reasoning can be applied.

Let there be two straight lines $\varepsilon_1$ and $\varepsilon_2$ forming an angle $\varphi$ at point $A$ and produced continuously. Let there be another straight line $\varepsilon_3$ of length $a$. By Proposition I.3, construct on $\varepsilon_1$ the segment $AB = a$ and on $\varepsilon_2$ the segment $AC = a$. Let $\beta > a$. By Proposition I.3, construct $AZ$ on $\varepsilon_1$ and $AH$ on $\varepsilon_2$. By *Common Notion 5*, $AB$ is part of $AZ$ and $AC$ is part of $AH$. By Definitions 4, 17, and 23, and by Postulates 1, 2, and 4, we know the nature of the straight line; and by Postulate 4 we know the definition of the plane. By Postulate 1, draw $ZC$, and by Postulate 2, draw $HB$. The rest of the proof then follows.

## Integrated Ontological Restatement

**Source formulation.** *In isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced, the angles under the base will be equal to one another.*

**Ontological role.** Given triangle ABC with AB = AC. Extend AB and AC to BD and CE (Postulate 2). Take point F on BD and cut AG equal to AF on AE (Proposition 3). Join FC and GB. Triangles AFC and AGB are congruent (Proposition 4). Therefore, FC = GB, and angle ACF = angle ABG. By subtraction of equal parts (Common Notion 3), BF = CG. Triangles BFC and CGB are congruent. Therefore, angle FBC = angle GCB (angles under the base) and angle ABC = angle ACB (angles at the base).

# Critical Apparatus: Alleged Assumption and Reply

**Objection.** A modern order-based reading may regard the choice of a point 'at random' on a produced line as an unaxiomatized existence and betweenness assumption.

**Reply within the reconstruction.** Postulate 2 makes production of the straight line continuously executable, so a further point on the produced line is not imported from the diagram. The subsequent equality construction is supplied by I.3. The position of the point is thus generated by operations already licensed in the system.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$ have $AB = AC$. [given]
2. Produce $AB$ to $D$ and $AC$ to $E$. [P2]
3. On $BD$ take any point $F$; cut off $AG$ on $AE$ equal to $AF$. [I.3]
4. Join $FC$ and $GB$. [P1]
5. Triangles $AFC$, $AGB$ have $AF = AG$, $AC = AB$, and the angle at $A$ common; so $FC = GB$ and the corresponding angles are equal. [I.4]
6. Subtracting the equal triangles $BFC$, $CGB$ leaves the base angles equal. [C.N. 3]

---

HOW DO YOU KNOW IT IS THERE?

**The points $F$ and $G$.** $F$ is taken freely on a line already produced by [P2]; $G$ is not free but determined, being cut off equal to $AF$ by [I.3]. The asymmetry is deliberate: one point may be arbitrary, the other must be earned.

---

Added to the repertoire by this proposition

**Warrant I.5.** An isosceles triangle has equal base angles, and equal angles below the base. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 6

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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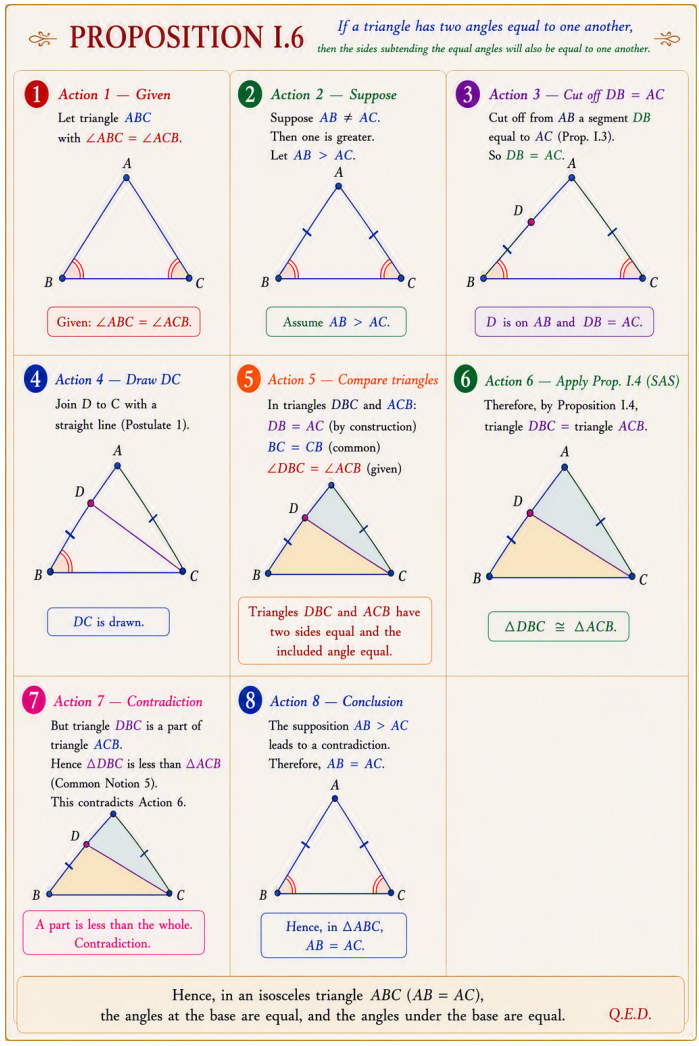


## Statement

*If two angles of a triangle are equal, then the sides subtending them are equal.*

## Actions

1. Let triangle ABC have angle ABC equal to angle ACB.
2. Assume, for contradiction, that AB is not equal to AC; let AB be greater.
3. Cut off DB from AB equal to AC and join DC.
4. Triangles DBC and ACB have DB=AC, BC common, and included angle DBC=ACB; by I.4 they are equal.
5. Then triangle DBC would equal triangle ACB although it is a proper part of it, which is impossible.
6. Therefore AB=AC.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** This proposition involves the subtraction of equal magnitudes and the use of inequality reasoning that Euclid never formalized explicitly. It also appears to require an order or betweenness axiom that is only implicit in the *Elements*.

**Critics:**

- **Hilbert (1899)** introduced explicit order axioms in his *Grundlagen der Geometrie* precisely to handle such arguments involving inequality and subtraction.
- Later formalists, such as **Tarski (1959)** and **Beeson (2019)**, have shown that Euclid's original language cannot justify statements of the form "greater than" without additional assumptions.

**Logical reconstruction:** I disagree with the above criticisms. Let there be a triangle $ABC$ such that $\widehat{ABC} = \widehat{ACB}$. By Proposition I.3, construct on side $CA$ a segment $CD$. Both $AD$ and $CD$ are parts of $AC$ and are therefore less than $AC$ by *Common Notion 3*. By the definition of the plane in *Postulate 5*, and of the straight line in *Definitions 4, 17,* and *23*, together with

*Postulates 1, 2, 4,* and *5*, we do not require any additional betweenness or order axioms. The reasoning follows entirely from the Euclidean postulates themselves.

## Integrated Ontological Restatement

**Source formulation.** *If a triangle has two angles equal to one another, the sides subtending the equal angles will also be equal to one another.*

**Ontological role.** Given triangle ABC with angle ABC = angle ACB. Suppose AB is not equal to AC. Then one is greater. Let AB > AC. Cut DB from AB equal to AC (Proposition 3). Join DC. Then triangle DBC has two sides equal to AC and BC, and angle DBC = angle ACB (given). By Proposition 4, triangle DBC equals triangle ACB, the part equal to the whole, impossible by Common Notion 5. Therefore, AB = AC.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The same family of criticism attributes the reductio to an unstated order or betweenness structure.

**Reply within the reconstruction.** The reductio uses the explicit whole–part ordering of Common Notion 5 together with the constructive subtraction supplied by I.3 and congruence already established at I.4. The reconstruction therefore treats the relevant order as operationally generated, not as an independent primitive.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$ have $\angle ABC = \angle ACB$; suppose $AB \neq AC$, and let $AB$ be greater. `[given]`
2. Cut off $DB$ equal to $AC$. `[I.3]`
3. Join $DC$. `[P1]`

4. Triangles $DBC$, $ACB$ have $DB = AC$, $BC$ common, and the included angles equal; so they are equal. [I.4]

5. But the lesser equals the greater, which is impossible; so $AB = AC$. [C.N. 5]

---

HOW DO YOU KNOW IT IS THERE?

**The point $D$.** Output of [I.3] on a line already given. The reductio does not conjure a point: it constructs one and then shows the construction is absurd.

---

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.6.** Equal base angles imply equal subtending sides. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 7

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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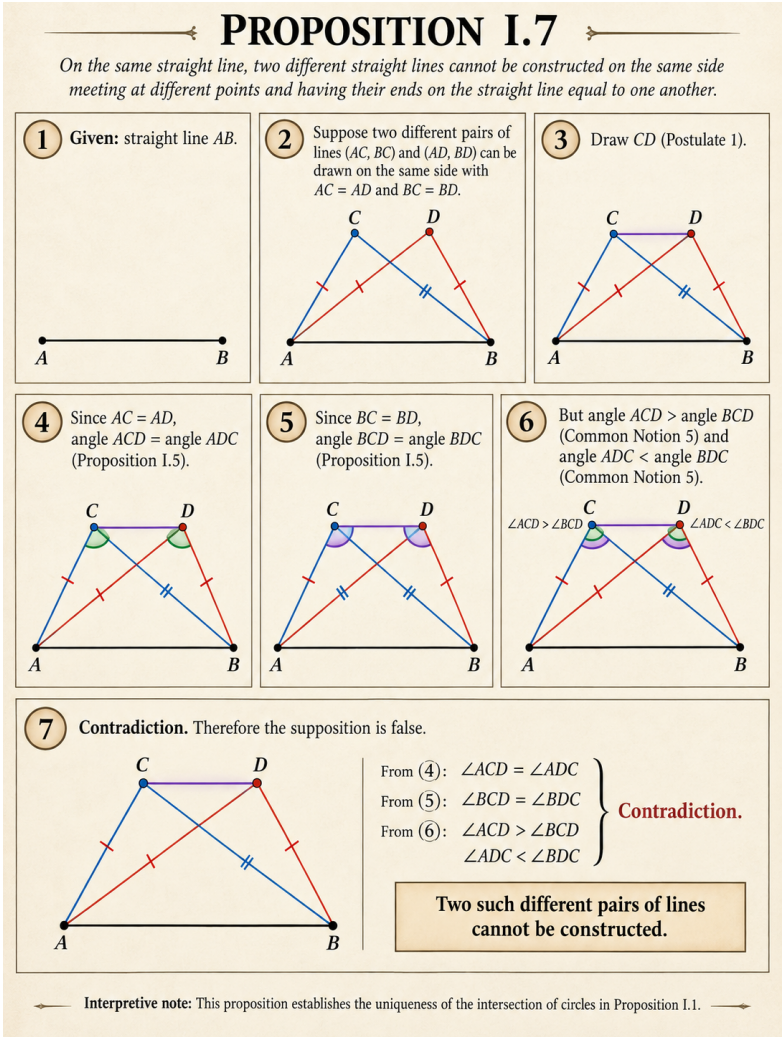


## Statement

*On the same base and on the same side of it there cannot be two distinct triangles having the two sides from the endpoints equal respectively.*

## Actions

1. Suppose two distinct points C and D lie on the same side of AB with AC=AD and BC=BD.
2. Join CD.
3. From AC=AD, triangle ACD is isosceles, so its base angles ACD and CDA are equal by I.5.
4. From BC=BD, triangle BCD is isosceles, so angles BCD and CDB are equal.
5. The resulting angle relations force a greater angle to equal a lesser angle, an impossibility.
6. Therefore the two vertices cannot be distinct.

## Integrated Ontological Restatement

**Source formulation.** *On the same straight-line and at the same point, two other straight-lines cannot be constructed on the same side having the same extremities as the initial straight-lines.*

**Ontological role.** Suppose triangles ACB and ADB are on the same base AB and on the same side, with AC = AD and BC = BD. If AC = AD, then angle ACD = angle ADC (Proposition 5). If BC = BD, then angle BCD = angle BDC (Proposition 5). But angle ACD is greater than angle BCD, while angle ADC is less than angle BDC—a contradiction. Therefore, the two triangles cannot exist; C and D must be the same point.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The objection combines two alleged gaps: uniqueness of straight lines and an unstated inside/outside ordering of the two candidate vertices.

**Reply within the reconstruction.** The uniqueness component is precisely what the cumulative reading of Definition 4 and Postulate 1 is designed to establish. The side on which the figures are constructed is part of the proposition's stated configuration, while the contradiction is obtained from already established angle equalities and whole–part comparison rather than from an independent topological axiom.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Suppose $AC = AD$ and $BC = BD$, with $C$, $D$ distinct and on the same side of $AB$. `[hypothesis]`
2. Join $CD$. `[P1]`
3. From $AC = AD$, the angles $ACD$, $ADC$ are equal. `[I.5]`
4. From $BC = BD$, the angles $BCD$, $BDC$ are equal. `[I.5]`
5. The two results cannot hold together, since one angle is part of the other and so less than it. `[C.N. 5]`

---

HOW DO YOU KNOW IT IS THERE?

**The point $D$.** Not constructed at all — assumed, in order to be refuted. This is the first proposition in Book I whose figure is deliberately impossible.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.7.** On the same base and the same side, two given lengths determine at most one apex. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 8

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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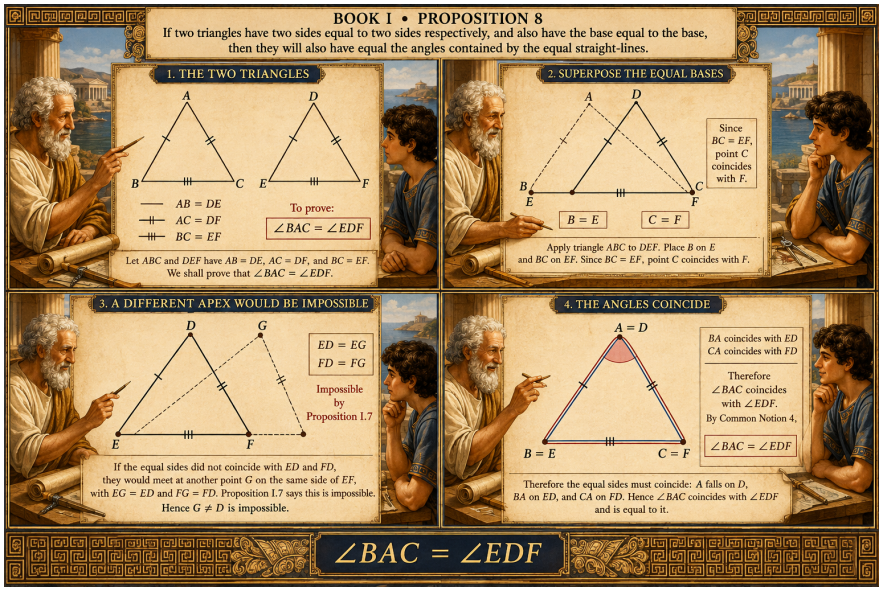


## Statement

> *If two triangles have all three corresponding sides equal, then the included corresponding angles are equal.*

## Actions

1. Let triangles ABC and DEF satisfy AB=DE, AC=DF, and BC=EF.
2. Apply triangle ABC to triangle DEF with B on E and C on F, so base BC coincides with EF.
3. By Proposition I.7, since the remaining two sides from the endpoints are equal respectively, the third vertex A must coincide with D.
4. Therefore BA lies on ED and CA on FD.
5. Hence angle BAC equals angle EDF; similarly the triangles coincide in all respects.

# Integrated Ontological Restatement

**Source formulation.** *If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have the angle encompassed by the equal straight-lines equal.*

**Ontological role.** Given triangles ABC and DEF with AB = DE, AC = DF, and BC = EF. Place triangle ABC on DEF such that B coincides with E and BC coincides with EF. If BA does not coincide with ED, then on the same base EF and on the same side, two different straight-lines would have the same extremities, contradicting Proposition 7. Therefore, angle BAC = angle EDF.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$, $DEF$ have $AB = DE$, $AC = DF$, $BC = EF$. `[given]`
2. Apply the base $BC$ to the base $EF$, so the extremities agree. `[Def. 4, C.N. 4]`
3. If the apexes did not agree, there would be two distinct apexes over one base with the same two lengths. `[hypothesis]`
4. That is impossible. `[I.7]`
5. Therefore the apexes agree and the contained angles are equal. `[C.N. 4]`

---

How do you know it is there?

**The coincidence of the bases.** As at I.4, this is Definition 4 doing the work, not motion. Two straight lines with the same extremities and the same length lie evenly with one another, hence coincide. **The coincidence of the apexes.** Forced by `[I.7]`, which is why I.7 had to come first. Book I proves uniqueness before it proves congruence.

---

Added to the repertoire by this proposition

**Warrant I.8.** Three equal sides force the equality of all corresponding angles. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 9

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [C5] still dormant

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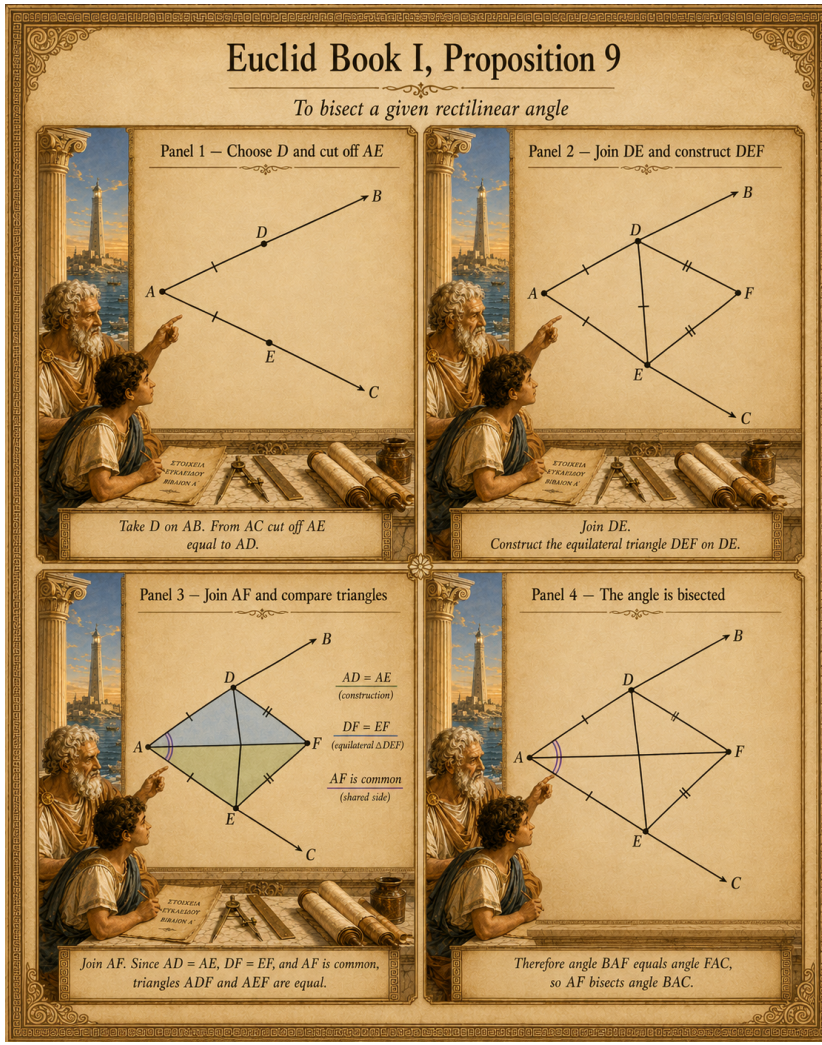


## Statement

*To bisect a given rectilinear angle.*

## Actions

1. Let BAC be the given angle.
2. Take D on AB and cut off AE on AC equal to AD.
3. Join DE and construct the equilateral triangle DEF on DE.
4. Join AF.
5. Triangles ADF and AEF have AD=AE, DF=EF, and AF common; by I.8 their included angles DAF and FAE are equal.
6. Therefore AF bisects angle BAC.

## Integrated Ontological Restatement

**Source formulation.** *To cut a given rectilinear angle in half.*

**Ontological role.** Given angle BAC. Take point D on AB. Cut AE equal to AD on AC (Proposition 3). Join DE. Construct equilateral triangle DEF on DE (Proposition 1). Join AF. Since AD = AE, AF is common, and DF = EF, triangles ADF and AEF are congruent (Proposition 8). Therefore, angle DAF = angle EAF. Thus, AF bisects the angle.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The criticism is that the auxiliary equilateral triangle must appear on the appropriate side of the segment so that the constructed line passes through the intended angular region.

**Reply within the reconstruction.** The construction is not licensed by a freehand diagram but by I.1 applied within the given rectilinear angle and by I.8. The selected intersections and joined straight lines are constrained by the already fixed rays of the angle. The intended region is therefore part of the constructional data rather than a visually guessed placement.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $\angle BAC$ be given. Take any point $D$ on $AB$. `[given, P1]`

**2.** Cut off $AE$ on $AC$ equal to $AD$. `[I.3]`

**3.** Join $DE$. `[P1]`

**4.** On $DE$ erect the equilateral triangle $DEF$, on the side away from $A$. `[I.1]`

**5.** Join $AF$. `[P1]`

**6.** Triangles $ADF$, $AEF$ have $AD = AE$, $DF = EF$, $AF$ common; so the angles at $A$ are equal. `[I.8]`

**7.** Therefore $AF$ bisects the angle.

---

HOW DO YOU KNOW IT IS THERE?

**The point $F$.** Output of `[I.1]`, which was earned eight propositions earlier and is here spent for the first time since I.2. The equilateral triangle is Book I's general-purpose symmetry device.

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---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

`I.9` Bisect a given rectilineal angle.

---

# Proposition 10

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- [I.1] erect an equilateral triangle on a given line
- [I.2] place at a given point a line equal to a given line
- [I.3] cut off from the greater a line equal to the less
- [I.9] bisect a given rectilineal angle
- [C5] still dormant

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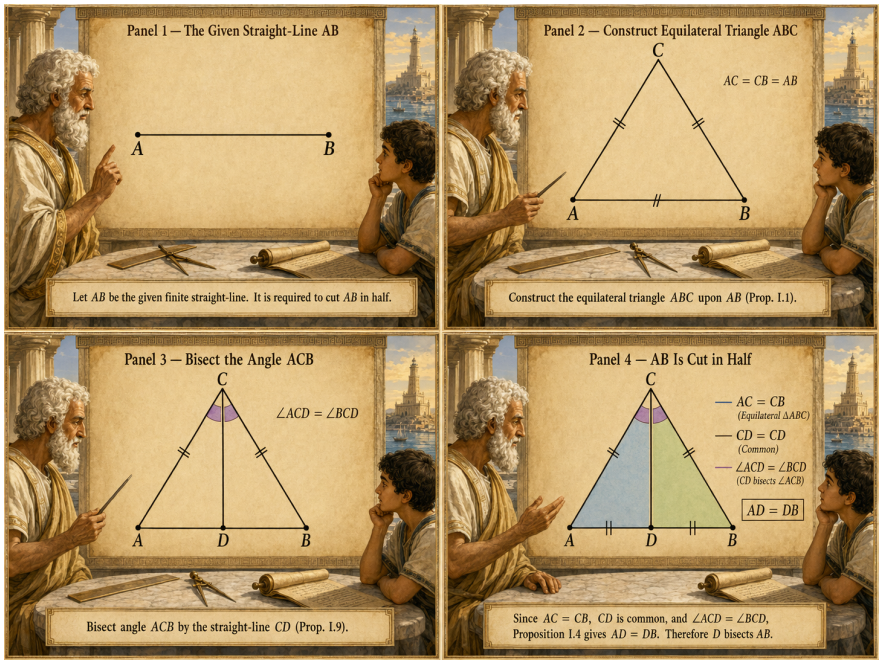


## Statement

*To bisect a given finite straight line.*

## Actions

1. Let AB be the given finite straight line.
2. Construct the equilateral triangle ABC on AB by I.1.
3. Bisect angle ACB by CD using I.9, meeting AB at D.
4. Triangles ACD and BCD have AC=BC, CD common, and angle ACD=BCD.
5. By I.4, AD=DB.
6. Therefore D bisects AB.

## Integrated Ontological Restatement

**Source formulation.** *To cut a given finite straight-line in half.*

**Ontological role.** Given AB. Construct equilateral triangle ABC on AB (Proposition 1). Bisect angle ACB (Proposition 9). The bisector CD meets AB at D. Triangles ACD and BCD are congruent (Proposition 4), so AD = DB. Thus, AB is bisected at D.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** As in I.9, critics may say that the auxiliary equilateral triangle and angle bisector are assumed to occupy the correct region and to meet the given segment.

**Reply within the reconstruction.** I.10 is a dependency chain: I.1 constructs the equilateral triangle, I.9 bisects its vertex angle, and I.4 proves equality of the resulting triangles. In the ontological reading, the relevant intersection occurs within the determined plane and is produced by the straight lines specified by those prior propositions.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $AB$ be the given line. Erect the equilateral triangle $ABC$ upon it. `[I.1]`
2. Bisect the angle $ACB$ by the line $CD$, meeting $AB$ at $D$. `[I.9]`
3. Triangles $ACD$, $BCD$ have $AC = BC$, $CD$ common, and the angles at $C$ equal; so $AD = DB$. `[I.4]`
4. Therefore $AB$ is bisected at $D$.

How do you know it is there?

**The point $D$.** Where the bisector of the apex angle meets the base. It exists because [I.9] returns a line through $C$ interior to the angle, and such a line meets the base of the triangle it divides. **Why not bisect the segment directly?.** Because no permission does that. The segment is bisected only by way of an angle bisection, which is why I.9 must precede I.10 — an order that looks arbitrary until the ledger is kept.

---

Added to the repertoire by this proposition

I.10 Bisect a given finite straight line.

---

# Proposition 11

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2]
- [I.3] cut off from the greater a line equal to the less
- [I.9] bisect a given rectilineal angle
- [I.10] bisect a given finite straight line
- [C5] still dormant

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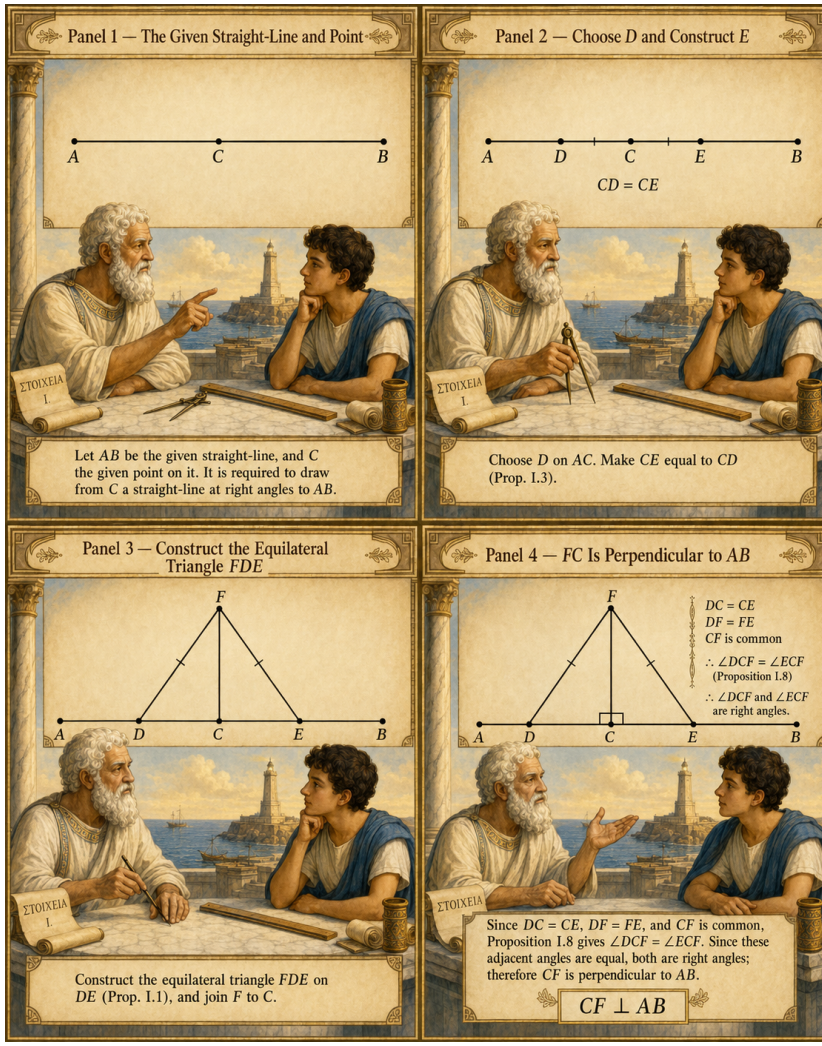


## Statement

*To draw a straight line at right angles to a given straight line from a given point on it.*

## Actions

1. Let C be the given point on the straight line AB.
2. Take D on one side of C and cut off CE on the other side equal to CD.
3. Construct the equilateral triangle FDE on DE and join FC.
4. Triangles FDC and FEC have FD=FE, DC=CE, and FC common; by I.8 their angles DCF and FCE are equal.
5. These adjacent equal angles lie on the straight line DE, so each is a right angle.
6. Therefore FC is perpendicular to AB at C.

## Integrated Ontological Restatement

**Source formulation.** *To draw a straight-line at right-angles to a given straight-line from a given point on it.*

**Ontological role.** Given line AB and point C on it. Take point D on AC. Make CE equal to CD (Proposition 3). Construct equilateral triangle FDE on DE (Proposition 1). Join FC. Triangles DCF and ECF are congruent (Proposition 8), so angle DCF = angle ECF. By Definition 10, these are right angles. Thus, FC is perpendicular to AB at C.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB$ be given and $C$ a point on it. Take $D$ on $CA$ freely. `[given]`

**2.** Cut off $CE$ on $CB$ equal to $CD$. `[I.3]`

**3.** On $DE$ erect the equilateral triangle $DEF$. `[I.1]`

**4.** Join $FC$. `[P1]`

**5.** Triangles $DCF$, $ECF$ have $DC = CE$, $CF$ common, $DF = EF$; so the angles at $C$ are equal. `[I.8]`

**6.** Being equal adjacent angles, each is right. [Def. 10]

---

HOW DO YOU KNOW IT IS THERE?

**The right angle.** Not measured against an instrument. Definition 10 defines a right angle as one of two equal adjacent angles, so the equality proved by [I.8] *is* the right angle. Postulate 4 then makes it the same magnitude anywhere.

---

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.11 Erect a perpendicular to a line from a point on it.

---

# Proposition 12

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3]
- [I.9] bisect a given rectilineal angle
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [C5] still dormant

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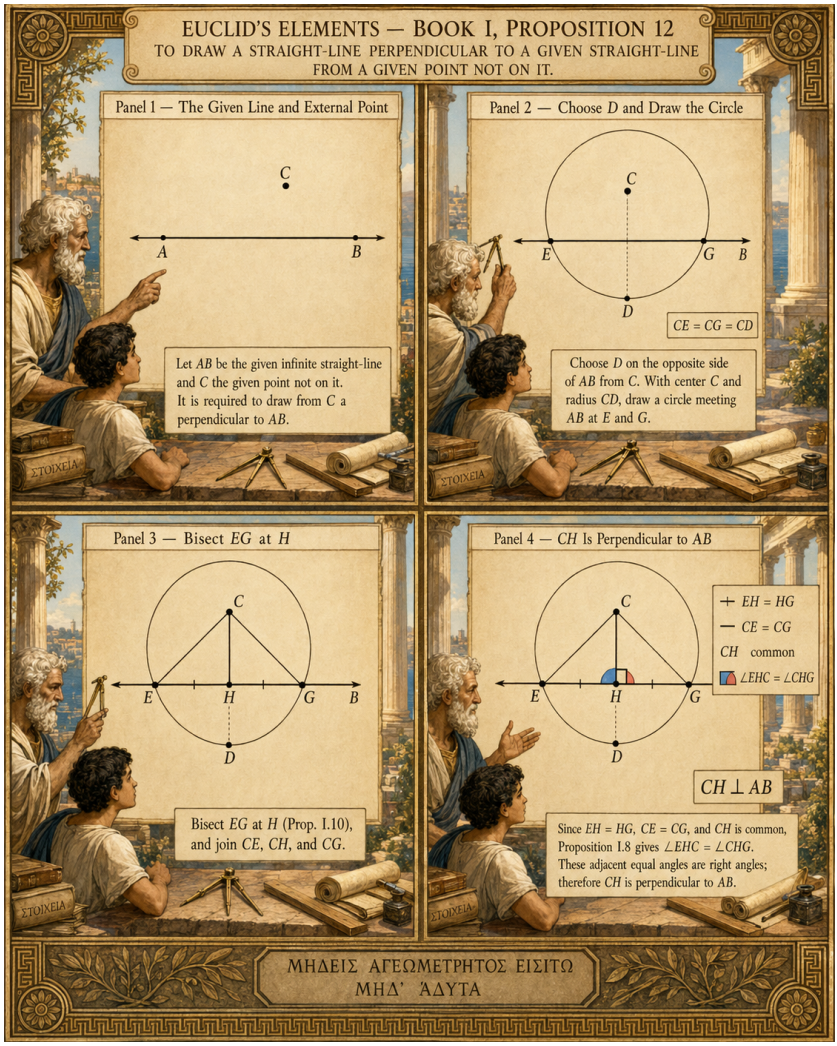


## Statement

*To draw a perpendicular to a given infinite straight line from a given point not on it.*

## Actions

1. Let AB be the given line and C the external point.
2. Choose D on the opposite side of AB and with centre C describe a circle through D cutting AB at E and F.
3. Bisect EF at G by I.10 and join CG.
4. Since CE=CF, EG=GF, and CG is common, triangles CEG and CFG are equal by I.8.
5. Therefore angles CGE and CGF are equal adjacent angles on the straight line EF.
6. Hence each is right, and CG is perpendicular to AB.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** The construction of a perpendicular to a given line at a point presupposes the completeness of the plane. This again involves the intersection of circles, whose existence is not explicitly guaranteed by the postulates.

**Critics:**

- **Beeson (2019)** argues that additional existence postulates are required to ensure the intersection of circles in the construction of the perpendicular.

**Response:** I disagree. The completeness and the definition of the plane are already established by *Postulate 5* (see Section **??**). Hence, the existence of the intersection points required for drawing a perpendicular follows logically from the nature of the plane as defined therein, without the need for any supplementary existence postulates.

## Integrated Ontological Restatement

**Source formulation.** *To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.*

**Ontological role.** Given infinite line AB and point C not on it. Take point D on the other side of AB. Draw circle with center C and radius CD (Postulate 3). The circle cuts AB at E and G. Bisect EG at H (Proposition 10). Join CH. Since EH = HG, CH is common, and CE = CG (radii of same circle), triangles CEH and CGH are congruent (Proposition 8). Therefore, angle CHE = angle CHG, which are adjacent and equal, hence right angles (Definition 10). Thus, CH is perpendicular to AB from C.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The standard continuity objection is that the circle centred at $C$ is assumed to cut the infinite straight line $AB$ in two points $E$ and $G$.

**Reply within the reconstruction.** The reconstruction treats this as an instance of the already established interaction of Postulates 2 and 3 within the plane determined by Postulate 5. The infinite straight line is continuously produced and the circle is the locus of constant radius; their two intersections in this configuration are therefore read as consequences of those determinations rather than as visual assumptions.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $AB$ be given and $C$ a point not on it. Take $D$ on the far side of $AB$. `[given]`
2. With centre $C$ and radius $CD$ describe a circle cutting $AB$ at $G$ and $E$. `[P3]`
3. Bisect $GE$ at $H$. `[I.10]`
4. Join $CH$. `[P1]`

**5.** Triangles $GHC$, $EHC$ have $GH = HE$, $HC$ common, $CG = CE$; so the angles at $H$ are equal, and each is right. [I.8, Def. 10]

---

HOW DO YOU KNOW IT IS THERE?

**The points $G$ and $E$.** The circle is executed by [P3] with a radius reaching across $AB$, since $D$ was taken on the far side. A locus at constant distance from $C$, passing from one side of $AB$ to the other, shares points with it. **Why the far-side point $D$?.** To guarantee the radius is long enough. Without it the circle might not reach $AB$ at all, and the construction would return nothing.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.12 Drop a perpendicular to a line from a point outside it.

---

# Proposition 13

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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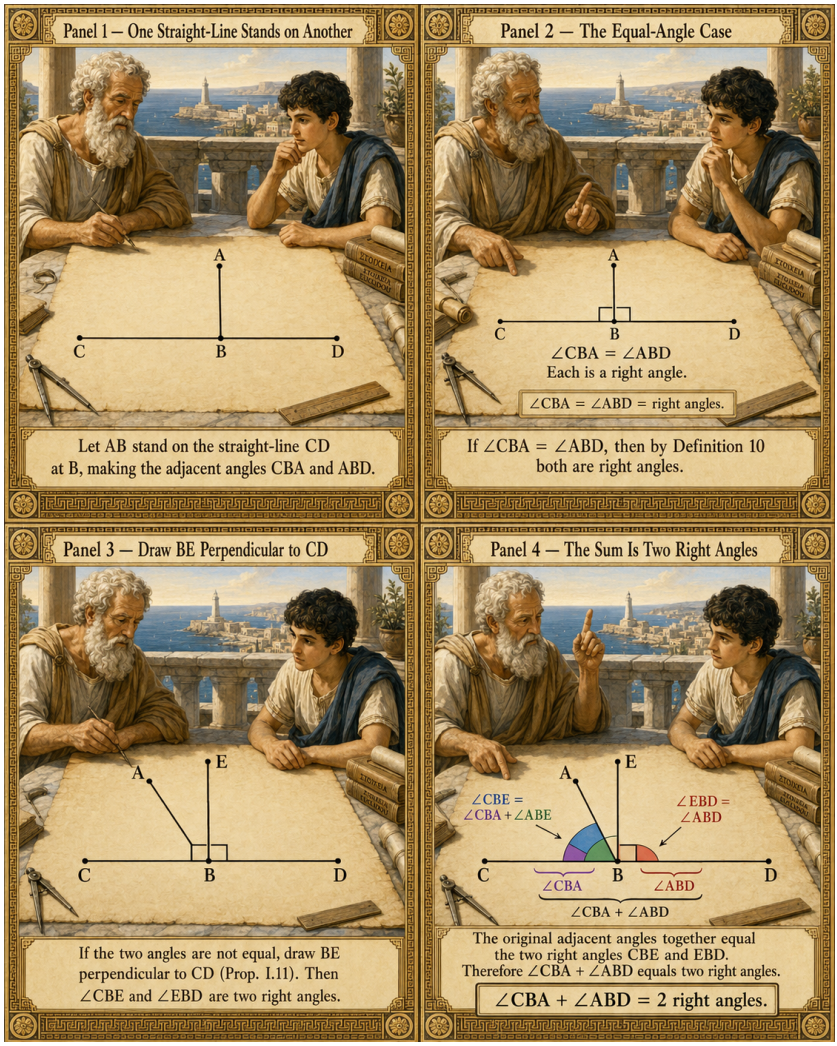


## Statement

*If a straight line standing on another straight line makes angles, it makes either two right angles or angles whose sum is two right angles.*

## Actions

1. Let AB stand on CD at B.
2. If angle CBA equals ABD, both are right angles by definition.
3. Otherwise draw BE perpendicular to CD at B by I.11.
4. Then CBE and EBD are right angles.
5. Add and subtract the common parts to show angles CBA and ABD together equal angles CBE and EBD together.
6. Therefore CBA+ABD equals two right angles.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** The construction of a perpendicular to a given line at a point presupposes the completeness of the plane. This again involves the intersection of circles, whose existence is not explicitly guaranteed by the postulates.

**Critics:**

- **Beeson (2019)** argues that additional existence postulates are required to ensure the intersection of circles in the construction of the perpendicular.

**Response:** I disagree. The completeness and the definition of the plane are already established by *Postulate 5* (see Section **??**). Hence, the existence of the intersection points required for drawing a perpendicular follows logically from the nature of the plane as defined therein, without the need for any supplementary existence postulates.

# Integrated Ontological Restatement

**Source formulation.** *If a straight-line stood upon another straight-line makes angles, it will certainly either make two right-angles, or angles whose sum is equal to two right-angles.*

**Ontological role.** Given line AB standing on CD at B, making angles CBA and ABD. If the angles are equal, they are right angles by Definition 10. If not, draw BE perpendicular to CD at B (Proposition 11). By Definition 10, CBE and EBD are right angles. Since CBE = CBA + ABE, and ABD = ABE + EBD, adding and subtracting equals yields CBA + ABD = CBE + EBD, which equals two right angles (Common Notion 2). Thus, the sum of the adjacent angles is two right angles.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $CD$ stand on $AB$ at $C$. `[given]`
2. If the angles are equal, each is right by definition. `[Def. 10]`
3. Otherwise erect $CE$ perpendicular to $AB$ at $C$. `[I.11]`
4. Then $\angle ACD + \angle DCB$ equals $\angle ACE + \angle ECB$, which is two right angles. `[C.N. 2]`

---

How do you know it is there?

**The perpendicular $CE$.** Output of `[I.11]`, earned two propositions earlier. I.13 is the first proposition to spend the perpendicular, and it spends it to create the standard against which all later angle sums are measured.

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---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.13.** Adjacent angles on a straight line sum to two right angles.

No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 14

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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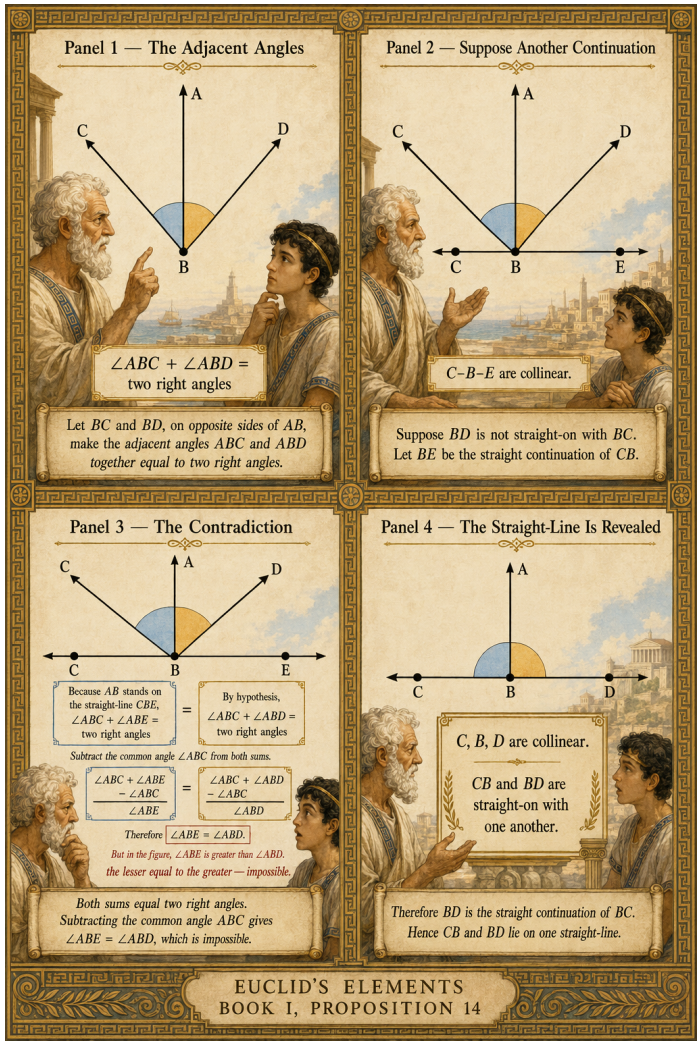


## Statement

*If two adjacent angles made with a straight line sum to two right angles, the outer sides form one straight line.*

## Actions

1. Let AB stand on BC, and suppose angles ABC and ABD together equal two right angles.
2. Assume BD is not in a straight line with BC, and let BE be the straight continuation of CB.
3. By I.13, angles ABC and ABE together equal two right angles.
4. By hypothesis, angles ABC and ABD together also equal two right angles.
5. Subtract the common angle ABC to obtain ABE=ABD.
6. This would make distinct rays BE and BD coincide, a contradiction. Therefore BD and BC form one straight line.

## Integrated Ontological Restatement

**Source formulation.** *If two straight-lines, not lying on the same side, make adjacent angles whose sum is equal to two right-angles with some straight-line at a point on it, then the two straight-lines will be straight-on with respect to one another.*

**Ontological role.** Given line AB with BC and BD on opposite sides such that angle ABC + angle ABD = two right angles. Suppose BD is not straight-on to BC. Let BE be straight-on to BC. Then angle ABC + angle ABE = two right angles (Proposition 13). Therefore, angle ABC + angle ABE = angle ABC + angle ABD. By subtraction (Common Notion 3), angle ABE = angle ABD, the lesser equal to the greater, impossible. Therefore, BD is straight-on to BC.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $BC$, $BD$ make with $AB$ at $B$ angles summing to two right angles. `[given]`

2. Suppose $BD$ is not in a straight line with $CB$; let $BE$ be so instead. [hypothesis]

3. Then $\angle ABC + \angle ABE$ is two right angles. [I.13]

4. Subtracting the common angle $ABC$ gives $\angle ABE = \angle ABD$, the lesser equal to the greater. [C.N. 3]

5. Impossible; therefore $BD$ is in a straight line with $CB$. [C.N. 5]

---

HOW DO YOU KNOW IT IS THERE?

**The line $BE$.** Assumed for refutation, as at I.7. Book I uses this device sparingly and always with a construction that can actually be carried out.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.14.** Angles summing to two right angles on opposite sides make one straight line. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 15

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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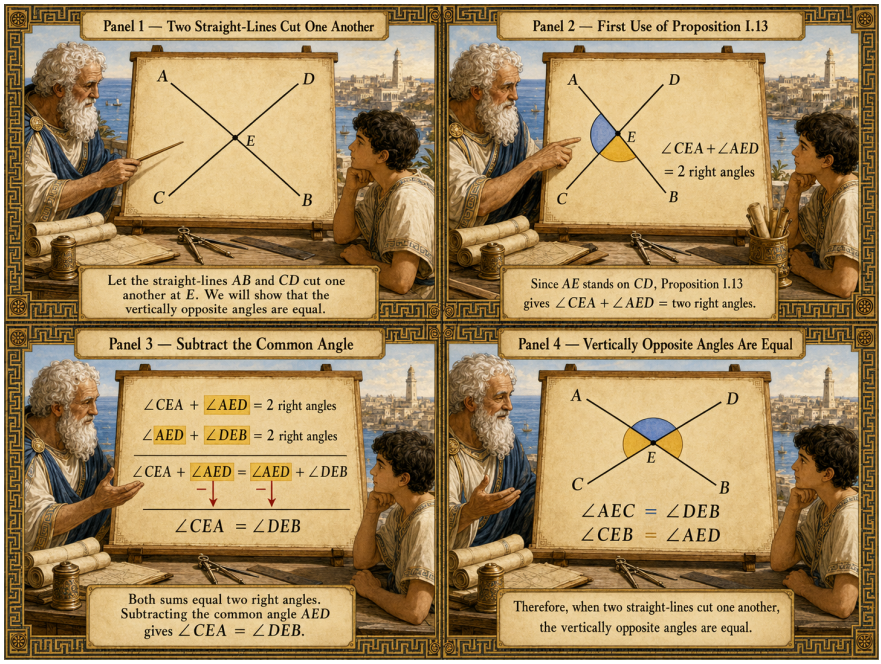


## Statement

> *If two straight lines cut one another, they make the vertically opposite angles equal.*

## Actions

1. Let AB and CD intersect at E.
2. By I.13, angles AEC and CEB sum to two right angles.
3. Also angles CEB and BED sum to two right angles.
4. Subtract the common angle CEB to get AEC=BED.
5. Similarly subtract the appropriate common angle to obtain CEA=DEB for the other vertical pair.
6. Thus vertically opposite angles are equal.

# Integrated Ontological Restatement

**Source formulation.** *If two straight-lines cut one another, they make the vertically opposite angles equal to one another.*

**Ontological role.** Given lines AB and CD cutting at E. By Proposition 13, angle AEC + angle CEB = two right angles, and angle AED + angle AEC = two right angles. Therefore, angle AEC + angle CEB = angle AED + angle AEC. By subtraction (Common Notion 3), angle CEB = angle AED. Similarly, angle AEC = angle DEB.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB$, $CD$ cut at $E$. `[given]`

**2.** $\angle AEC + \angle AED$ is two right angles. `[I.13]`

**3.** $\angle AED + \angle DEB$ is two right angles. `[I.13]`

**4.** Therefore $\angle AEC = \angle DEB$. `[C.N. 1, C.N. 3]`

---

HOW DO YOU KNOW IT IS THERE?

**The point $E$.** Given by hypothesis: the lines are stated to cut one another. I.15 does not have to prove that they meet — it is told that they do.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.15.** Vertical angles are equal. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 16

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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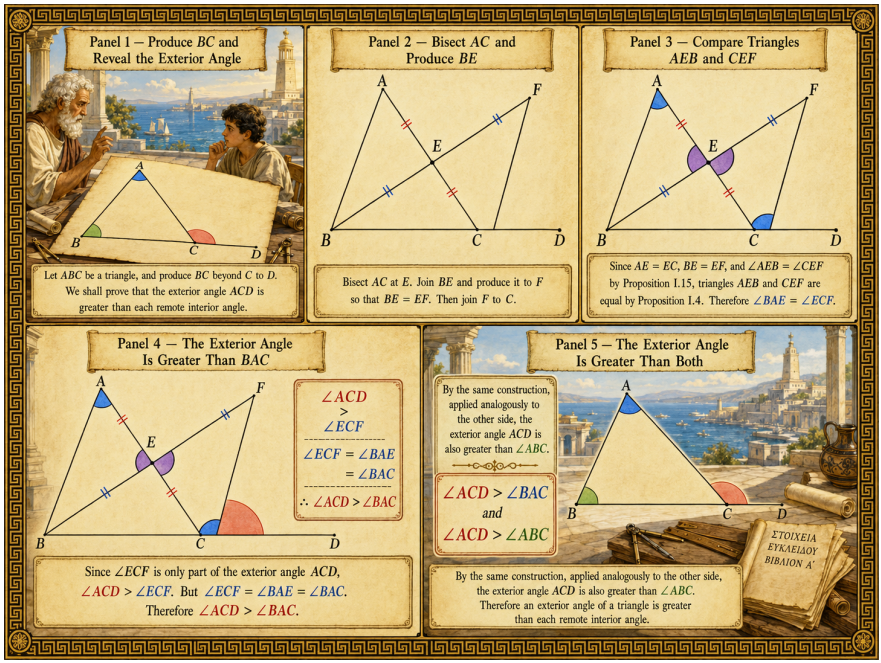


## Statement

> *In any triangle, if one side is produced, the exterior angle is greater than either interior opposite angle.*

## Actions

1. Let side BC of triangle ABC be produced to D.
2. Bisect AC at E and join BE; produce BE to F so that EF=BE; join FC.
3. Triangles AEB and CEF have AE=EC, BE=EF, and vertical angles AEB=CEF; by I.4 they are equal.
4. Hence angle BAE equals angle ECF.
5. But exterior angle ACD contains angle ECF and is therefore greater than angle BAE.

6. By an analogous construction, ACD is also greater than angle ABC.

## Integrated Ontological Restatement

**Source formulation.** *In any triangle, if one of the sides is produced, the external angle is greater than each of the internal and opposite angles.*

**Ontological role.** Given triangle ABC with side BC produced to D. Bisect AC at E (Proposition 10). Join BE and extend to F such that EF = BE (Proposition 3). Join FC. Triangles AEB and CEF are congruent (Proposition 4), so angle ECF = angle EAB = angle BAC. Since angle ACD is greater than angle ECF (the whole is greater than the part, Common Notion 5), angle ACD > angle BAC. Similarly, angle ACD > angle ABC.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** A frequently cited order objection is that the constructed point $F$, obtained after bisecting and extending, is taken to lie in the relevant interior region, so that one angle is properly part of another.

**Reply within the reconstruction.** The construction fixes $F$ by equality on the produced straight line, and the comparison is then governed by Common Notion 5: the whole angle is greater than its proper part. The reconstruction takes the relevant incidence and order from the produced lines and their established intersections rather than from an additional inside/outside axiom.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $ABC$ have $BC$ produced to $D$. `[P2]`

**2.** Bisect $AC$ at $E$. `[I.10]`

**3.** Join $BE$ and produce it to $F$, making $EF = BE$. `[P1, P2, I.3]`

**4.** Join $FC$. `[P1]`

5. Triangles $AEB, CEF$ have $AE = EC$, $BE = EF$, and vertical angles equal; so $\angle BAE = \angle ECF$. [I.15, I.4]

6. But $\angle ECD$ is greater than $\angle ECF$; so the exterior angle exceeds the interior opposite. [C.N. 5]

---

HOW DO YOU KNOW IT IS THERE?

**The point $F$.** Constructed, not assumed: $BE$ is produced by [P2] and the equal length laid off by [I.3]. The doubling of the median is the characteristic move of this proof.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.16.** An exterior angle exceeds either remote interior angle. *Proved without Postulate 5.* No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 17

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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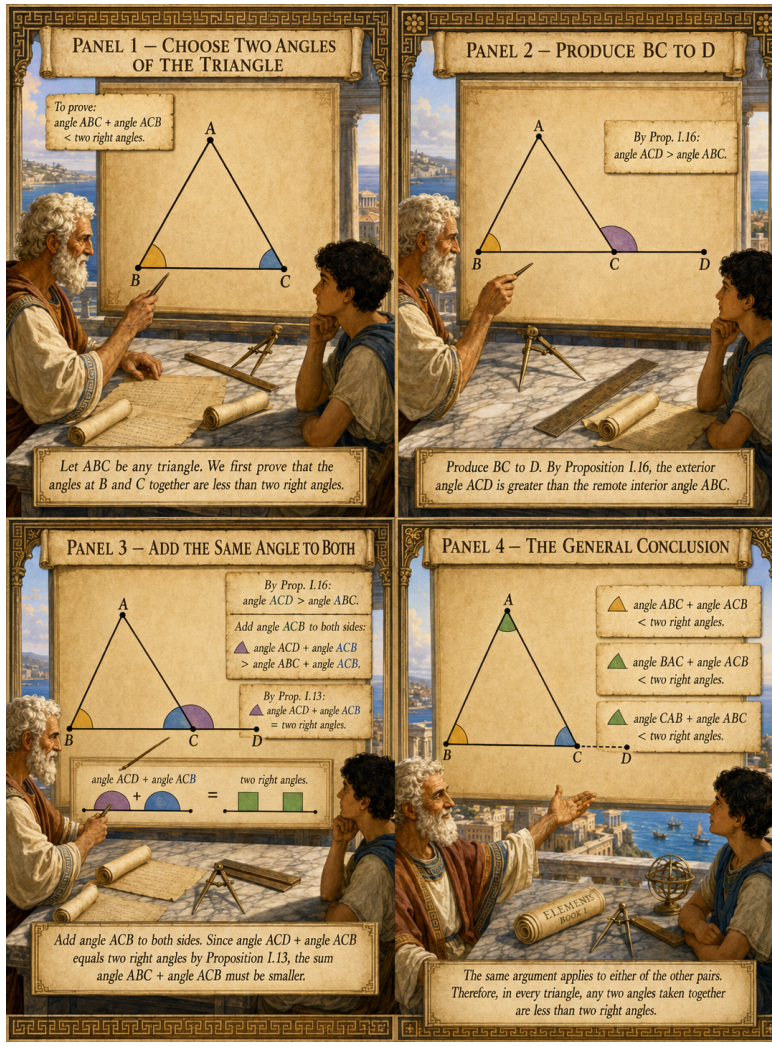


## Statement

*In any triangle, any two angles together are less than two right angles.*

## Actions

1. Let triangle ABC have side BC produced to D.
2. By I.16, exterior angle ACD is greater than interior opposite angle ABC.
3. Add angle ACB to both sides.
4. Angles ACD and ACB together equal two right angles by I.13.
5. Therefore angles ABC and ACB together are less than two right angles.
6. The same reasoning applies to any other pair of angles.

## Integrated Ontological Restatement

**Source formulation.** *For any triangle, the sum of two angles taken together in any way is less than two right-angles.*

**Ontological role.** Given triangle ABC. Extend BC to D. By Proposition 16, angle ACD > angle ABC. Add angle ACB to both sides. Then angle ACD + angle ACB > angle ABC + angle ACB. But angle ACD + angle ACB = two right angles (Proposition 13). Therefore, angle ABC + angle ACB < two right angles. Similarly for the other pairs.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Produce $BC$ to $D$. `[P2]`

**2.** The exterior angle $ACD$ exceeds the interior opposite angle $ABC$. `[I.16]`

**3.** Add $\angle ACB$ to each: $\angle ACD + \angle ACB$ is two right angles. `[I.13]`

**4.** Therefore $\angle ABC + \angle ACB$ is less than two right angles. `[C.N. 5]`

How do you know it is there?

**The bound.** Notice it is *less than*, never *equal to*. The exact angle sum must wait for I.32, which needs the parallel constraint. Until then Book I can only bound.

Added to the repertoire by this proposition

**Warrant I.17.** Any two angles of a triangle are together less than two right angles. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 18

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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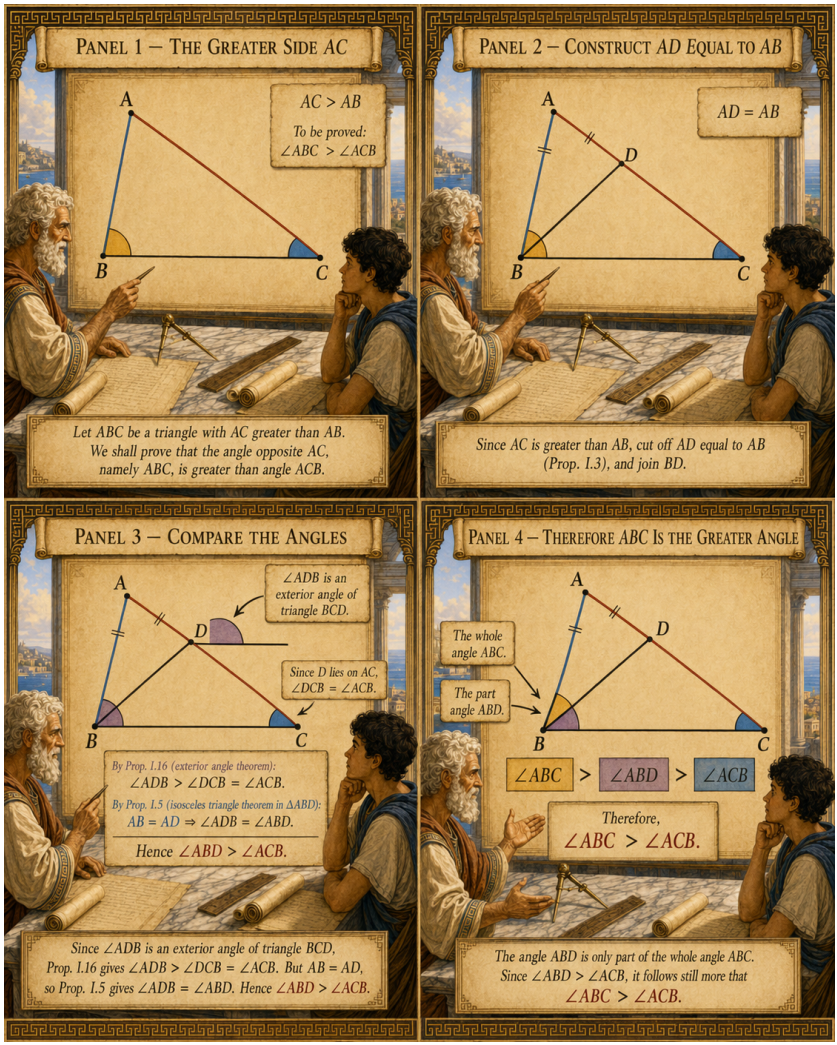


## Statement

*In any triangle, the greater side subtends the greater angle.*

## Actions

1. Let triangle ABC have side AC greater than AB.
2. Cut off AD from AC equal to AB and join BD.
3. Triangle ABD is isosceles, so angle ABD=ADB by I.5.
4. Angle ADB, being an exterior angle of triangle BDC, is greater than angle DCB by I.16.
5. Thus angle ABD is greater than angle ACB.
6. Since angle ABC contains angle ABD, angle ABC is greater than angle ACB.

## Integrated Ontological Restatement

**Source formulation.** *In any triangle, the greater side subtends the greater angle.*

**Ontological role.** Given triangle ABC with AC > AB. Cut AD from AC equal to AB (Proposition 3). Join BD. Since AD = AB, angle ABD = angle ADB (Proposition 5). Since angle ADB > angle DCB (Proposition 16, external to triangle BCD), angle ABD > angle DCB. But angle ABC > angle ABD, so angle ABC > angle ACB.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AC$ be greater than $AB$ in triangle $ABC$. `[given]`

**2.** Cut off $AD$ on $AC$ equal to $AB$. `[I.3]`

**3.** Join $BD$. `[P1]`

**4.** $\angle ADB = \angle ABD$. `[I.5]`

**5.** $\angle ADB$, being exterior to triangle $BDC$, exceeds $\angle DCB$. `[I.16]`

**6.** Therefore $\angle ABC$ exceeds $\angle ACB$. [C.N. 5]

---

How do you know it is there?

**The point $D$.** Output of [I.3] — the cut-off action, earned at the third proposition, is still the workhorse fifteen propositions later.

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Added to the repertoire by this proposition

**Warrant I.18.** The greater side subtends the greater angle. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

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# Proposition 19

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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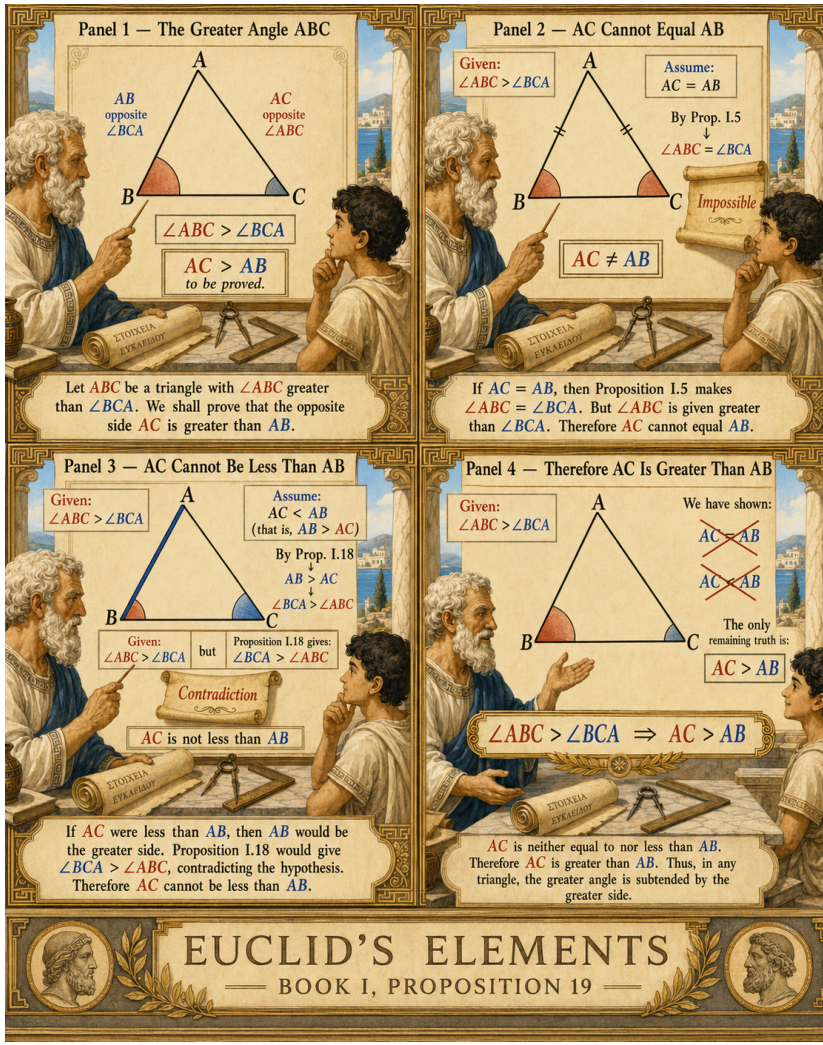


## Statement

*In any triangle, the greater angle is subtended by the greater side.*

## Actions

1. Let triangle ABC have angle ABC greater than angle ACB.
2. Suppose AC is not greater than AB.
3. If AC=AB, then the base angles would be equal by I.5, contrary to the hypothesis.
4. If AC<AB, then by I.18 the angle opposite AB, namely ACB, would be greater than angle ABC, also impossible.
5. Therefore AC>AB.

## Integrated Ontological Restatement

**Source formulation.** *In any triangle, the greater angle is subtended by the greater side.*

**Ontological role.** Given triangle ABC with angle ABC > angle ACB. Suppose AC is not greater than AB. If AC = AB, then angle ABC = angle ACB (Proposition 5), contradicting the given. If AC < AB, then angle ABC < angle ACB (Proposition 18), also contradicting. Therefore, AC > AB.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $\angle ABC$ exceed $\angle ACB$. `[given]`

**2.** If $AC = AB$ the angles would be equal. `[I.5]`

**3.** If $AC$ were less than $AB$, the angle at $B$ would be less than that at $C$. `[I.18]`

**4.** Both are contrary to hypothesis; therefore $AC$ exceeds $AB$.

How do you know it is there?

**No new object.** I.19 introduces nothing. It is a pure exclusion argument on the two warrants already banked at I.5 and I.18.

Added to the repertoire by this proposition

**Warrant I.19.** The greater angle is subtended by the greater side. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 20

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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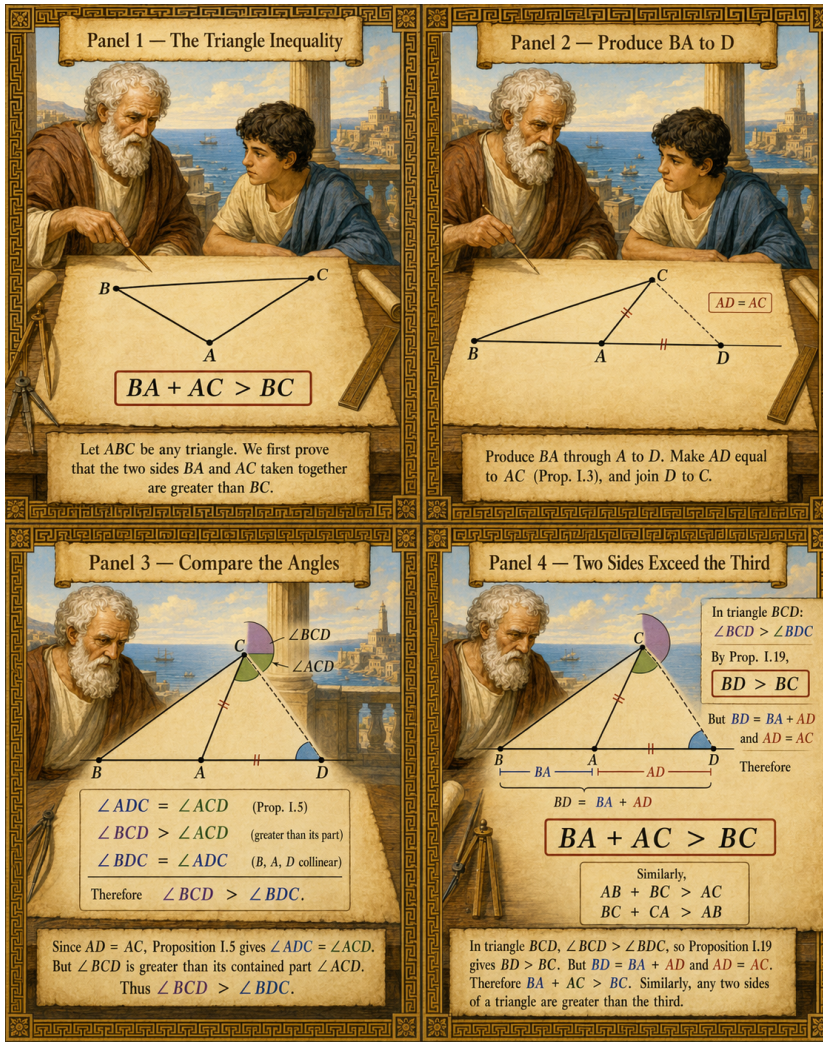


## Statement

*In any triangle, the sum of any two sides is greater than the remaining side.*

## Actions

1. Let triangle ABC be given.
2. Produce BA to D and make AD=AC; join DC.
3. Then triangle ACD is isosceles, so angle ACD=ADC by I.5.
4. Angle BCD is greater than ACD and therefore greater than BDC.
5. By I.19, in triangle BCD the side BD is greater than BC.
6. But BD=BA+AD=BA+AC. Therefore BA+AC>BC; similarly for the other pairs.

## Integrated Ontological Restatement

**Source formulation.** *In any triangle, the sum of two sides taken together in any way is greater than the remaining side.*

**Ontological role.** Given triangle ABC. Extend BA to D such that AD = AC (Proposition 3). Join DC. Since AD = AC, angle ADC = angle ACD (Proposition 5). But angle BCD > angle ACD, so angle BCD > angle ADC. Therefore, BD > BC (Proposition 19). But BD = BA + AD = BA + AC. Thus, BA + AC > BC. Similarly for the other pairs.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The criticism says that producing $BA$ to $D$ assumes the betweenness relation $B–A–D$.

**Reply within the reconstruction.** Postulate 2 is precisely the operation of producing a finite straight line continuously in the same straight line. To produce $BA$ beyond $A$ is therefore already to determine $A$ as lying between the original endpoint $B$ and the newly constructed point $D$. No separate betweenness assertion is added after the construction.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Produce $BA$ to $D$, making $AD = AC$. `[P2, I.3]`

**2.** Join $DC$. `[P1]`

**3.** $\angle ADC = \angle ACD$. `[I.5]`

**4.** $\angle BCD$ exceeds $\angle ACD$, hence exceeds $\angle ADC$. `[C.N. 5]`

**5.** Therefore $BD$ exceeds $BC$. `[I.19]`

**6.** But $BD = BA + AC$.

---

HOW DO YOU KNOW IT IS THERE?

**The point $D$.** `[P2]` extends, `[I.3]` measures. Two of the earliest permissions, combined, give the proposition the Epicureans mocked as obvious to a donkey — and which Euclid nonetheless proves.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.20.** Any two sides of a triangle exceed the third. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 21

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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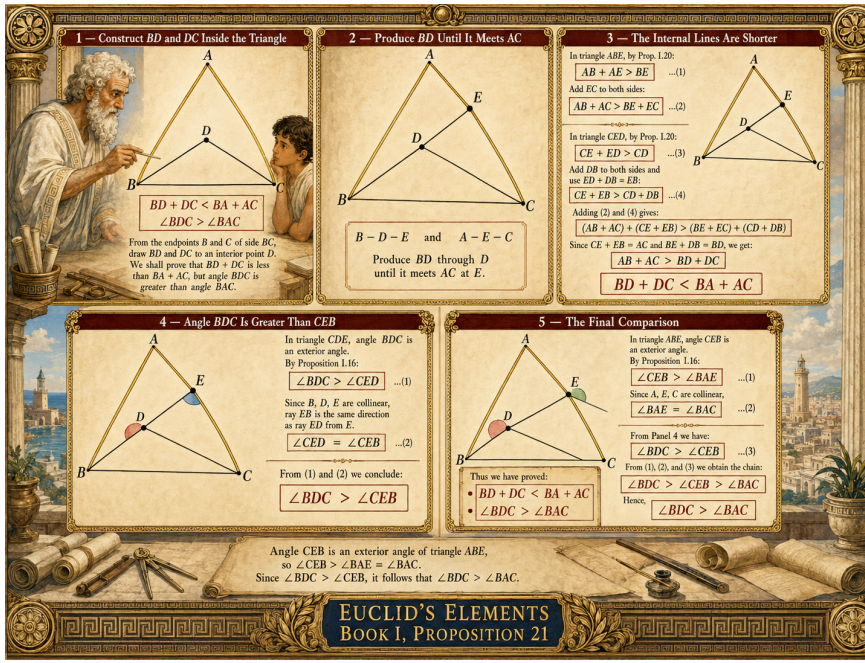


## Statement

*If two straight lines are drawn from the ends of one side of a triangle to a point inside, their sum is less than the other two sides, but the angle between them is greater than the angle at the third vertex.*

## Actions

1. Let D be inside triangle ABC and join BD and CD.
2. Produce BD to meet AC at E.
3. By I.20 in triangle ABE, AB+AE>BE; add EC to both sides.
4. Again by I.20 in triangle CDE, CE+ED>CD.
5. Combining gives AB+AC>BD+CD.

6. By I.16, angle BDC is greater than angle BEC, and angle BEC is greater than angle BAC; hence angle BDC>angle BAC.

## Integrated Ontological Restatement

**Source formulation.** *If two internal straight-lines are constructed on one of the sides of a triangle from its ends, the constructed lines will be less than the two remaining sides of the triangle, but will encompass a greater angle.*

**Ontological role.** Given triangle ABC with internal lines BD and DC constructed from B and C on BC. Extend BD to E. By Proposition 20, in triangle ABE, AB + AE > BE. Adding EC gives BA + AC > BE + EC. In triangle CED, CE + ED > CD. Adding DB gives CE + EB > CD + DB. Therefore, BA + AC > CD + DB. For the angle: angle BDC > angle DEC (Proposition 16), and angle DEC > angle BAC (Proposition 16), so angle BDC > angle BAC.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $BD$, $DC$ be drawn to the interior point $D$ of triangle $ABC$. `[P1]`

**2.** Produce $BD$ to meet $AC$ at $E$. `[P2]`

**3.** $BA + AE > BE$. `[I.20]`

**4.** $EC + ED > DC$; adding and cancelling gives $BD + DC < BA + AC$. `[I.20, C.N.]`

**5.** $\angle BDC$ exceeds $\angle BEC$, which exceeds $\angle BAC$. `[I.16]`

---

HOW DO YOU KNOW IT IS THERE?

**The point $E$.** Where $BD$ produced meets $AC$. It exists because $D$ is interior: a line from a vertex through an interior point leaves the triangle across the opposite side.

Added to the repertoire by this proposition

**Warrant I.21.** Interior cevians are shorter in sum but enclose a greater angle.

No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 22

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9]
- [I.10] bisect a given finite straight line
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [C5] still dormant

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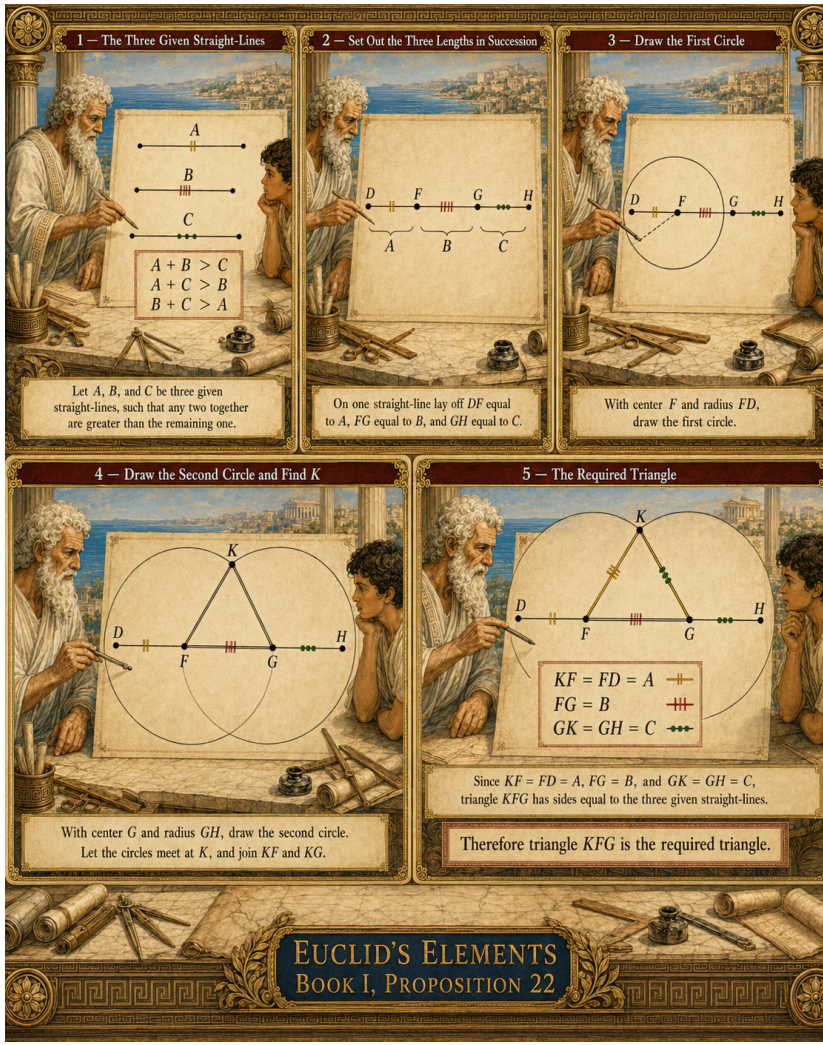


## Statement

*To construct a triangle from three given straight lines, each pair of which is greater than the remaining one.*

## Actions

1. Let the three given lengths be A, B, C satisfying the triangle inequalities.
2. Take a straight line DE and mark DF=A, FG=B, and GH=C successively.
3. With centre F and radius A describe a circle; with centre G and radius C describe another circle meeting at K.
4. Join KF and KG.
5. By construction KF=A, FG=B, and KG=C.
6. Therefore triangle KFG has its three sides equal to the three given straight lines.

## Integrated Ontological Restatement

**Source formulation.** *To construct a triangle from three straight-lines which are equal to three given straight-lines. It is necessary for the sum of any two to be greater than the remaining one.*

**Ontological role.** Given lines A, B, C with any two summing to greater than the third. Set out line DE. Make DF = A, FG = B, GH = C (Proposition 3). Draw circle with center F and radius FD. Draw circle with center G and radius GH. These circles intersect at K (guaranteed by Postulate 5). Join KF and KG. Then KF = A, KG = C, FG = B. Thus, triangle KFG is constructed from three lines equal to A, B, C.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The two circles centred at $F$ and $G$ are said to meet at $K$. Modern formalizations isolate this step as a circle–circle continuity requirement.

**Reply within the reconstruction.** The triangle inequalities are not decorative hypotheses: they are the conditions under which the two constant-radius loci can meet in the Euclidean plane. The ontological reconstruction reads

the intersection as guaranteed by the plane-and-circle structure already established; the formalist's continuity axiom is interpreted as an explicit recoding of that content.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $A$, $B$, $C$ be the given lines, any two together exceeding the third. `[I.20]`

**2.** Set out a line $DE$, terminated at $D$ and unbounded toward $E$. `[P1, P2]`

**3.** Cut off $DF = A$, $FG = B$, $GH = C$. `[I.3]`

**4.** With centre $F$ and radius $FD$ describe a circle; with centre $G$ and radius $GH$ describe another. `[P3]`

**5.** Let them cut at $K$; join $KF$ and $KG$. `[P1]`

**6.** Then $KFG$ is the required triangle. `[Def. 15]`

---

HOW DO YOU KNOW IT IS THERE?

**The point $K$.** The two circles meet because of the condition imported from `[I.20]`: each circle reaches past the other's centre precisely when any two of the three lengths exceed the third. The necessary condition is not a caveat appended to the construction — it is the guarantee that the construction returns anything at all.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

`I.22` Construct a triangle on three given lengths, when the triangle inequality permits.

---

# Proposition 23

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10]
- [I.11] erect a perpendicular from a point on a line
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [C5] still dormant

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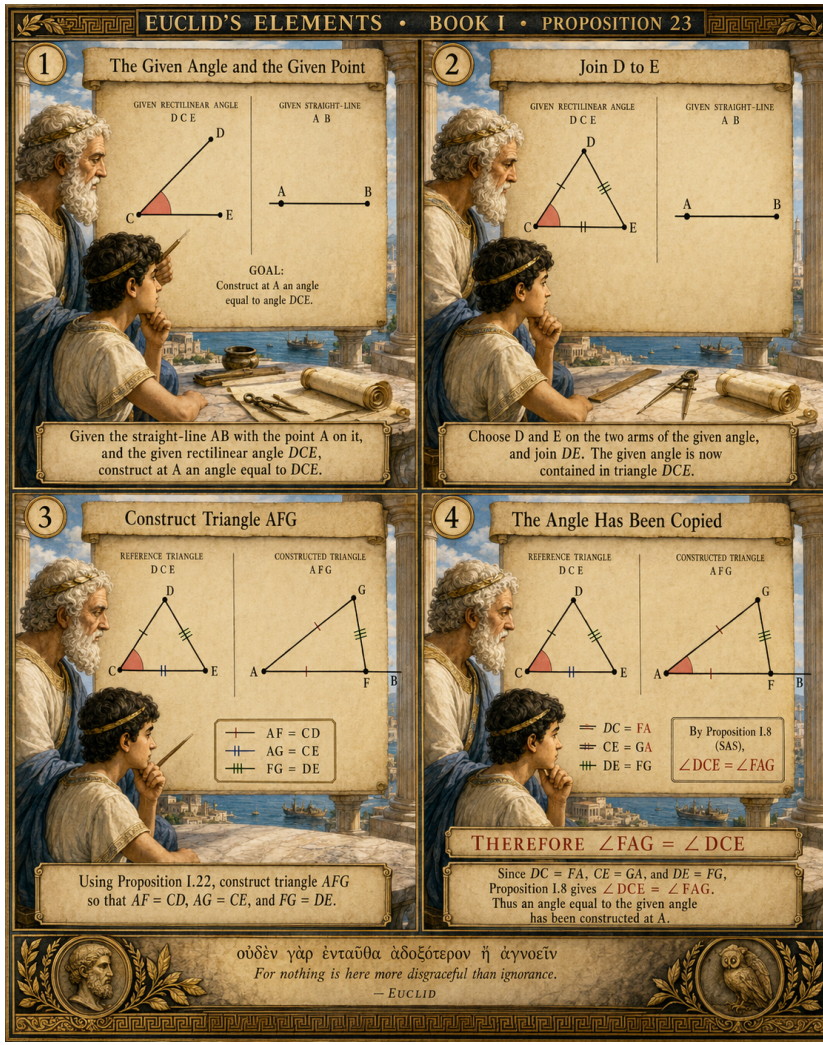


## Statement

*At a given point on a given straight line, to construct a rectilinear angle equal to a given rectilinear angle.*

## Actions

1. Let AB be the given line, A the given point, and DCE the given angle.
2. Choose points D and E on its sides and join DE.
3. Construct on AB a triangle AFG whose three sides correspond to CD, CE, and DE by I.22.
4. Triangles DCE and FAG have all three corresponding sides equal.
5. By I.8, angle DCE equals angle FAG.
6. Thus angle FAG equal to the given angle has been constructed at A on AB.

## Integrated Ontological Restatement

**Source formulation.** *To construct a rectilinear angle equal to a given rectilinear angle at a given point on a given straight-line.*

**Ontological role.** Given line AB, point A, and angle DCE. Take points D and E on the sides of the angle. Join DE. Construct triangle AFG (Proposition 22) such that AF = CD, AG = CE, FG = DE. By Proposition 8, triangle DCE is congruent to triangle FAG. Therefore, angle FAG = angle DCE.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $\angle DCE$ be given, and $AB$ with the point $A$ on it. `[given]`

**2.** Take $D$, $E$ on the two sides and join $DE$. `[P1]`

**3.** Construct the triangle $AFG$ from three lines equal to $CD$, $DE$, $CE$. `[I.22]`

**4.** Then $\angle FAG = \angle DCE$. `[I.8]`

How do you know it is there?

**The triangle** $AFG$**.** Output of [I.22], which is why I.22 precedes I.23. The angle is copied by copying a triangle — Book I has no way to carry an angle directly, so it carries three lengths and rebuilds.

---

Added to the repertoire by this proposition

I.23 Copy a given rectilineal angle to a given point on a given line.

---

# Proposition 24

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] still dormant

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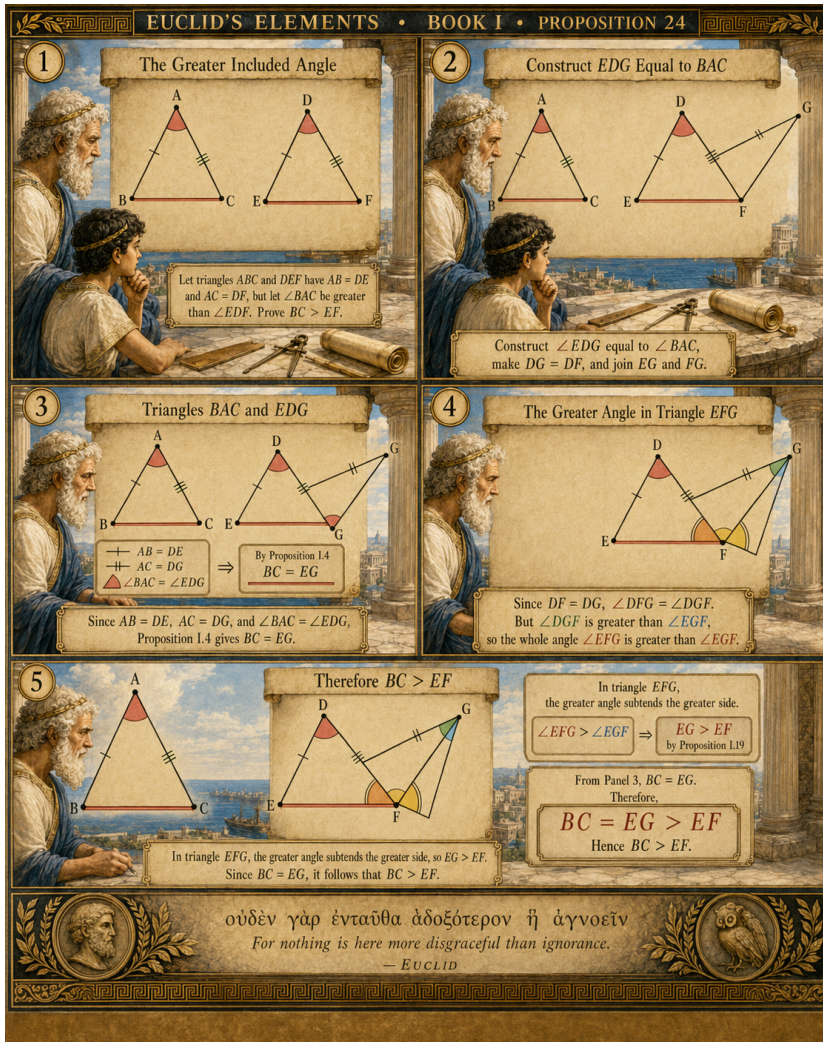


## Statement

*If two triangles have two corresponding sides equal but one included angle greater, then the base opposite the greater angle is greater.*

## Actions

1. Let AB=DE and AC=DF, but angle BAC greater than angle EDF.
2. At D construct angle EDG equal to BAC by I.23 and make DG=AC; join EG and FG as needed.
3. Triangles BAC and EDG have two sides and included angle equal; by I.4, BC=EG.
4. Because DF=DG, triangle DFG is isosceles and its base-angle relations show angle EFG is greater than EGF.
5. By I.19, EG>EF.
6. Since EG=BC, it follows that BC>EF.

## Integrated Ontological Restatement

**Source formulation.** *If two triangles have two sides equal to two sides, respectively, but one has the angle encompassed by the equal straight-lines greater than the corresponding angle in the other, then the former triangle will also have a base greater than the base of the latter.*

**Ontological role.** Given triangles ABC and DEF with AB = DE, AC = DF, and angle BAC > angle EDF. Copy angle EDF to angle EDG at D (Proposition 23), with DG = AC = DF. Join EG and FG. Triangles ABC and DEG are congruent (Proposition 4), so BC = EG. Since DF = DG, angle DFG = angle DGF (Proposition 5). Angle DFG > angle EGF, so angle EFG > angle EGF. Therefore, EG > EF (Proposition 19). Thus, BC > EF.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB = DE$, $AC = DF$, and $\angle BAC$ exceed $\angle EDF$. `[given]`

**2.** At $D$ construct $\angle EDG$ equal to $\angle BAC$. `[I.23]`

**3.** Make $DG = AC = DF$ and join $EG$, $FG$. `[I.3, P1]`

**4.** $EG = BC$. [I.4]

**5.** $DG = DF$, so $\angle DFG = \angle DGF$. [I.5]

**6.** Hence $\angle EFG$ exceeds $\angle EGF$, and $EG$ exceeds $EF$. [I.19]

---

HOW DO YOU KNOW IT IS THERE?

**The point $G$.** Built by [I.23] and [I.3]. The copied angle is the whole method: the larger angle is reproduced at $D$ so the two configurations can be compared in one figure.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.24.** The greater included angle subtends the greater base. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 25

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] still dormant

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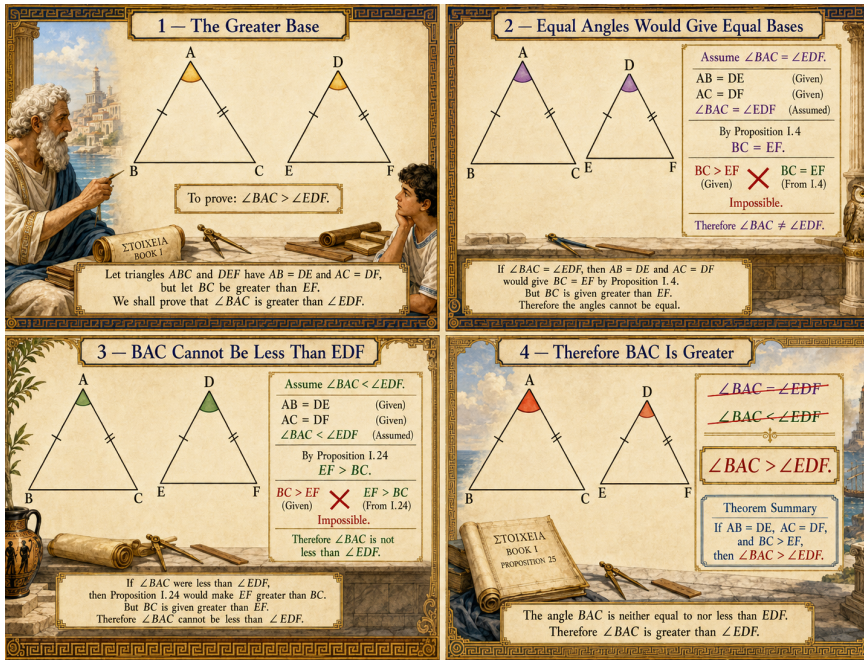


## Statement

> *If two triangles have two corresponding sides equal but one base greater, then the angle subtended by the greater base is greater.*

## Actions

1. Let AB=DE, AC=DF, and BC>EF.
2. Suppose angle BAC is not greater than angle EDF.
3. If the angles were equal, I.4 would give BC=EF, contradiction.
4. If angle BAC were less, then by I.24 BC would be less than EF, contradiction.
5. Therefore angle BAC is greater than angle EDF.

# Integrated Ontological Restatement

**Source formulation.** *If two triangles have two sides equal to two sides, respectively, but one has a base greater than the base of the other, then the former triangle will also have the angle encompassed by the equal straight-lines greater than the corresponding angle in the latter.*

**Ontological role.** Given triangles ABC and DEF with AB = DE, AC = DF, and BC > EF. If angle BAC is not greater than angle EDF, then it is either equal or less. If equal, then BC = EF (Proposition 4), contradicting the given. If less, then BC < EF (Proposition 24), also contradicting. Therefore, angle BAC > angle EDF.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Suppose the included angles were equal; the bases would be equal. `[I.4]`

**2.** Suppose the first were less; the first base would be less. `[I.24]`

**3.** Both contradict the hypothesis; so the first angle is greater.

---

HOW DO YOU KNOW IT IS THERE?

**No new object.** A second pure exclusion argument, on the warrants of I.4 and I.24.

---

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.25.** The greater base implies the greater included angle. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 26

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] still dormant

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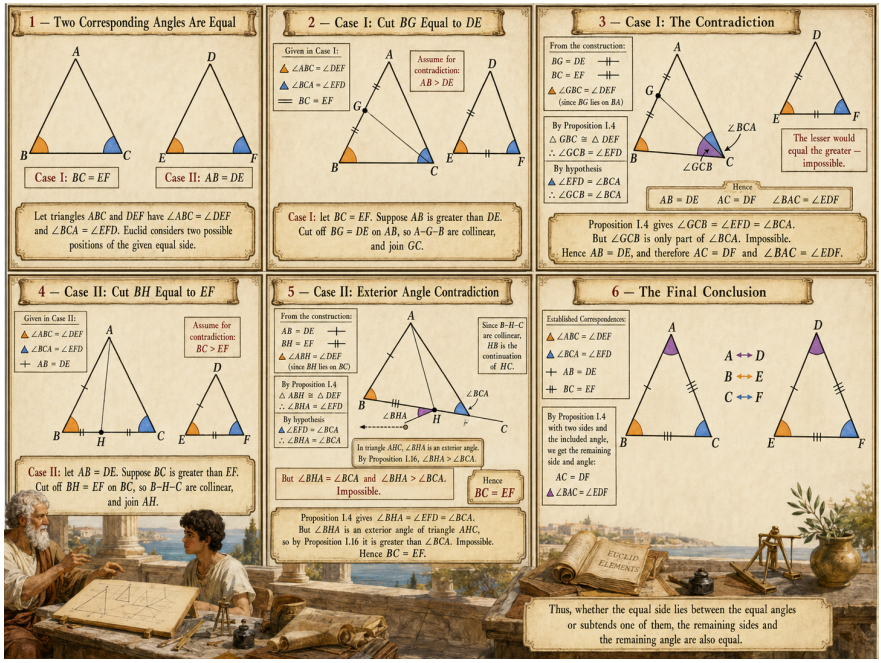


## Statement

> *If two triangles have two corresponding angles equal and one corresponding side equal, then the remaining sides and angle are equal.*

## Actions

1. Let triangles ABC and DEF have two corresponding angles equal, together with one corresponding side equal.
2. First consider the case where the equal side lies between the equal angles.
3. Assume one remaining corresponding side is unequal; cut off an equal part and compare the resulting triangles by I.4.
4. The comparison forces an angle equal to a proper part of itself, impossible.

5. Hence the corresponding sides are equal, and then the remaining side and angle follow from I.4.

6. The same contradiction argument handles the case where the equal side subtends one of the equal angles.

## Integrated Ontological Restatement

**Source formulation.** *If two triangles have two angles equal to two angles, respectively, and one side equal to one side—either that by the equal angles or that subtending one of the equal angles—then the triangles will also have the remaining sides equal to the corresponding remaining sides and the remaining angle equal to the remaining angle.*

**Ontological role.** Given triangles ABC and DEF with angle ABC = angle DEF, angle ACB = angle DFE, and one side equal (first case: BC = EF). If AB $\neq$ DE, let AB > DE. Cut BG = DE (Proposition 3). Join GC. Triangles GBC and DEF are congruent (Proposition 4), so angle GCB = angle DFE = angle ACB, the greater equal to the lesser, impossible. Therefore, AB = DE. The rest follows (Proposition 4). Second case: given AB = DE (the side subtending one of the equal angles). If BC $\neq$ EF, let BC > EF. Cut BH = EF (Proposition 3). Join AH. Triangles ABH and DEF are congruent (Proposition 4), so angle BHA = angle EFD = angle BCA, which is impossible by Proposition 16. Therefore, BC = EF. The rest follows.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let the angles at $B$, $C$ equal those at $E$, $F$, and $BC = EF$. `[given]`

**2.** Suppose $AB \neq DE$; let $AB$ be greater, and cut off $BG = DE$. `[I.3]`

**3.** Join $GC$; then triangles $GBC$, $DEF$ are equal. `[I.4]`

**4.** So $\angle GCB = \angle DFE = \angle ACB$, a part equal to the whole. `[C.N. 5]`

**5.** Impossible; so $AB = DE$, and the triangles are equal. `[I.4]`

How do you know it is there?

**The point** $G$**.** Constructed by [I.3] for refutation. The same shape of argument as I.6 and I.26 throughout: build the offending object, then show it cannot stand.

Added to the repertoire by this proposition

**Warrant I.26.** Two angles and a corresponding side determine the triangle.

No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 27

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] still dormant

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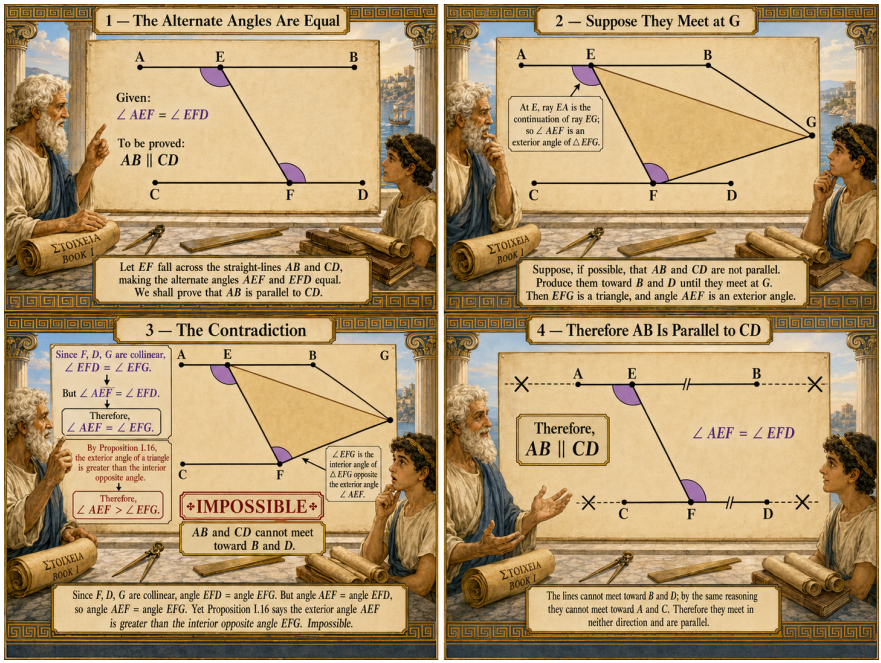


## Statement

> *If a straight line falling on two straight lines makes alternate angles equal, the two straight lines are parallel.*

## Actions

1. Let EF cut AB and CD at G and H, and suppose angle AGH equals GHD.
2. Assume AB and CD are not parallel; then, if produced, they meet on one side, say at K.
3. In triangle KGH, the exterior angle AGH would be greater than the interior opposite angle GHD by I.16.
4. But AGH=GHD by hypothesis, contradiction.

5. Therefore AB and CD do not meet on that side; the same reasoning applies on the other side.

6. Hence AB is parallel to CD.

## Integrated Ontological Restatement

**Source formulation.** *If a straight-line falling across two straight-lines makes the alternate angles equal to one another, then the two straight-lines will be parallel to one another.*

**Ontological role.** Given lines AB and CD crossed by EF, with angle AEF = angle EFD. Suppose AB and CD are not parallel. Then they meet on one side, say at G. In triangle GEF, angle AEF is an exterior angle and angle EFG is an interior opposite angle. But angle AEF = angle EFD = angle EFG, so an exterior angle equals an interior opposite angle, contradicting Proposition 16. Therefore, AB and CD cannot meet; they are parallel (Definition 23).

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The criticism is that Definition 23 already presupposes indefinitely extended straight lines in a flat plane and that the proof tacitly relies on this background geometry.

**Reply within the reconstruction.** Definition 23 is deliberately read cumulatively with Postulate 2 and Postulate 5. Infinite production and planar behaviour are not hidden background conditions: they are explicit stages in the determination of what 'parallel straight lines' means in Book I.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $EF$ fall on $AB$, $CD$ making alternate angles $AEF$, $EFD$ equal. `[given]`

**2.** Suppose $AB$, $CD$ meet when produced, at $G$. `[hypothesis, P2]`

3. Then in triangle $GEF$ the exterior angle $AEF$ equals the interior opposite $EFG$. [given]

4. But the exterior angle must be greater. [I.16]

5. Impossible; therefore they do not meet, and are parallel. [Def. 23]

---

How do you know it is there?

**The point $G$.** Assumed for refutation only. It is never constructed, and the proof is precisely that it cannot exist.

---

**Postulate 5 is not used here.** The existence of parallels is established from [I.16] alone. This is the first of three propositions — I.27, I.28, I.31 — that build the entire positive theory of parallels while the fifth postulate remains dormant.

---

Added to the repertoire by this proposition

**Warrant I.27.** Equal alternate angles imply parallelism. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 28

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] still dormant

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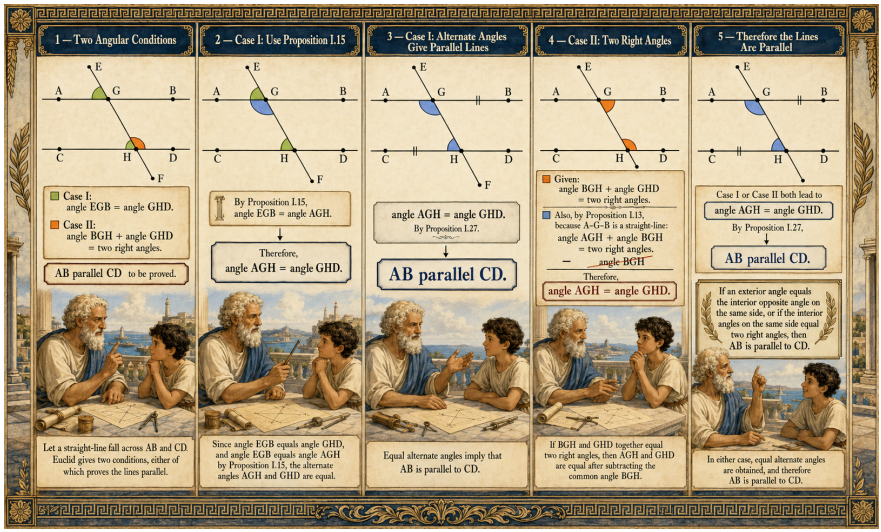


## Statement

> *If a straight line falling on two straight lines makes a corresponding exterior angle equal to an interior opposite angle, or makes interior angles on the same side sum to two right angles, the lines are parallel.*

## Actions

1. Let EF cut AB and CD.
2. Case 1: suppose an exterior angle equals the interior opposite angle. By I.15 the vertically opposite angle is equal, so a pair of alternate angles is equal.
3. Then by I.27 the two lines are parallel.
4. Case 2: suppose the two interior angles on the same side sum to two right angles.
5. By I.13 the adjacent exterior angle with one of them also sums to two right angles.

6. Subtracting gives equality of an exterior and opposite interior angle; hence by the first case the lines are parallel.

## Integrated Ontological Restatement

**Source formulation.** *If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or makes the sum of the internal angles on the same side equal to two right-angles, then the two straight-lines will be parallel to one another.*

**Ontological role.** Given lines AB and CD crossed by EF. Case 1: angle EGB = angle GHD. Since angle EGB = angle AGH (Proposition 15), angle AGH = angle GHD. These are alternate angles, so AB ∥ CD (Proposition 27). Case 2: angle BGH + angle GHD = two right angles. Since angle AGH + angle BGH = two right angles (Proposition 13), angle AGH = angle GHD. These are alternate angles, so AB ∥ CD (Proposition 27).

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The same objection applies to the second criterion for parallelism: it is said to rely on an unstated global flatness and infinite extension.

**Reply within the reconstruction.** The reconstruction again places those properties in the explicit dependency chain Definition 23 + Postulate 2 + Postulate 5, while I.13, I.15, and I.27 provide the local angle relations used in the proof.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $EF$ fall on $AB$, $CD$ with $\angle EGB = \angle GHD$. `[given]`

**2.** $\angle EGB = \angle AGH$. `[I.15]`

**3.** So $\angle AGH = \angle GHD$, and the lines are parallel. `[I.27]`

4. For the second case, $\angle BGH + \angle GHD$ is two right angles, and so is $\angle AGH + \angle BGH$. [I.13]

5. Subtracting gives $\angle AGH = \angle GHD$. [C.N. 3, I.27]

---

HOW DO YOU KNOW IT IS THERE?

**No new object.** I.28 constructs nothing. It converts one criterion for parallelism into two more, using only vertical angles and the straight-line sum.

---

**Postulate 5 is still not used.**

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.28.** Equal corresponding angles, or co-interior angles summing to two right angles, imply parallelism. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 29

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] the meeting condition (dormant — woken here)

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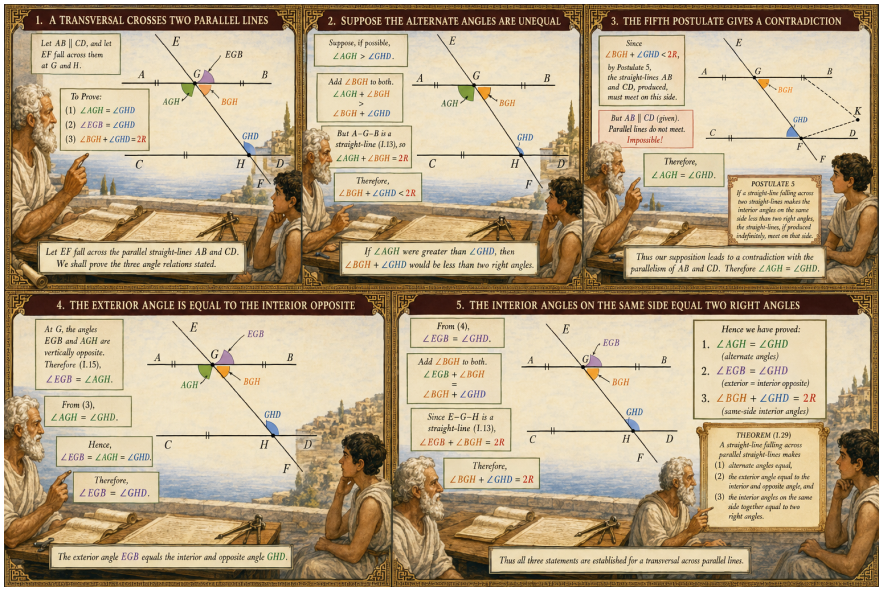


## Statement

> *A straight line falling on parallel straight lines makes alternate angles equal, corresponding angles equal, and interior angles on the same side sum to two right angles.*

## Actions

1. Let EF cut parallel lines AB and CD at G and H.
2. If alternate angles AGH and GHD were unequal, one would be greater.
3. Adding an adjacent angle would make a pair of interior angles on one side sum to less than two right angles, forcing the parallel lines to meet by Postulate 5, contradiction.
4. Therefore the alternate angles are equal.
5. By I.15 and substitution, the corresponding exterior angle equals the opposite interior angle.

6. By I.13, the two interior angles on the same side sum to two right angles.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** These propositions depend upon the *Fifth Postulate*. When interpreted on non-Euclidean surfaces, such as spherical or hyperbolic geometries, the results no longer hold. This dependence has historically been regarded as evidence that the Fifth Postulate cannot be derived from the preceding axioms.

**Critics:**

- **Saccheri (1733)**, *Euclides ab omni naevo vindicatus*.
- **Lambert (1766)**, *Theorie der Parallellinien*.
- **Gauss**, **Lobachevsky**, **Bolyai**, and **Riemann** later developed non-Euclidean geometries that contradicted the Euclidean parallel postulate.

**Response:** I disagree with this interpretation. The Fifth Postulate is not an empirical assumption but a definitional axiom that determines the nature of the plane itself. It specifies the geometric structure in which all preceding propositions operate. Hence, the validity of Propositions I.29–I.31 remains intact, since within Euclidean geometry the Fifth Postulate defines the plane rather than emerging from it.

## Integrated Ontological Restatement

**Source formulation.** *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external angle equal to the internal and opposite angle, and the sum of the internal angles on the same side equal to two right-angles.*

**Ontological role.** Given AB || CD crossed by EF. Suppose angle AGH $\neq$ angle GHD. Let angle AGH > angle GHD. Add angle BGH to both. Then angle AGH + angle BGH > angle BGH + angle GHD. But angle AGH +

angle BGH = two right angles (Proposition 13), so angle BGH + angle GHD < two right angles. By Postulate 5, AB and CD would meet on that side, contradicting Definition 23. Therefore, angle AGH = angle GHD. The rest follows from Proposition 15 and Proposition 13.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** I.29 is the first proposition that explicitly invokes Postulate 5. Historically, the fifth postulate has often been treated as a troublesome assumption that Euclid postponed as long as possible.

**Reply within the reconstruction.** The ontological reading changes the status of the dependency without denying it. I.29 depends on Postulate 5 because Postulate 5 selects the plane in which the theorem is being proved. The proposition unfolds the consequences of the domain-defining condition; it does not purport to derive that condition from the previous four postulates.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $EF$ fall on the parallels $AB$, $CD$. `[given]`
2. Suppose the alternate angles $AGH$, $GHD$ were unequal; let $AGH$ be greater. `[hypothesis]`
3. Add $\angle BGH$ to each: then $\angle BGH + \angle GHD$ is less than two right angles. `[C.N., I.13]`
4. Therefore $AB$ and $CD$, produced, meet on that side. `[C5]`
5. But they are parallel, and do not meet. `[Def. 23]`
6. Therefore the alternate angles are equal; the other two results follow. `[I.15, I.13]`

How do you know it is there?

**The meeting point.** The only object in Book I whose existence is asserted by [C5], and it is asserted only to be contradicted. Postulate 5 is used exactly once, in a reductio, to license the inference that unequal co-interior angles *force* a meeting.

---

**Postulate 5 wakes here, at the twenty-ninth proposition of forty-eight.** Everything before this point — all the congruence theorems, all the inequalities, the existence of parallels at I.27 and I.28 — was proved without it. What C5 supplies is not the existence of parallels but the *converse* inference: from parallelism back to the equality of angles. A constraint held in reserve for twenty-eight propositions and then spent on a single reductio is not a blemish on the system. It is the most carefully rationed assumption in the book.

---

Added to the repertoire by this proposition

**Warrant I.29.** Parallels cut by a transversal give equal alternate angles, equal corresponding angles, and co-interior angles summing to two right angles. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 30

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] the meeting condition (in force since I.29)

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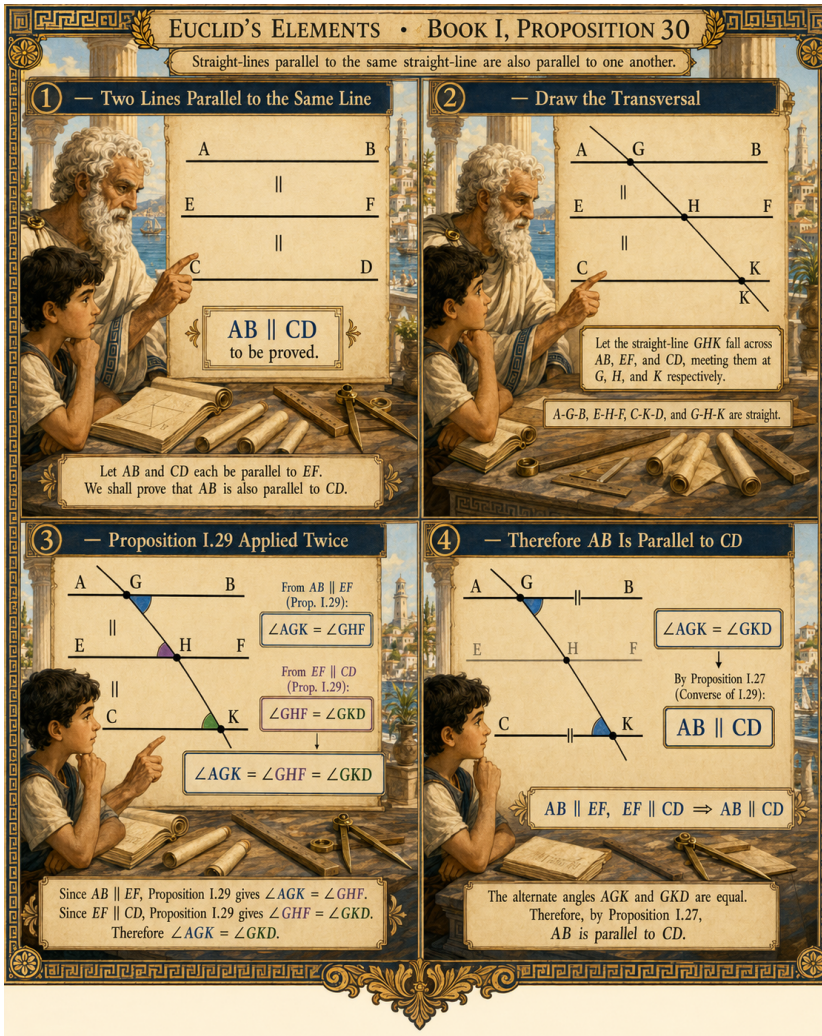


## Statement

*Straight lines parallel to the same straight line are parallel to one another.*

## Actions

1. Let AB and CD each be parallel to EF.
2. Draw a transversal GH crossing all three lines.
3. By I.29, the alternate angle made with AB equals the corresponding alternate angle made with EF.
4. Likewise the alternate angle made with CD equals that with EF.
5. Therefore the alternate angles for AB and CD are equal to one another.
6. By I.27, AB is parallel to CD.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** These propositions depend upon the *Fifth Postulate*. When interpreted on non-Euclidean surfaces, such as spherical or hyperbolic geometries, the results no longer hold. This dependence has historically been regarded as evidence that the Fifth Postulate cannot be derived from the preceding axioms.

**Critics:**

- **Saccheri (1733)**, *Euclides ab omni naevo vindicatus*.
- **Lambert (1766)**, *Theorie der Parallellinien*.
- **Gauss**, **Lobachevsky**, **Bolyai**, and **Riemann** later developed non-Euclidean geometries that contradicted the Euclidean parallel postulate.

**Response:** I disagree with this interpretation. The Fifth Postulate is not an empirical assumption but a definitional axiom that determines the nature of the plane itself. It specifies the geometric structure in which all preceding propositions operate. Hence, the validity of Propositions I.29–I.31 remains intact, since within Euclidean geometry the Fifth Postulate defines the plane rather than emerging from it.

# Integrated Ontological Restatement

**Source formulation.** *Straight-lines parallel to the same straight-line are also parallel to one another.*

**Ontological role.** Given AB ∥ EF and CD ∥ EF. Let GK cross AB, CD, and EF. By Proposition 29, angle AGK = angle GHF (since AB ∥ EF), and angle GHF = angle GKD (since EF ∥ CD). Therefore, angle AGK = angle GKD. These are alternate angles, so AB ∥ CD (Proposition 27).

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $AB$ and $CD$ each be parallel to $EF$, and let $GK$ fall across all three. `[P1]`
2. Since $AB \parallel EF$, the alternate angles $AGK$, $GHF$ are equal. `[I.29]`
3. Since $EF \parallel CD$, the angles $GHF$, $HKD$ are equal. `[I.29]`
4. Therefore $\angle AGK = \angle HKD$, and $AB \parallel CD$. `[C.N. 1, I.27]`

---

HOW DO YOU KNOW IT IS THERE?

**The transversal $GK$.** Drawn by `[P1]` between points taken on two of the lines. The proof needs a transversal to compare angles, and one is always available.

---

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.30.** Parallelism is transitive. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 31

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11]
- [I.12] drop a perpendicular from a point outside a line
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [C5] the meeting condition (in force since I.29)

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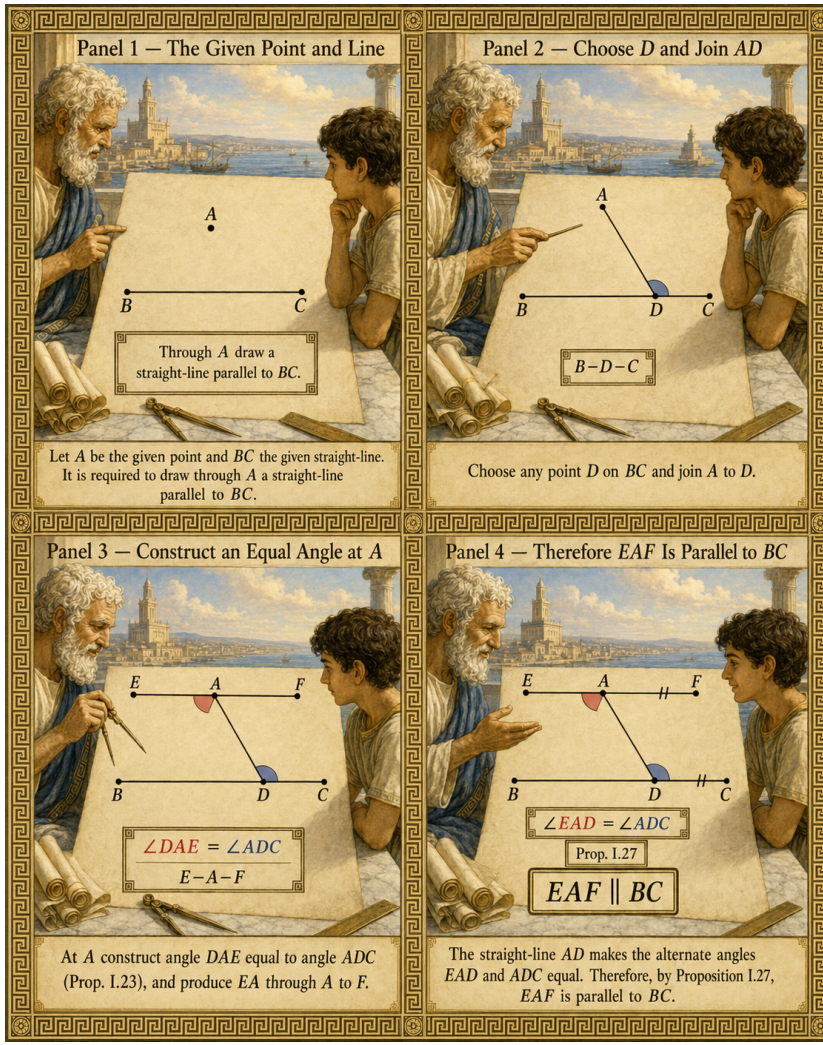


## Statement

*Through a given point, to draw a straight line parallel to a given straight line.*

## Actions

1. Let A be the given point and BC the given straight line.
2. Choose D on BC and join AD.
3. At A construct angle DAE equal to angle ADC by I.23.
4. Produce AE as a straight line through A.
5. Since AD is a transversal and alternate angles DAE and ADC are equal, AE is parallel to BC by I.27.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** These propositions depend upon the *Fifth Postulate*. When interpreted on non-Euclidean surfaces, such as spherical or hyperbolic geometries, the results no longer hold. This dependence has historically been regarded as evidence that the Fifth Postulate cannot be derived from the preceding axioms.

**Critics:**

- **Saccheri (1733)**, *Euclides ab omni naevo vindicatus*.
- **Lambert (1766)**, *Theorie der Parallellinien*.
- **Gauss**, **Lobachevsky**, **Bolyai**, and **Riemann** later developed non-Euclidean geometries that contradicted the Euclidean parallel postulate.

**Response:** I disagree with this interpretation. The Fifth Postulate is not an empirical assumption but a definitional axiom that determines the nature of the plane itself. It specifies the geometric structure in which all preceding propositions operate. Hence, the validity of Propositions I.29–I.31 remains intact, since within Euclidean geometry the Fifth Postulate defines the plane rather than emerging from it.

# Integrated Ontological Restatement

**Source formulation.** *To draw a straight-line parallel to a given straight-line through a given point.*

**Ontological role.** Given point A and line BC. Take point D on BC. Join AD. Copy angle ADC to angle DAE at A (Proposition 23). Extend AF straight-on with EA (Postulate 2). Since angle EAD = angle ADC, and these are alternate angles, EF ∥ BC (Proposition 27).

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $A$ be the given point and $BC$ the given line. Take $D$ on $BC$ and join $AD$. `[P1]`
2. At $A$ on the line $AD$ construct $\angle DAE$ equal to $\angle ADC$. `[I.23]`
3. Produce $EA$ to $F$. `[P2]`
4. The alternate angles being equal, $EF \parallel BC$. `[I.27]`

---

HOW DO YOU KNOW IT IS THERE?

**The line $EF$.** Built entirely from `[I.23]` and `[P2]`, and certified parallel by `[I.27]` — which does not use the fifth postulate. *The construction of a parallel does not need C5.* Only the converse inference at I.29 does.

---

This is the last purely linear construction of Book I. Everything after it is about area.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

`I.31` Draw, through a given point, a parallel to a given line.

# Proposition 32

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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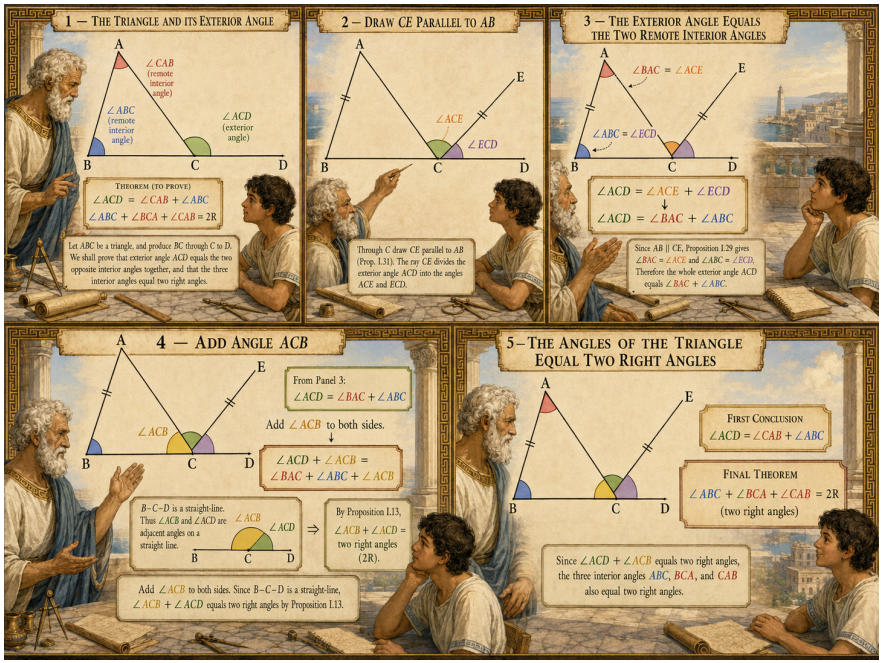


## Statement

*In any triangle, an exterior angle equals the sum of the two interior opposite angles, and the three interior angles equal two right angles.*

## Actions

1. Let side BC of triangle ABC be produced to D.
2. Through C draw CE parallel to AB by I.31.
3. By I.29, angle ACE equals angle CAB, and angle ECD equals angle ABC.
4. Therefore exterior angle ACD=ACE+ECD=angle CAB+angle ABC.
5. Add angle ACB to both sides.

6. Since ACD+ACB equals two right angles by I.13, the three interior angles ABC+BCA+CAB equal two right angles.

## Integrated Ontological Restatement

**Source formulation.** *In any triangle, if one of the sides is produced, the external angle is equal to the sum of the two internal and opposite angles, and the sum of the three internal angles of the triangle is equal to two right-angles.*

**Ontological role.** Given triangle ABC with BC produced to D. Draw CE through C parallel to AB (Proposition 31). By Proposition 29, angle BAC = angle ACE (alternate), and angle ABC = angle ECD (external to internal). Therefore, angle ACD = angle ACE + angle ECD = angle BAC + angle ABC. Also, angle BCA + angle ACD = two right angles (Proposition 13), so angle BCA + angle BAC + angle ABC = two right angles.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $ABC$ have $BC$ produced to $D$. `[P2]`

**2.** Through $C$ draw $CE$ parallel to $AB$. `[I.31]`

**3.** $\angle ACE = \angle BAC$, alternate angles. `[I.29]`

**4.** $\angle ECD = \angle ABC$, corresponding angles. `[I.29]`

**5.** Hence the exterior angle $ACD$ equals the two interior opposite angles. `[C.N. 2]`

**6.** Adding $\angle ACB$ to each side gives two right angles. `[I.13]`

---

HOW DO YOU KNOW IT IS THERE?

**The line $CE$.** Output of `[I.31]`. This is where the ledger repays its investment: the angle sum, bounded but not determined at I.17, becomes exact the moment a parallel can be drawn and I.29 can be cited.

Added to the repertoire by this proposition

**Warrant I.32.** The exterior angle equals the sum of the remote interior angles; the three interior angles sum to two right angles. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 33

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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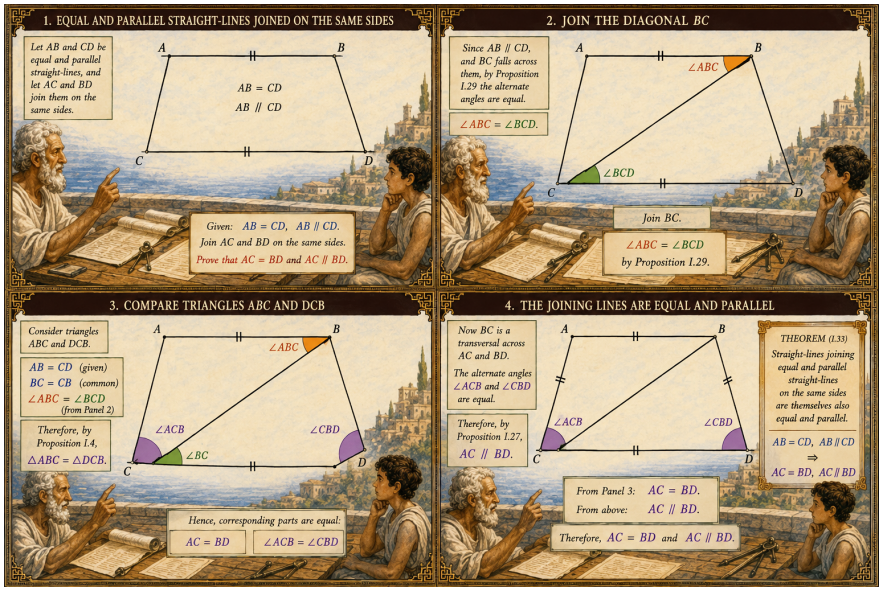


## Statement

> *Straight lines joining the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.*

## Actions

1. Let AB and CD be equal and parallel, with A joined to C and B to D.
2. Join BC.
3. Because AB is parallel to CD, alternate angles ABC and BCD are equal by I.29.
4. With AB=CD and BC common, triangles ABC and DCB are equal by I.4.
5. Hence AC=BD and angle ACB=CBD.
6. By I.27, AC is parallel to BD. Thus the joining lines are equal and parallel.

## Integrated Ontological Restatement

**Source formulation.** *Straight-lines joining equal and parallel straight-lines on the same sides are themselves also equal and parallel.*

**Ontological role.** Given AB = CD and AB ∥ CD, with AD and BC joining them on the same sides. Join BD. Since AB ∥ CD, angle ABD = angle BDC (Proposition 29). Since AB = CD and BD is common, triangles ABD and CDB are congruent (Proposition 4). Therefore, AD = BC and angle ADB = angle DBC. These are alternate angles, so AD ∥ BC (Proposition 27).

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The objection is that joining equal parallel straight lines is assumed to produce a closed rectilinear figure with the expected transversal-angle relations.

**Reply within the reconstruction.** Closure is obtained by the explicitly drawn joining segments, while the angle relations follow from I.29 and the equality follows from I.4. The proof therefore invokes the already determined behaviour of parallels rather than a new closure axiom.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $AB$, $CD$ be equal and parallel; join $AC$ and $BD$. [P1]
2. Join $BC$. [P1]
3. $\angle ABC = \angle BCD$, alternate. [I.29]
4. Triangles $ABC$, $DCB$ have $AB = CD$, $BC$ common, included angles equal; so $AC = BD$. [I.4]
5. The alternate angles being equal, $AC \parallel BD$. [I.27]

How do you know it is there?

**The diagonal $BC$.** Drawn by [P1]. Splitting a quadrilateral by a diagonal into two triangles is the standard device of the area sequence that follows.

---

---

Added to the repertoire by this proposition

**Warrant I.33.** Equal parallels are joined by equal parallels. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 34

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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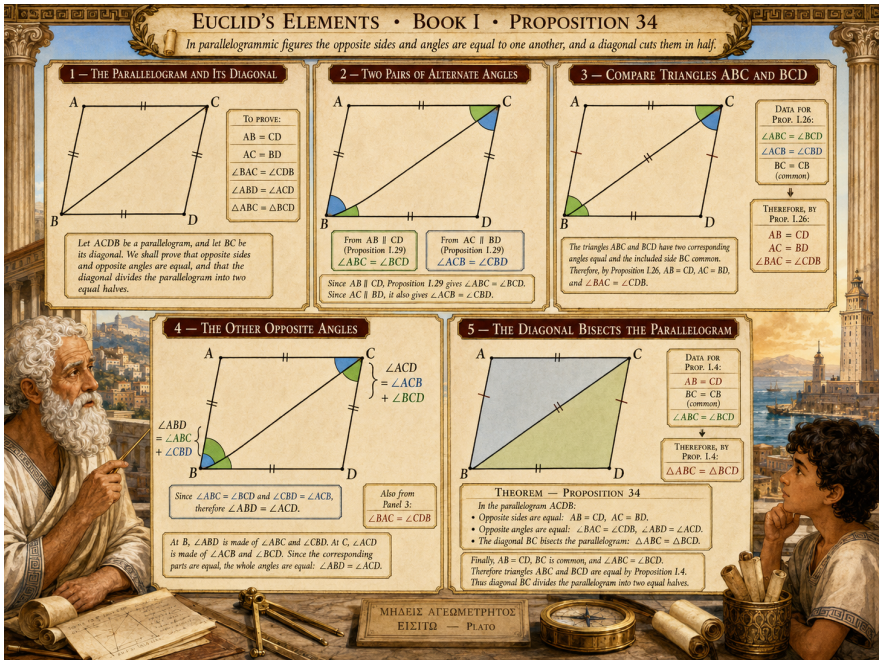


## Statement

> *In a parallelogram, opposite sides and opposite angles are equal, and a diagonal bisects the area.*

## Actions

1. Let ACDB be a parallelogram and draw diagonal BC.
2. Because opposite sides are parallel, alternate angles produced by BC are equal by I.29.
3. With BC common, triangles ABC and DCB are equal by I.26.
4. Therefore opposite sides AB=CD and AC=BD, and opposite angles are equal.

5. Since the two triangles are equal, diagonal BC bisects the parallelogram into equal areas.

## Integrated Ontological Restatement

**Source formulation.** *In parallelogrammic figures, the opposite sides and angles are equal to one another, and a diagonal cuts them in half.*

**Ontological role.** Given parallelogram ACDB with diagonal BC. Since AB ‖ CD, angle ABC = angle BCD (Proposition 29). Since AC ‖ BD, angle ACB = angle CBD (Proposition 29). With BC common, triangles ABC and BCD are congruent (Proposition 26). Therefore, AB = CD, AC = BD, and angle BAC = angle CDB. Also, triangle ABC = triangle BCD, so the diagonal cuts the parallelogram in half.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The same family of criticism regards the parallelogram properties as depending on an unspoken flatness assumption and on exact same-side angle relations.

**Reply within the reconstruction.** At this stage flatness is no longer background intuition: I.29 has already derived the relevant alternate-angle relations from Postulate 5, and I.26 converts them into triangle congruence. I.34 consequently depends on proved propositions, not on a fresh diagrammatic assumption.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $ACDB$ be a parallelogram with diameter $BC$. `[given]`

**2.** $\angle ABC = \angle BCD$ and $\angle ACB = \angle CBD$, alternate angles. `[I.29]`

**3.** Triangles $ABC$, $DCB$ have two angles and the common side $BC$; so they are equal in all respects. `[I.26]`

4. Hence the opposite sides and angles are equal.

5. The two triangles together are the whole parallelogram, and they are equal; so each measures the parallelogram exactly twice. [C.N. 2]

6. That is what it is for the diameter to bisect the area: the triangle is a *part* of the parallelogram in the sense of [B0] — the lesser that measures the greater. [B0]

---

How do you know it is there?

**The word *half*.** Introduced here for the first time in Book I, and it is not a new notion. By the prior layer a part is the lesser magnitude that measures the greater; a half is the part that measures it exactly twice. Nothing is assumed about halves that part and multiple did not already supply.

---

**Note.** Keep this reading of *half* in view: it is what makes I.37, I.38 and I.41 go through without a sixth Common Notion.

---

Added to the repertoire by this proposition

**Warrant I.34.** Opposite sides and angles of a parallelogram are equal; the diameter bisects it — that is, each triangle measures the parallelogram twice. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 35

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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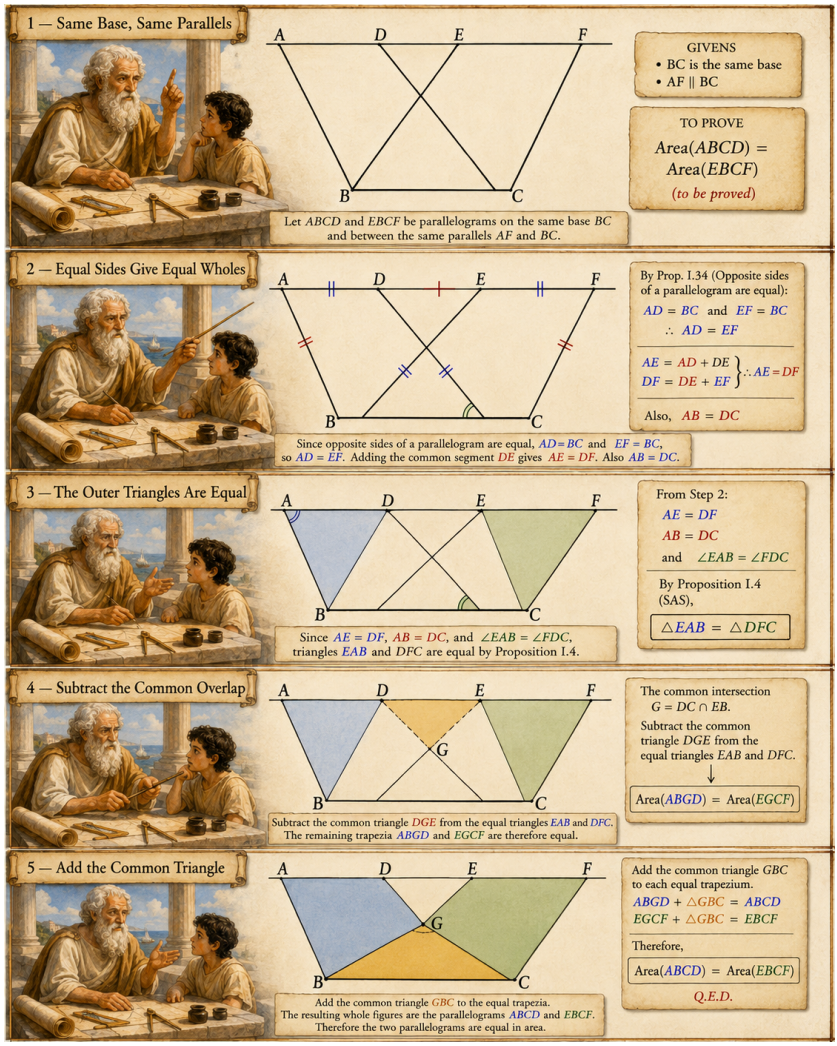


# Statement

*Parallelograms on the same base and between the same parallels are equal in area.*

## Actions

1. Let parallelograms ABCD and EBCF stand on the same base BC and between the same parallels BC and AF.
2. Opposite sides in each parallelogram are equal by I.34: AD=BC and EF=BC, hence AD=EF.
3. Using the collinear upper line, subtract common segments to obtain the required equality of the outer top segments.
4. Compare the outer triangles cut off from the two parallelograms; they have equal sides and corresponding angles and are equal.
5. Subtract these equal triangles from the same encompassing figure.
6. The remainders are the two parallelograms, hence they are equal in area.

## Integrated Ontological Restatement

**Source formulation.** *Parallelograms which are on the same base and between the same parallels are equal to one another.*

**Ontological role.** Given parallelograms ABCD and EBCF on base BC between parallels AF and BC. Since ABCD is a parallelogram, AD = BC (Proposition 34). Since EBCF is a parallelogram, EF = BC (Proposition 34). Therefore, AD = EF. Since AB = DC and angle FDC = angle EAB (Proposition 29, external-internal), triangles EAB and FDC are congruent (Proposition 4). Subtracting common triangle DGE and adding common triangle GBC (Common Notions 2 and 3) yields parallelogram ABCD = parallelogram EBCF.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** For I.35–I.38 the standard concern shifts from incidence to equality of figures: subtraction and addition of triangles are said to require an implicit additivity principle for geometrical regions.

**Reply within the reconstruction.** The reconstruction applies Common Notions 2 and 3 directly to geometrical figures already proved equal. The equality involved is not numerical area but equivalence under composition and decomposition. The sequence I.33–I.45 develops these substitutions before they are used in I.47.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABCD$, $EBCF$ be on the base $BC$ and between the same parallels. `[given]`
2. $AD = BC = EF$, opposite sides. `[I.34]`
3. Triangles $ABE$, $DCF$ are equal. `[I.4]`
4. Subtracting each in turn from the trapezium $ABCF$ leaves the parallelograms equal. `[C.N. 3]`

---

HOW DO YOU KNOW IT IS THERE?

**Equality without measurement.** No number appears. *Equal* here means equal in area by decomposition and subtraction — Common Notions 2 and 3 doing the work a formula would do later.

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---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.35.** Parallelograms on the same base and in the same parallels are equal. No new action: nothing is drawable now that was not drawable before.

The second ledger grows; the first does not.

---

# Proposition 36

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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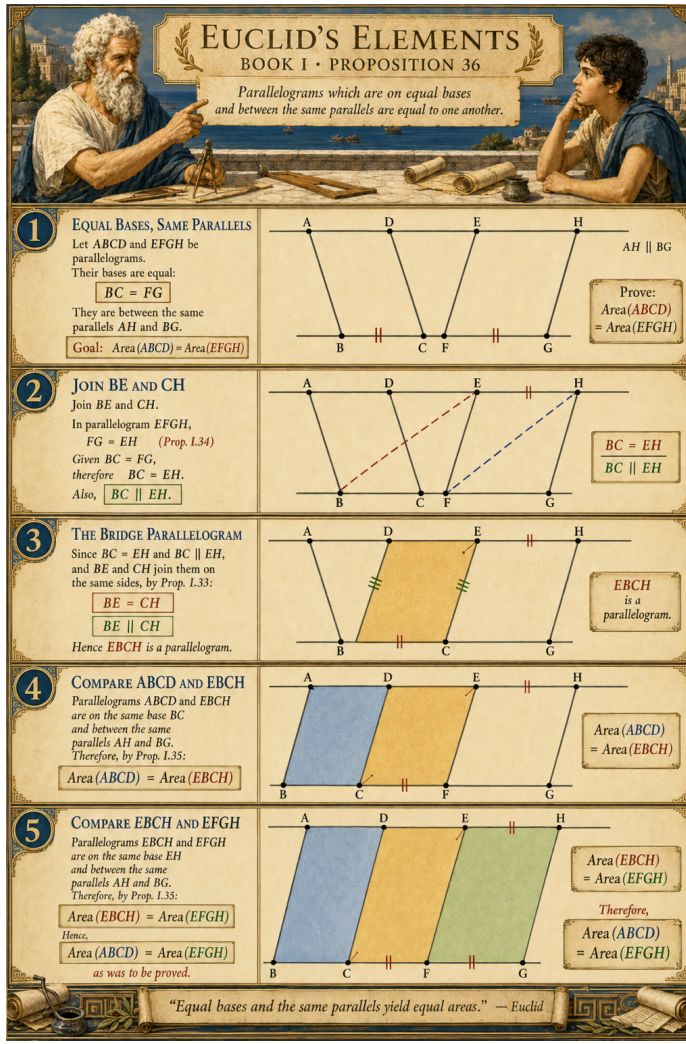


## Statement

*Parallelograms on equal bases and between the same parallels are equal in area.*

## Actions

1. Let ABCD and EFGH be parallelograms on equal bases BC and FG and between the same parallels.
2. Join suitable endpoints to form a third parallelogram spanning the equal bases.
3. By I.33 the connecting sides are equal and parallel, so the auxiliary quadrilateral is a parallelogram.
4. By I.35, one given parallelogram equals the auxiliary parallelogram because they share a base and parallels.
5. Likewise the other given parallelogram equals the same auxiliary parallelogram.
6. Therefore the two given parallelograms are equal in area.

## Integrated Ontological Restatement

**Source formulation.** *Parallelograms which are on equal bases and between the same parallels are equal to one another.*

**Ontological role.** Given parallelograms ABCD and EFGH on equal bases BC and FG between parallels AH and BG. Join BE and CH. Since BC = FG and BC = EH (opposite sides of parallelogram), FG = EH. Construct parallelogram EBCH. By Proposition 35, ABCD = EBCH (same base BC) and EBCH = EFGH (same base EH). Therefore, ABCD = EFGH (Common Notion 1).

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The objection is that equality of parallelograms on equal bases relies on a tacit measure of area or on an unstated rule that equal figures remain equal under recombination.

**Reply within the reconstruction.** I.36 reduces the claim to I.35 twice and then uses transitivity (Common Notion 1). The proof is therefore formulated entirely in terms of previously established figure-equalities rather than an independently quantified area.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABCD$, $EFGH$ be on the equal bases $BC$, $FG$, in the same parallels. `[given]`
2. Join $BE$ and $CH$. `[P1]`
3. $BC = FG = EH$, and $EH \parallel BC$; so $EB$, $HC$ are equal and parallel. `[I.34, I.33]`
4. Hence $EBCH$ is a parallelogram equal to each of the others. `[I.35]`

---

HOW DO YOU KNOW IT IS THERE?

**The parallelogram $EBCH$.** An intermediary, built to reduce the equal-bases case to the same-base case already settled at I.35.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.36.** Parallelograms on equal bases in the same parallels are equal.
No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 37

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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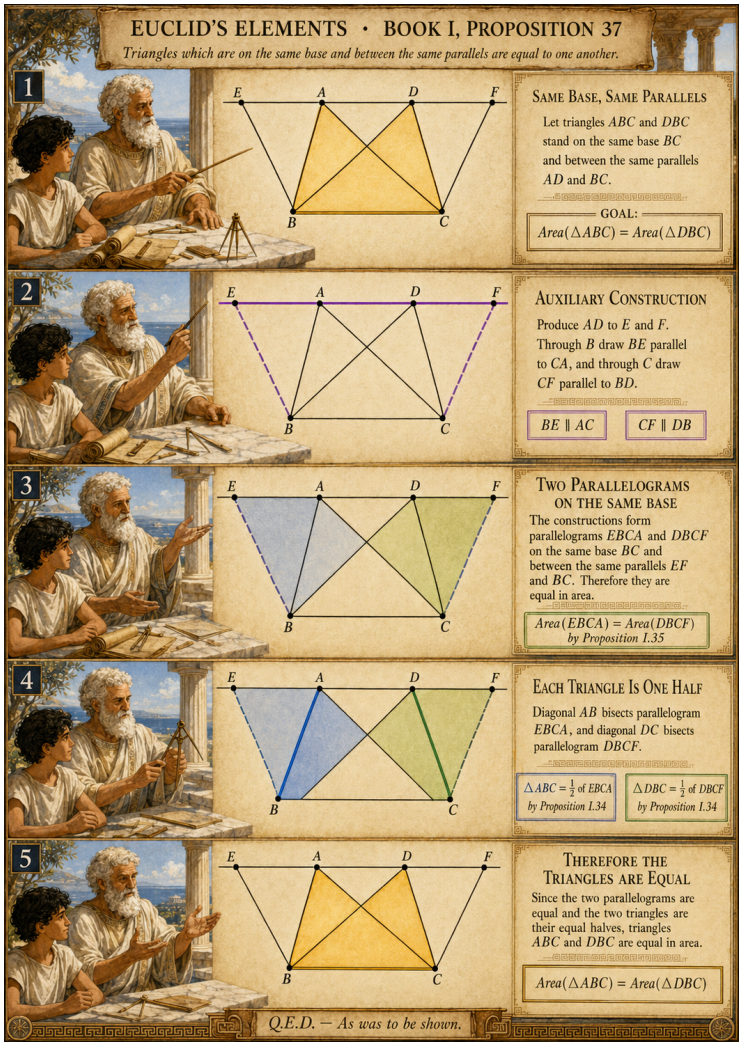


## Statement

*Triangles on the same base and between the same parallels are equal in area.*

## Actions

1. Let triangles ABC and DBC stand on the same base BC and between the same parallels BC and AD.
2. Through B and C draw lines parallel to AC and BD to complete parallelograms on the common base BC.
3. The completed parallelograms are on the same base BC and between the same parallels, so by I.35 they are equal.
4. Each parallelogram is double its corresponding triangle because a diagonal bisects a parallelogram by I.34.
5. Halves of equal areas are equal.
6. Therefore triangle ABC equals triangle DBC.

## Integrated Ontological Restatement

**Source formulation.** *Triangles which are on the same base and between the same parallels are equal to one another.*

**Ontological role.** Given triangles ABC and DBC on base BC between parallels AD and BC. Draw lines through A and D parallel to BC and through B and C parallel to AD, forming parallelograms EBCA and DBCF. By Proposition 35, these parallelograms are equal. By Proposition 34, triangle ABC is half of EBCA, and triangle DBC is half of DBCF. By Common Notion 1 and the implicit rule that halves of equals are equal, triangle ABC = triangle DBC.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The supplied assumption catalogue singles out the maxim 'halves of equal things are equal' as an apparently additional common notion.

**Reply within the reconstruction.** This is an important test for the no-hidden-assumptions thesis. The reconstruction treats 'half' not as a new numerical operator but as the result of the same proved bisection relation: I.34 establishes that a diagonal divides each parallelogram into two equal triangles. Equal wholes subjected to the same established bisection relation yield corresponding equal halves within the existing equality-and-substitution calculus. Thus the maxim is taken as a derived corollary of the already proved equalities and Common Notions 1–3, not as a new primitive axiom.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$, $DBC$ be on the same base $BC$ and between the same parallels $AD$, $BC$. `[given]`
2. Produce $AD$ in both directions. `[P2]`
3. Through $B$ draw $BE$ parallel to $CA$. `[I.31]`
4. Through $C$ draw $CF$ parallel to $BD$. `[I.31]`
5. $EBCA$ and $DBCF$ are parallelograms on the same base $BC$ and in the same parallels; therefore they are equal. `[I.35]`
6. The diameter $AB$ bisects $EBCA$: triangle $ABC$ measures it exactly twice. `[I.34]`
7. The diameter $DC$ bisects $DBCF$: triangle $DBC$ measures it exactly twice. `[I.34]`
8. The two parallelograms are equal, and each is measured exactly twice by its triangle; therefore the two triangles are equal. `[B0 — see panel]`
9. Therefore triangle $ABC$ equals triangle $DBC$. `[C.N. 1]`

---

HOW DO YOU KNOW IT IS THERE?

**The lines $BE$ and $CF$.** Outputs of `[I.31]`. Triangles are compared by completing them to parallelograms, which is possible only because the parallel construction has been earned. **Step 8 — and the oldest patch in the**

***Elements.*** This is the step at which readers have felt the floorboard give, and it is exactly where the manuscript tradition inserted an extra axiom. Some traditions carry, beside Euclid's five, further Common Notions about doubles and halves, including *halves of equals are equal*. Heiberg treated them as interpolations; Heath reports that the halves-of-equals principle entered the tradition early, before Theon, and discusses it precisely here, at I.37.

The reason for the patch is plain. A reader who begins at Book I meets, at this proposition, two figures said to be halves of two equal figures, and is asked to conclude that the halves are equal — having been granted nothing whatever about halves. Common Notions 1–5 speak of equals added, equals subtracted, coincidence, and whole and part. Not one of them mentions a half.

But the patch answers the wrong question. What was missing is not an axiom. It is a book. **Why no axiom is needed.** Book 0 places the quantitative science before the geometry. The primitive term *meros* receives its two formal definitions at VII.3 and V.1 — *a part is the lesser that measures the greater* — and Book V supplies part, multiple, ratio and proportion for homogeneous magnitudes generally.

Once a magnitude may be taken as a multiple of another, *double* and *half* are not fresh notions needing fresh axioms. They are names of particular multiples and parts: a half is the part that measures its whole exactly twice, and a double is that whole. Their equality properties follow from what part and multiple already are, together with Common Notions 2 and 3 and the whole–part order of Common Notion 5.

So step 8 cites the prior layer, not a sixth Common Notion. Written out: triangle $ABC$ measures parallelogram $EBCA$ exactly twice; triangle $DBC$ measures parallelogram $DBCF$ exactly twice; the two parallelograms are equal; therefore the two triangles are equal, since equal magnitudes measured the same number of times by their parts have equal parts. **What this costs, and what it buys.** It costs the reader a book: Book V must be read before Book I, which is the whole argument of Book 0. It buys the removal of an interpolation that stood in the text for two thousand years. Book I was never missing an axiom. It was missing a *book* — and the book was sitting four hundred pages further on, waiting.

**The same step recurs at I.38 and, inverted, at I.41.** Each is discharged the same way, and none of them needs an axiom of halves.

Added to the repertoire by this proposition

**Warrant I.37.** Triangles on the same base and in the same parallels are equal. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 38

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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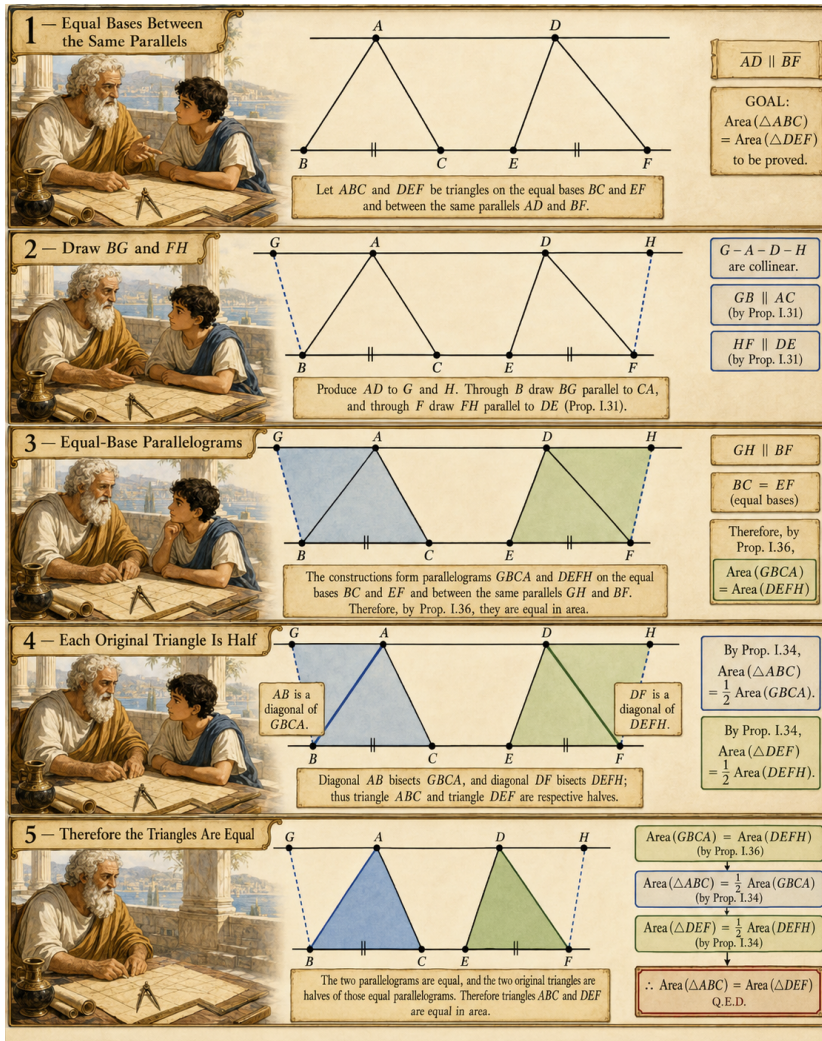


## Statement

*Triangles on equal bases and between the same parallels are equal in area.*

## Actions

1. Let triangles ABC and DEF stand on equal bases BC and EF and between the same parallels.
2. Complete parallelograms on BC and EF using parallels through the opposite vertices.
3. By I.36 the completed parallelograms are equal because they are on equal bases and between the same parallels.
4. Each triangle is half of its corresponding parallelogram by I.34.
5. Therefore the two triangles are equal in area.

## Integrated Ontological Restatement

**Source formulation.** *Triangles which are on equal bases and between the same parallels are equal to one another.*

**Ontological role.** Given triangles ABC and DEF on equal bases BC and EF between parallels BF and AD. Through B draw BG ∥ CA, and through F draw FH ∥ DE, forming parallelograms GBCA and DEFH. By Proposition 36, these parallelograms are equal. By Proposition 34, triangle ABC is half of GBCA, and triangle FED is half of DEFH. Therefore, triangle ABC = triangle DEF.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** As in I.37, the modern objection is that passing from equal parallelograms to their triangular halves requires an extra additivity or halving principle.

**Reply within the reconstruction.** The response is the same: the halves are geometrically characterized by diagonals and the relevant parallelogram equalities have already been proved. Equality is transferred through those proved decompositions rather than through a numerical area function.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$, $DEF$ be on the equal bases $BC$, $EF$ and in the same parallels. `[given]`
2. Complete each triangle to a parallelogram, as at I.37. `[P2, I.31]`
3. The two parallelograms are on equal bases in the same parallels; therefore they are equal. `[I.36]`
4. Each triangle measures its own parallelogram exactly twice. `[I.34]`
5. Equal magnitudes measured the same number of times by their parts have equal parts. `[B0]`
6. Therefore the triangles are equal. `[C.N. 1]`

---

HOW DO YOU KNOW IT IS THERE?

**Step 5.** The identical move to step 8 of I.37, and discharged identically: not by a Common Notion about halves, but by part and multiple from the prior layer. Written out once at I.37; cited here.

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---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.38.** Triangles on equal bases in the same parallels are equal. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 39

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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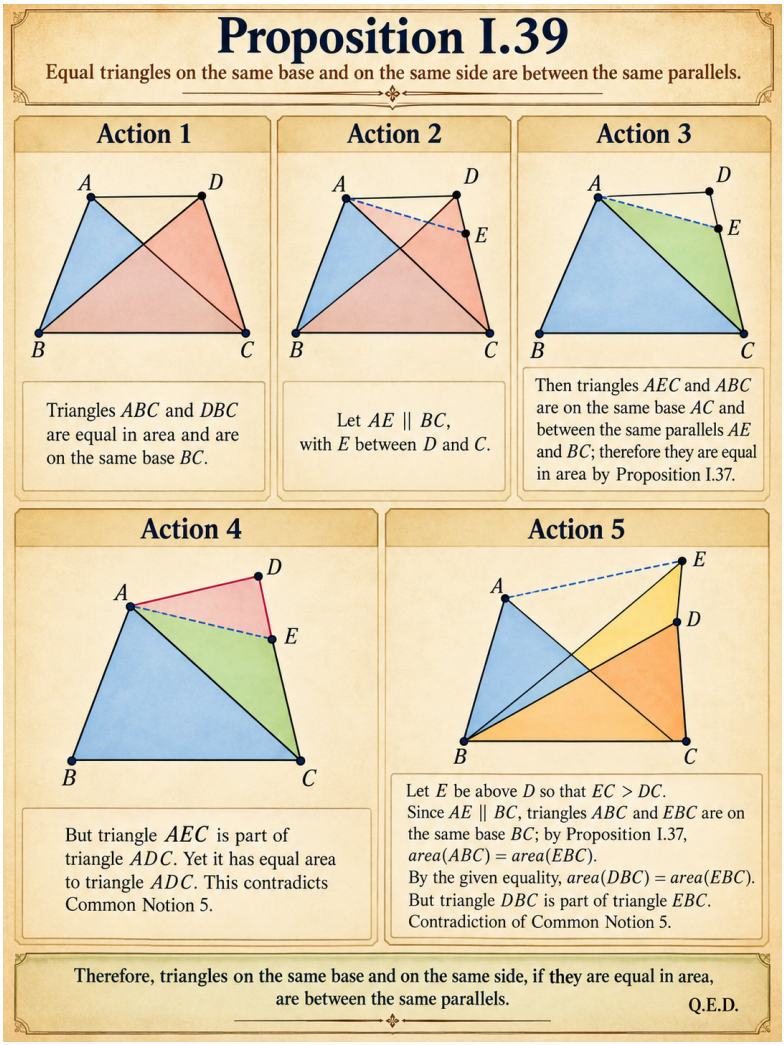


## Statement

*Equal triangles on the same base and on the same side are between the same parallels.*

## Actions

1. Let equal triangles ABC and DBC stand on the same base BC and same side.
2. Join AD.
3. Suppose AD is not parallel to BC; through A draw AE parallel to BC meeting DB at E.
4. Then triangles ABC and EBC are equal by I.37 because they share base BC and lie between the same parallels.
5. But triangle ABC also equals triangle DBC by hypothesis, so EBC=DBC.
6. This is impossible because one is a proper part of the other. Therefore AD is parallel to BC.

## Integrated Ontological Restatement

**Source formulation.** *Equal triangles which are on the same base and on the same side are also between the same parallels.*

**Ontological role.** Given triangles ABC and DBC equal, on base BC on the same side. Join AD. Suppose AD is not parallel to BC. Draw AE ∥ BC through A (Proposition 31) and join EC. By Proposition 37, triangle ABC = triangle EBC. But triangle ABC = triangle DBC (given), so triangle DBC = triangle EBC, the greater equal to the lesser (Common Notion 5), impossible. Therefore, AD ∥ BC.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $ABC$, $DBC$ be equal, on $BC$, on the same side. Join $AD$. `[P1]`

**2.** Suppose $AD$ not parallel to $BC$; draw $AE$ parallel to $BC$ meeting $BD$ at $E$, and join $EC$. `[I.31]`

**3.** Then triangle $ABC$ equals triangle $EBC$. `[I.37]`

**4.** So $EBC$ equals $DBC$, the part equal to the whole. [C.N. 5]

**5.** Impossible; therefore $AD \parallel BC$.

---

How do you know it is there?

**The point $E$.** Constructed by [I.31] for refutation. Another built-then-refuted object.

---

---

Added to the repertoire by this proposition

**Warrant I.39.** Equal triangles on the same base and side lie in the same parallels. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 40

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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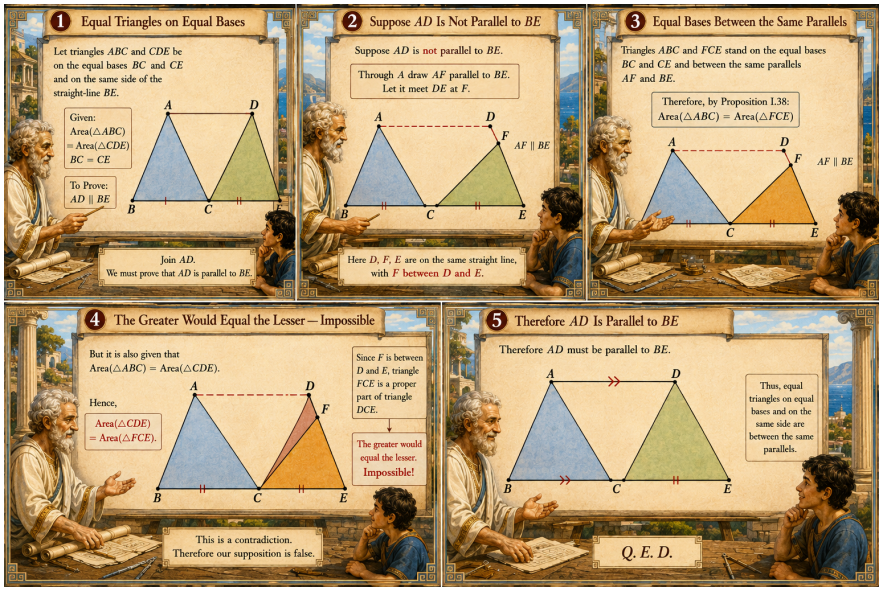


## Statement

> *Equal triangles on equal bases and on the same side are between the same parallels.*

## Actions

1. Let equal triangles ABC and DEF stand on equal bases BC and EF in the same straight line and on the same side.
2. Join AD through their opposite vertices as appropriate.
3. Suppose the line through the opposite vertices is not parallel to the line of the bases.
4. Draw through one vertex a parallel to the bases and form an auxiliary triangle on an equal base.
5. By I.38 the auxiliary triangle equals the corresponding given triangle, hence equals the other given triangle.

6. A proper-part contradiction follows unless the opposite vertices lie on one parallel. Therefore the triangles are between the same parallels.

## Integrated Ontological Restatement

**Source formulation.** *Equal triangles which are on equal bases and on the same side are also between the same parallels.*

**Ontological role.** Given triangles ABC and CDE equal, on equal bases BC and CE on the same side. Join AD. Suppose AD is not parallel to BE. Draw AF ∥ BE through A (Proposition 31) and join FE. By Proposition 38, triangle ABC = triangle FCE. But triangle ABC = triangle DCE (given), so triangle DCE = triangle FCE, impossible. Therefore, AD ∥ BE.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** As in I.39, assuming a non-parallel join and drawing a parallel instead. `[I.31]`

**2.** The resulting triangles are equal. `[I.38]`

**3.** A part equals the whole; impossible. `[C.N. 5]`

---

HOW DO YOU KNOW IT IS THERE?

**The construction.** The I.39 argument with `[I.38]` substituted for `[I.37]`.

---

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.40.** Equal triangles on equal bases and the same side lie in the same parallels. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 41

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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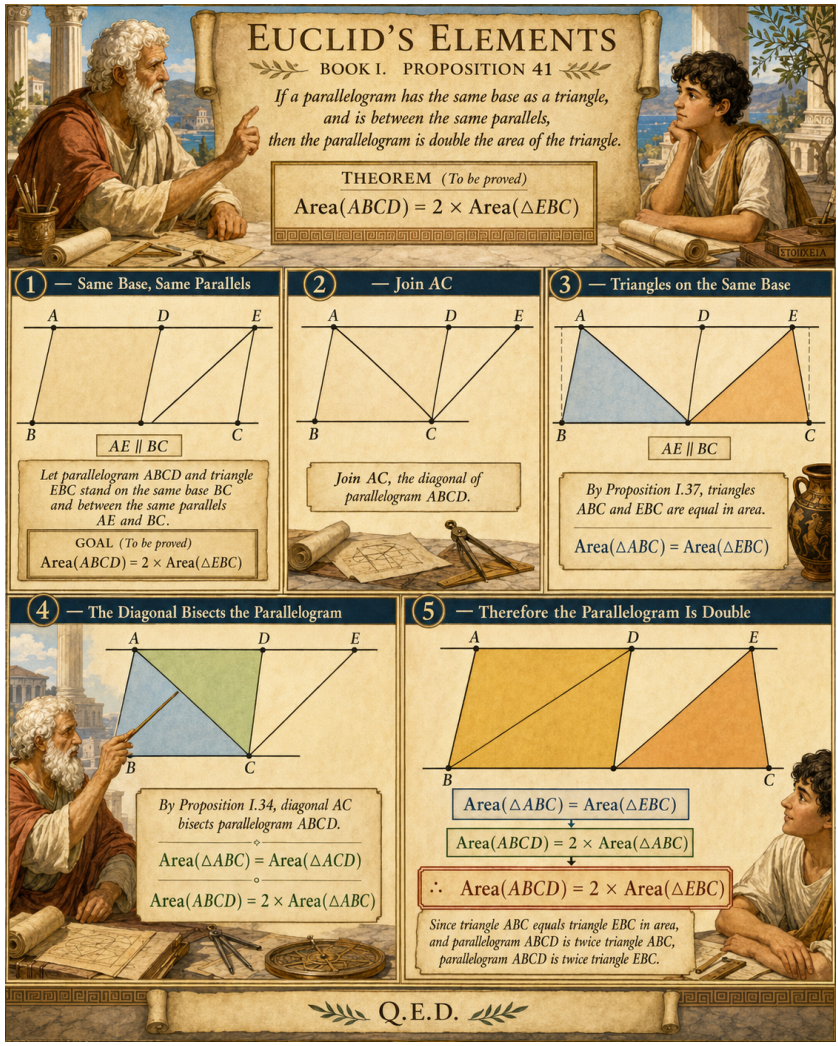


## Statement

*If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double the triangle.*

## Actions

1. Let parallelogram ABCD and triangle EBC stand on the same base BC and between the same parallels BC and AE.
2. Join AC to divide the parallelogram into triangles ABC and ACD.
3. By I.34, triangle ABC is half the parallelogram.
4. Triangles ABC and EBC are on the same base BC and between the same parallels, so by I.37 they are equal.
5. Therefore the parallelogram, being double triangle ABC, is also double triangle EBC.

## Integrated Ontological Restatement

**Source formulation.** *If a parallelogram has the same base as a triangle and is between the same parallels, then the parallelogram is double the area of the triangle.*

**Ontological role.** Given parallelogram ABCD and triangle EBC on base BC between parallels BC and AE. Join AC. Triangle ABC = triangle EBC (Proposition 37). But parallelogram ABCD is double triangle ABC (Proposition 34, diagonal AC bisects it). Therefore, parallelogram ABCD is double triangle EBC.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let the parallelogram $ABCD$ and the triangle $EBC$ share the base $BC$ and lie in the same parallels. `[given]`

**2.** Join $AC$. `[P1]`

**3.** Triangles $ABC$, $EBC$ are equal. `[I.37]`

4. The diameter $AC$ bisects the parallelogram, so triangle $ABC$ measures it exactly twice — that is, the parallelogram is the double of the triangle. `[I.34, B0]`

5. Therefore the parallelogram is double triangle $EBC$ also. `[C.N. 1]`

---

HOW DO YOU KNOW IT IS THERE?

**The word *double*.** The inverse of the half, and no more primitive than it. A double is the multiple that its part measures exactly twice. The prior layer defines multiple before Book I begins, so the word arrives here already determined — as does the inference at step 5. **Why this matters two propositions later.** I.41 is what I.47 consumes: the square on a side is shown double a triangle, and the rectangle on the hypotenuse double another. If *double* required an unstated axiom, the theorem of Pythagoras would rest on it. It does not.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.41.** A parallelogram is double any triangle on the same base in the same parallels. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 42

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12]
- [I.22] construct a triangle from three given straight lines
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [C5] the meeting condition (in force since I.29)

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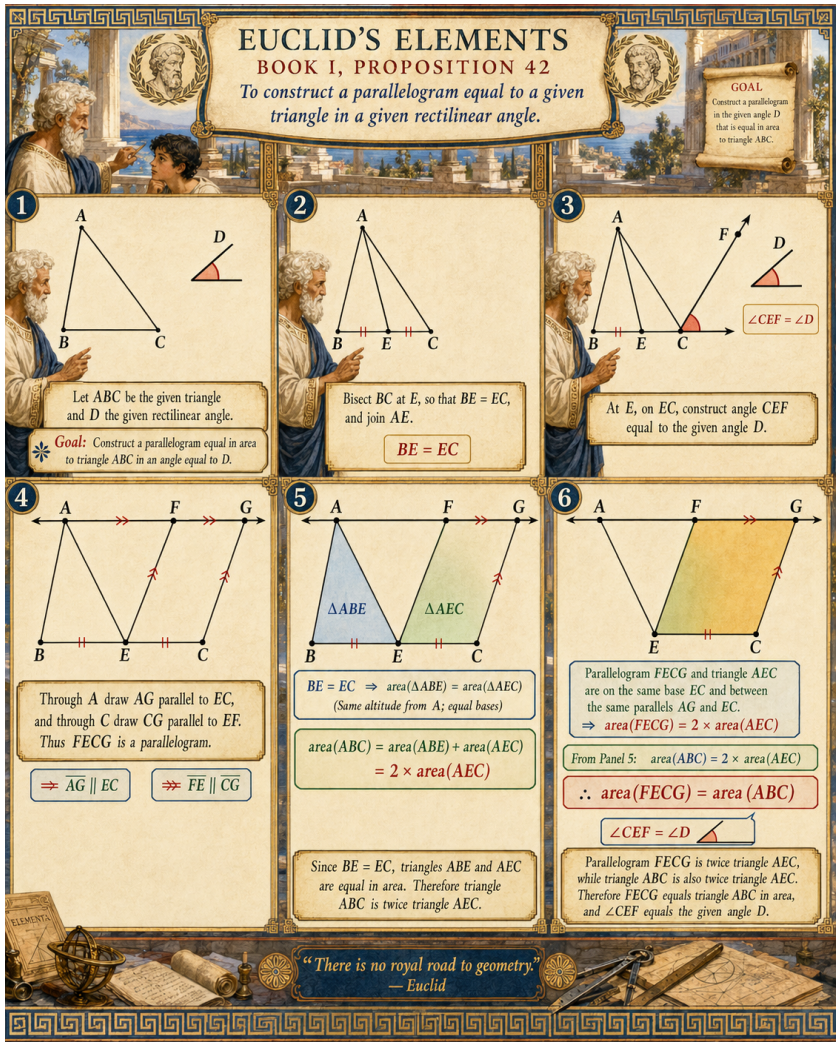


## Statement

*To construct a parallelogram equal in area to a given triangle and having a given angle.*

## Actions

1. Let ABC be the given triangle and D the given angle.
2. Bisect BC at E and join AE.
3. At E construct angle CEF equal to the given angle D, and through A draw AG parallel to EC; through C draw CG parallel to EF.
4. The resulting quadrilateral EFCG is a parallelogram with angle CEF equal to D.
5. Triangles ABE and AEC are equal because they have equal bases BE=EC and lie between the same parallels.
6. By I.41 the parallelogram on EC between the same parallels is double triangle AEC, hence equal to the whole triangle ABC.
7. Thus the required parallelogram has been constructed.

## Integrated Ontological Restatement

**Source formulation.** *To construct a parallelogram equal to a given triangle in a given rectilinear angle.*

**Ontological role.** Given triangle ABC and angle D. Bisect BC at E (Proposition 10). Join AE. Copy angle D to angle CEF at E on EC (Proposition 23). Draw AG ∥ EC through A (Proposition 31) and CG ∥ EF through C (Proposition 31). Then FECG is a parallelogram. Since BE = EC, triangle ABE = triangle AEC (Proposition 38). Therefore, triangle ABC = 2 × triangle AEC. Also, parallelogram FECG = 2 × triangle AEC (Proposition 41). Thus, parallelogram FECG = triangle ABC, and angle CEF = D.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $ABC$ be the triangle and $D$ the given angle. Bisect $BC$ at $E$. `[I.10]`

**2.** Join $AE$. `[P1]`

3. At $E$ construct $\angle CEF$ equal to $D$. [I.23]

4. Through $A$ draw $AG$ parallel to $EC$, and through $C$ draw $CG$ parallel to $EF$. [I.31]

5. $FECG$ is a parallelogram on the base $EC$, in the same parallels as triangle $AEC$. [Def. 23]

6. Triangle $ABE$ equals triangle $AEC$, so the parallelogram equals the whole triangle $ABC$. [I.38, I.41]

---

HOW DO YOU KNOW IT IS THERE?

**The point $E$.** [I.10]. The bisection of the base is what makes the parallelogram equal to the *whole* triangle rather than half of it. **The angle at $E$.** [I.23]. The given angle is not approximated; it is copied exactly.

---

The first proposition in which area becomes constructible rather than merely comparable. The application of areas begins here.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.42 Construct a parallelogram equal to a given triangle, in any given angle.

---

# Proposition 43

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22]
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [I.42] construct a parallelogram equal to a given triangle, in a given angle
- [C5] the meeting condition (in force since I.29)

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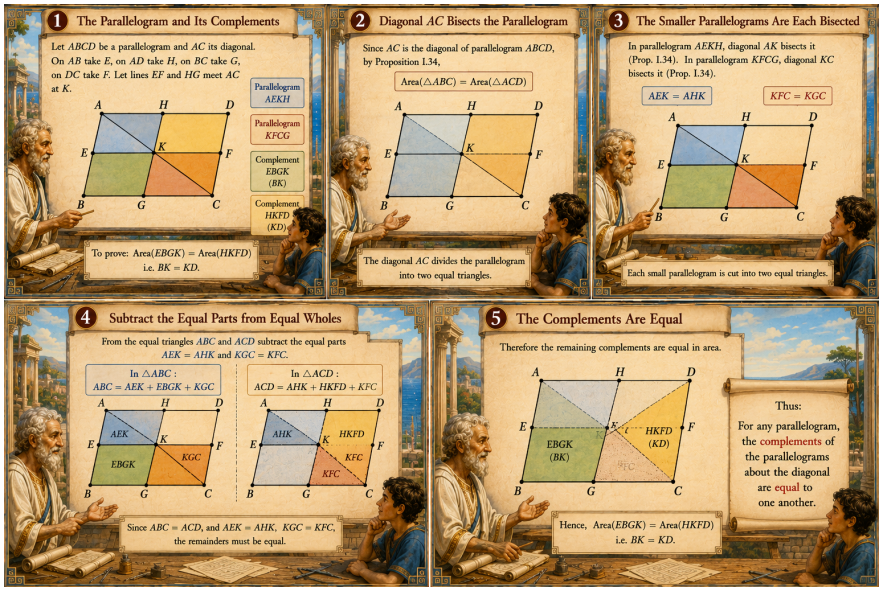


## Statement

> *In any parallelogram, the complements of the parallelograms about a diagonal are equal.*

## Actions

1. Let ABCD be a parallelogram with diagonal AC, and let smaller parallelograms lie about the diagonal, leaving complements BK and KD.
2. By I.34, diagonal AC bisects the whole parallelogram, so triangle ABC equals triangle CDA.
3. The diagonal also bisects each smaller parallelogram about AC.
4. Subtract the equal halves of the smaller parallelograms from the equal large triangles.
5. The remaining complements BK and KD are equal.

# Integrated Ontological Restatement

**Source formulation.** *For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.*

**Ontological role.** Given parallelogram ABCD with diagonal AC, and parallelograms EH and FG about AC, with complements BK and KD. By Proposition 34, triangle ABC = triangle ADC. Also, triangle AEK = triangle AHK (Proposition 34), and triangle KFC = triangle KGC (Proposition 34). Therefore, triangle AEK + triangle KGC = triangle AHK + triangle KFC. Subtracting these from the equal triangles ABC and ADC (Common Notion 3) yields BK = KD.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABCD$ have diameter $AC$, with $EH$, $FG$ the parallelograms about it and $BK$, $KD$ the complements. `[given]`
2. The diameter bisects the whole, and each of the two small parallelograms. `[I.34]`
3. Subtracting the equal triangles from the equal halves leaves the complements equal. `[C.N. 3]`

---

How do you know it is there?

**The complements.** Not new objects: they are what remains of the parallelogram once the two about the diameter are removed. The proposition is entirely a subtraction.

---

---

Added to the repertoire by this proposition

**Warrant I.43.** The complements about the diameter are equal. No new ac-

tion: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

# Proposition 44

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22]
- [I.23] copy a given rectilineal angle at a given point
- [I.31] draw a parallel to a given line through a given point
- [I.42] construct a parallelogram equal to a given triangle, in a given angle
- [C5] the meeting condition (in force since I.29)

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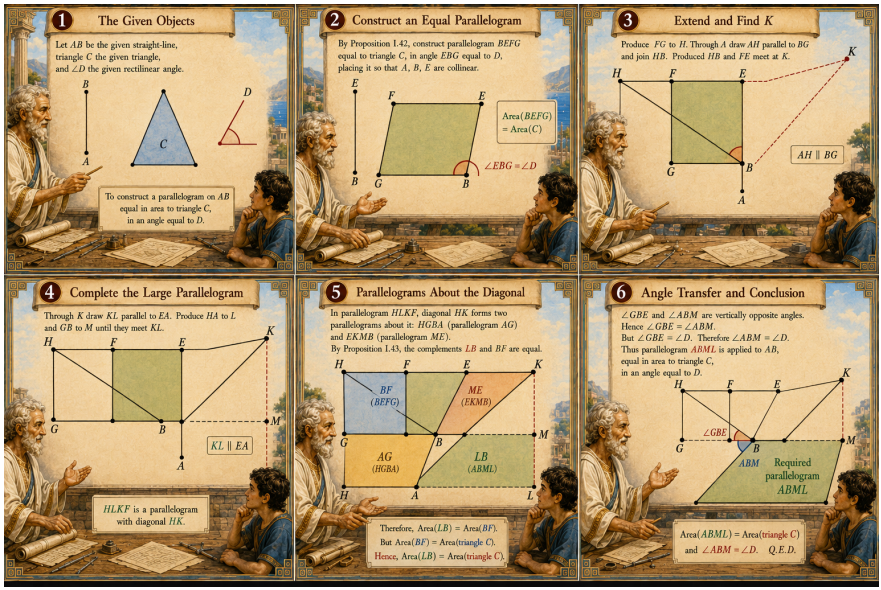


## Statement

*To apply to a given straight line a parallelogram equal to a given triangle and having a given angle.*

## Actions

1. Let AB be the given straight line, C the given triangle, and D the given angle.
2. By I.42 construct a parallelogram equal to triangle C and having angle D.
3. Place an equal side along AB and extend the construction with parallels so that the desired parallelogram is applied to AB.
4. Use I.43 on the resulting larger parallelogram to identify equal complements.

5. Transfer the area of the auxiliary parallelogram to the complement lying on AB.

6. The resulting parallelogram on AB is equal to triangle C and has an angle equal to D.

## Integrated Ontological Restatement

**Source formulation.** *To apply a parallelogram equal to a given triangle to a given straight-line in a given rectilinear angle.*

**Ontological role.** Given line AB, triangle C, and angle D. Construct parallelogram BEFG equal to C in angle EBG = D (Proposition 42). Place BE on AB. Draw AH ∥ BG through A (Proposition 31). Join HB. Extend HB and FE to meet at K (guaranteed by Postulate 5). Draw KL ∥ FH through K (Proposition 31). Extend HA and GB to L and M. Then LB is a parallelogram equal to triangle C (by Proposition 43), and angle ABM = angle GBE = D. Thus, parallelogram LB is applied to AB in angle D.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The produced straight lines $HB$ and $FE$ must meet at $K$. The criticism is that this depends on indefinite extension and the flatness of the plane.

**Reply within the reconstruction.** Euclid does not leave this intersection to the figure: the relevant angle condition is precisely the type covered by Postulate 5, and the lines are produced by Postulate 2. On the present reading, the cited dependence is explicit rather than hidden.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Let $AB$ be the line, $C$ the triangle, $D$ the angle. `[given]`

**2.** Construct the parallelogram $BEFG$ equal to $C$ in the angle $D$, placed so that $BE$ is in a straight line with $AB$. `[I.42]`

3. Produce $FG$ to $H$; through $A$ draw $AH$ parallel to $BG$; join $HB$ and produce it. [P2, I.31]

4. $HB$ produced meets $FE$ produced at $K$, since the co-interior angles are less than two right angles. [C5]

5. Complete the figure; the complements $AG$ and $ME$ are equal. [I.43]

6. Hence the parallelogram on $AB$ equals the triangle $C$. [C.N. 1]

---

HOW DO YOU KNOW IT IS THERE?

**The point $K$.** The *second and last* use of [C5] in Book I, and here it is used positively rather than in a reductio: the fifth postulate is what guarantees the two produced lines actually meet, so that the figure closes and the complement can be taken.

---

**Postulate 5 is spent a second time here.** Twice in forty-eight propositions: once at I.29 to refute, once at I.44 to construct. That is the whole of its employment in Book I.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.44 Apply to a given line a parallelogram equal to a given triangle, in a given angle.

---

# Proposition 45

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22], [I.23]
- [I.31] draw a parallel to a given line through a given point
- [I.42] construct a parallelogram equal to a given triangle, in a given angle
- [I.44] apply to a given line a parallelogram equal to a given triangle
- [C5] the meeting condition (in force since I.29)

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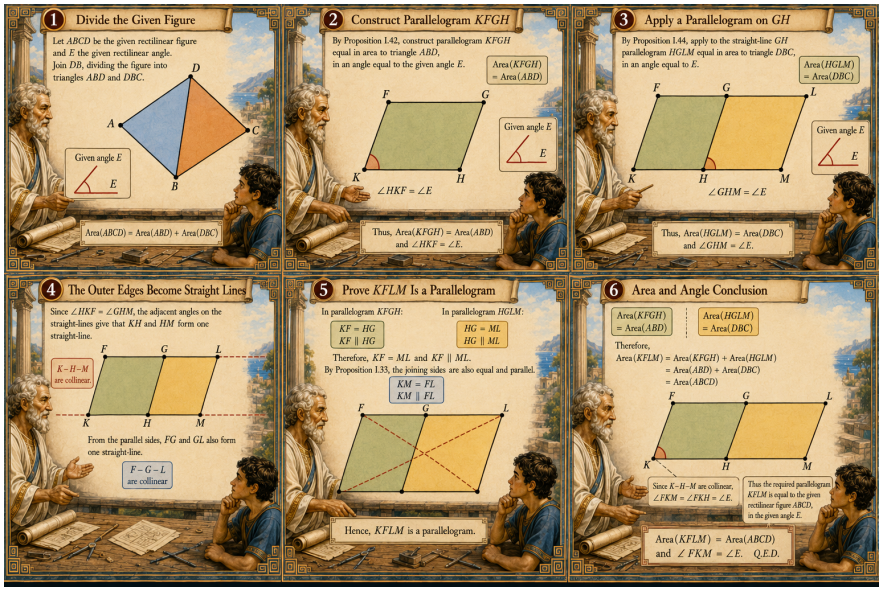


## Statement

*To construct a parallelogram equal in area to a given rectilinear figure and having a given angle.*

## Actions

1. Divide the given rectilinear figure into triangles by drawing diagonals from one vertex.
2. By I.42 construct a parallelogram equal to the first triangle and having the given angle.
3. Successively apply parallelograms equal to the remaining triangles along a common straight line using I.44.
4. Because adjacent angles are supplementary, the assembled boundaries form straight lines by I.14.

5. The union of the constructed parallelograms is itself a parallelogram with the prescribed angle.

6. Its area is the sum of the triangle areas and therefore equals the original rectilinear figure.

## Integrated Ontological Restatement

**Source formulation.** *To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.*

**Ontological role.** Given rectilinear figure ABCD and angle E. Join DB. Construct parallelogram FH equal to triangle ABD in angle HKF = E (Proposition 42). Apply parallelogram GM equal to triangle DBC to line GH in angle GHM = E (Proposition 44). Since angle E equals both HKF and GHM, HKF = GHM. Adding KHG shows KH is straight-on to HM (Proposition 14). Similarly, FG is straight-on to GL. By Proposition 33, KFLM is a parallelogram. Since triangle ABD = FH and DBC = GM, the whole figure ABCD = parallelogram KFLM in angle FKM = E.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

**1.** Divide the given figure into triangles by joining its vertices. `[P1]`

**2.** On a line construct a parallelogram equal to the first triangle in the given angle. `[I.42]`

**3.** Apply to the resulting side a parallelogram equal to the next triangle, in the same angle. `[I.44]`

**4.** Repeat until every triangle is used. `[I.44]`

**5.** The successive figures join into a single parallelogram, the adjacent sides lying in a straight line. `[I.29, I.14]`

How do you know it is there?

**The joining of the pieces.** [I.14] is what certifies that the successive parallelograms line up into one figure rather than a chain of separate ones: the angles at each junction sum to two right angles, so the sides lie in a straight line.

---

Any polygon whatever is now equal to a constructible parallelogram. This is the culmination of the application of areas, and the reason Book II can speak of rectangles as freely as it does.

---

Added to the repertoire by this proposition

I.45 Reduce any rectilineal figure to a single parallelogram in a given angle.

---

# Proposition 46

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Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22], [I.23], [I.31]
- [I.42] construct a parallelogram equal to a given triangle, in a given angle
- [I.44] apply to a given line a parallelogram equal to a given triangle
- [I.45] reduce any rectilineal figure to a parallelogram in a given angle
- [C5] the meeting condition (in force since I.29)

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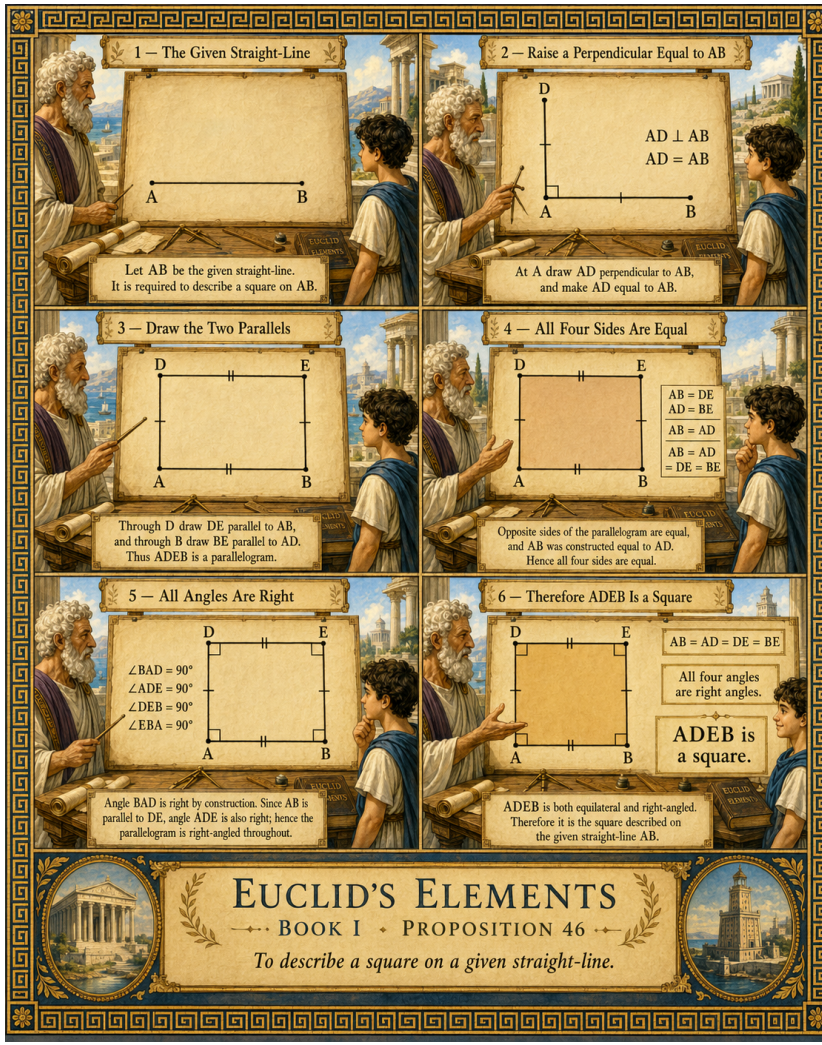


## Statement

*To describe a square on a given straight line.*

## Actions

1. Let AB be the given straight line.
2. At A draw AC perpendicular to AB by I.11 and make AC=AB.
3. Through C draw CD parallel to AB and through B draw BD parallel to AC by I.31.
4. Then ACDB is a parallelogram, so opposite sides are equal by I.34; hence AB=CD and AC=BD.
5. Since AB=AC, all four sides are equal.
6. Because one angle is right and opposite sides are parallel, all four angles are right by I.29 and I.34.
7. Therefore ACDB is a square described on AB.

## Integrated Ontological Restatement

**Source formulation.** *To describe a square on a given straight-line.*

**Ontological role.** Given AB. Draw AC perpendicular to AB at A (Proposition 11). Make AD = AB (Proposition 3). Draw DE ∥ AB through D (Proposition 31) and BE ∥ AD through B (Proposition 31). Then ADEB is a parallelogram. By Proposition 34, AB = DE and AD = BE. Since AB = AD, all four sides are equal. By Proposition 29, since AD crosses parallels AB and DE, angle BAD + angle ADE = two right angles. Angle BAD is right, so angle ADE is right. Thus, all angles are right. Therefore, ADEB is a square.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The construction of the square is said to assume that the two parallels close into the expected right-angled quadrilateral and preserve right angularity throughout the plane.

**Reply within the reconstruction.** I.46 invokes I.11 for the perpendicular, I.3 for the equal side, I.31 for the parallels, I.34 for opposite-side equality, and I.29 for the angle relations. Postulate 4 supplies the invariance of right angles. The square is therefore a synthesis of earlier proved operations rather than a diagrammatically assumed closure.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $AB$ be given. Erect $AC$ at right angles to $AB$ at $A$. [I.11]
2. Cut off $AD$ equal to $AB$. [I.3]
3. Through $D$ draw $DE$ parallel to $AB$, and through $B$ draw $BE$ parallel to $AD$. [I.31]
4. $ADEB$ is a parallelogram, so $DE = AB$ and $BE = AD$; all four sides are equal. [I.34]
5. The angles are right, since co-interior angles with a right angle are right. [I.29]
6. Therefore $ADEB$ is a square. [Def. 22]

---

HOW DO YOU KNOW IT IS THERE?

**The right angle at $A$.** [I.11], earned thirty-five propositions earlier. **The equal side $AD$.** [I.3], earned at the third proposition. The square — the figure Book II is built from — is assembled from the two oldest earned actions and the parallel.

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ADDED TO THE REPERTOIRE BY THIS PROPOSITION

I.46 Describe a square on a given straight line.

# Proposition I.47

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22], [I.23], [I.31], [I.42]
- [I.44] apply to a given line a parallelogram equal to a given triangle
- [I.45] reduce any rectilineal figure to a parallelogram in a given angle
- [I.46] describe a square on a given straight line
- [C5] the meeting condition (in force since I.29)

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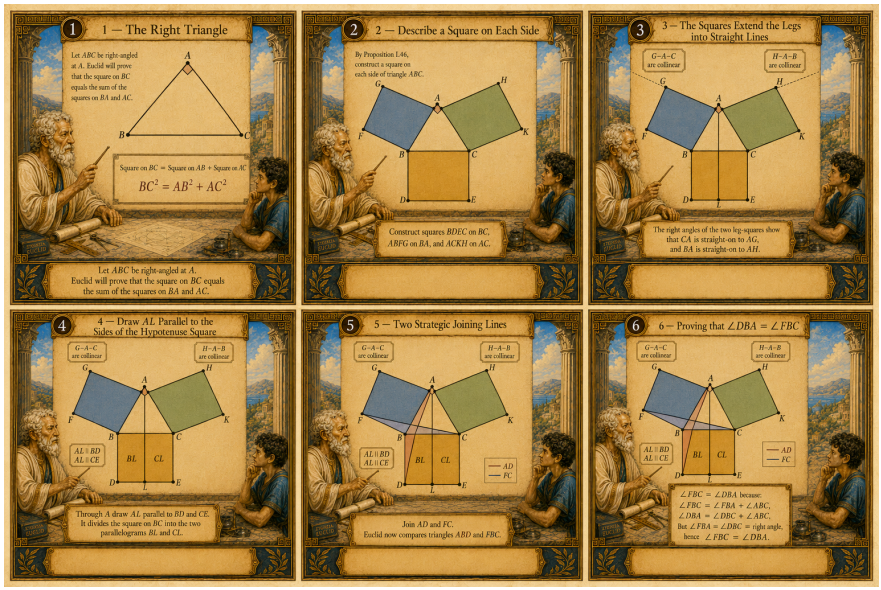
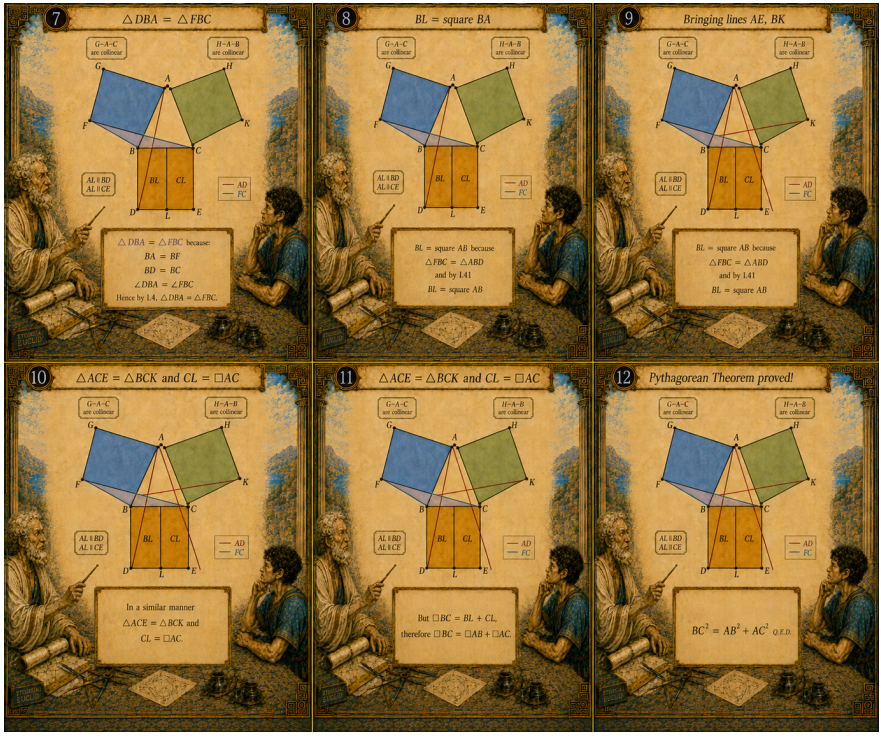

1 — The Right Triangle
Let ABC be right-angled at A. Euclid will prove that the square on BC equals the sum of the squares on BA and AC.
Square on BC = Square on AB + Square on AC
$BC^2 = AB^2 + AC^2$
Let ABC be right-angled at A. Euclid will prove that the square on BC equals the sum of the squares on BA and AC.
2 — Describe a Square on Each Side
By Proposition I.46, construct a square on each side of triangle ABC.
Construct squares BDEC on BC, ABFG on BA, and ACKH on AC.
3 — The Squares Extend the Legs into Straight Lines
G–A–C are collinear
H–A–B are collinear
The right angles of the two leg-squares show that CA is straight-on to AG, and BA is straight-on to AH.
4 — Draw AL Parallel to the Sides of the Hypotenuse Square
AL ∥ BD
AL ∥ CE
BL
CL
Through A draw AL parallel to BD and CE. It divides the square on BC into the two parallelograms BL and CL.
5 — Two Strategic Joining Lines
AD
FC
Join AD and FC. Euclid now compares triangles ABD and FBC.
6 — Proving that ∠DBA = ∠FBC
∠FBC = ∠DBA because: ∠FBC = ∠FBA + ∠ABC; ∠DBA = ∠DBC + ∠ABC. But ∠FBA = ∠DBC = right angle, hence ∠FBC = ∠DBA.


7 — △DBA = △FBC
△DBA = △FBC because: BA = BF, BD = BC, ∠DBA = ∠FBC. Hence by I.4, △DBA = △FBC.
8 — BL = square BA
BL = square AB because △FBC = △ABD and by I.41 BL = square AB
9 — Bringing lines AE, BK
BL = square AB because △FBC = △ABD and by I.41 BL = square AB
10 — △ACE = △BCK and CL = □AC
In a similar manner △ACE = △BCK and CL = □AC.
11 — △ACE = △BCK and CL = □AC
But □BC = BL + CL, therefore □BC = □AB + □AC.
12 — Pythagorean Theorem proved!
$BC^2 = AB^2 + AC^2$ Q.E.D.

## Statement

*In right-angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.*

## Actions

1. Given: right triangle ABC with right angle BAC.
2. Construct square BDEC on BC (Proposition I.46).
3. Construct squares GB on BA and HC on AC (Proposition I.46).
4. Draw AL □ BD or CE through A (Proposition I.31).
5. Draw AD and FC (Postulate 1).
6. $\angle DBC = \angle FBA =$ right angle (Definition 22).
7. Add $\angle ABC$: $\angle DBA = \angle FBC$ (Common Notion 2).
8. DB = BC and FB = BA (Definition 22).
9. Therefore triangle ABD = triangle FBC (Proposition I.4).
10. Parallelogram BL = 2 × triangle ABD (Proposition I.41).
11. Square GB = 2 × triangle FBC (Proposition I.41).
12. Therefore BL = GB (Common Notion 1).
13. Similarly, CL = HC.
14. Therefore BDEC = GB + HC (Common Notion 2).
15. Therefore $BC^2 = AB^2 + AC^2$.

## Interpretation from the Commentary

This proof does not rely on any unaxiomatized notion of area. It uses congruence and equivalence of rectilinear figures established in Propositions I.33–I.45. The equality of areas is established through the equality of triangles and parallelograms.

## Expanded Commentary from *Explaining Fully the First Book of Elements*

**Problem:** This proposition employs reasoning based on the equality of areas of squares and parallelograms. However, Euclid never formally defined the notion of "area" or its additivity. Formally, the statement "equal to the same area" presupposes a measure or congruence relation for plane figures not explicitly defined in Book I.

**Critics:**

- **Heath (1908)**, in *The Thirteen Books of Euclid's Elements*, discusses the absence of formal axioms of area in Book I.
- **Hartshorne (2000)** formalizes Book I and observes that Proposition I.47 relies on unaxiomatized notions of area congruence.

**Response:** I disagree. There is no necessity to define "area" in Euclid's framework. Instead of areas, Euclid employs congruence and equivalence of rectilinear figures. He proves results through the equality of triangles, the equivalence of triangles and parallelograms, and the equivalence of parallelograms with rectilinear figures (Propositions I.33–I.45). Thus, the reasoning of Proposition I.47 stands entirely upon the established congruence relations and does not rely on any unaxiomatized notion of area.

# Integrated Ontological Restatement

**Source formulation.** *In right-angled triangles, the square on the side subtending the right-angle is equal to the sum of the squares on the sides containing the right-angle.*

**Ontological role.** Given right-angled triangle ABC with right angle at A. Construct squares BCED on BC, ABFG on AB, and ACKH on AC (Proposition 46). Draw AL through A parallel to BD (Proposition 31). Join AD and FC. Triangles ABD and FBC are congruent (Proposition 4: AB = FB, BD = BC, angle ABD = 90° + angle ABC = angle FBC). Therefore, triangle ABD = triangle FBC. Rectangle BDML is double triangle ABD (Proposition 41, same base BD, between parallels). Square ABFG is double triangle FBC (Proposition 41, same base FB, between parallels). Thus, rectangle BDML = square ABFG. Similarly, rectangle MLEC = square ACKH. Adding these equalities (Common Notion 2) yields square BCED = square ABFG + square ACKH.

## The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$ be right-angled at $A$. Describe squares $BDEC$, $GFBA$, $HKCA$ on the three sides. `[I.46]`
2. Through $A$ draw $AL$ parallel to $BD$, meeting $DE$ at $L$; join $AD$ and $FC$. `[I.31, P1]`
3. $\angle DBC = \angle FBA$, each being a right angle plus $\angle ABC$. `[C.N. 2]`
4. Triangles $ABD$ and $FBC$ have $BD = BC$, $BA = BF$, and equal included angles; so they are equal. `[I.4]`
5. The parallelogram $BDLM$ is double triangle $ABD$; the square $GFBA$ is double triangle $FBC$. `[I.41]`
6. Therefore $BDLM$ equals the square on $AB$; similarly $MLEC$ equals the square on $AC$. `[C.N. 1]`
7. Adding, the square on $BC$ equals the sum of the squares on $AB$ and $AC$. `[C.N. 2]`

HOW DO YOU KNOW IT IS THERE?

**The three squares.** Each an output of [I.46], the action earned one proposition earlier. I.47 is immediately preceded by the construction it consumes. **The line $AL$.** [I.31]. It divides the square on the hypotenuse into the two rectangles that the proof matches, one to each smaller square. **The collinearity of $G$, $A$, $H$.** [I.14]: the angles at $A$ sum to two right angles, so the sides lie in a straight line. Without this the figure would not close.

Every permission spent here was earned earlier in the book, and the newest of them — [I.46] — is one proposition old. The most celebrated theorem in the corpus is the point at which the ledger balances exactly.

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.47.** The square on the hypotenuse equals the sum of the squares on the other two sides. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

# Proposition 48

---

Tools in hand before this proposition

- [B0] the prior layer — part, multiple, magnitude, and the five Common Notions (Book 0; warrant only, no action)
- [P1] join [P2] extend [P3] sweep
- Earned earlier: [I.1], [I.2], [I.3], [I.9], [I.10], [I.11], [I.12], [I.22], [I.23], [I.31], [I.42]
- [I.44] apply to a given line a parallelogram equal to a given triangle
- [I.45] reduce any rectilineal figure to a parallelogram in a given angle
- [I.46] describe a square on a given straight line
- [C5] the meeting condition (in force since I.29)

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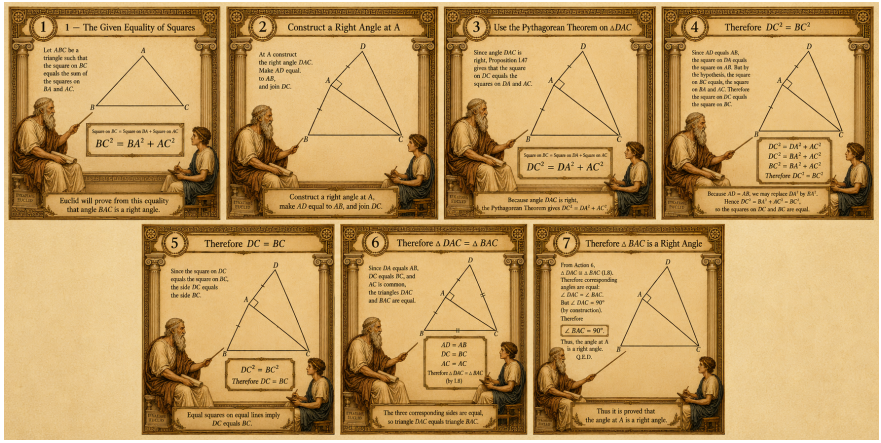


## Statement

> *If the square on one side of a triangle equals the sum of the squares on the other two sides, then the angle contained by those two sides is a right angle.*

## Actions

1. Let triangle ABC satisfy $BC^2=BA^2+AC^2$. The goal is to prove angle BAC is a right angle.
2. At A construct AD perpendicular to AC, make AD=AB, and join DC.
3. Since angle DAC is right, Proposition I.47 gives $DC^2=DA^2+AC^2$.
4. Because AD=AB and by hypothesis $BC^2=BA^2+AC^2$, it follows that $DC^2=BC^2$.
5. Therefore DC=BC.
6. Now DA=AB, DC=BC, and AC is common; by Proposition I.8 triangle DAC equals triangle BAC.
7. Hence angle DAC=angle BAC. But angle DAC is right by construction; therefore angle BAC is also a right angle. Q.E.D.

## Integrated Ontological Restatement

**Source formulation.** *If the square on one of the sides of a triangle is equal to the sum of the squares on the two remaining sides, then the angle contained by the two remaining sides is a right-angle.*

**Ontological role.** Given triangle ABC with square on BC equal to sum of squares on BA and AC. Draw AD perpendicular to AC at A (Proposition 11). Make AD = AB (Proposition 3). Join DC. Since DA = AB, the square on DA equals the square on AB. Add the square on AC to both. Then the squares on DA and AC equal the squares on BA and AC. By Proposition 47, since angle DAC is right, the square on DC equals the squares on DA and AC. But the square on BC equals the squares on BA and AC (given). Therefore, the square on DC equals the square on BC, so DC = BC. Since DA = AB, AC is common, and DC = BC, triangles DAC and BAC are congruent (Proposition 8). Therefore, angle DAC = angle BAC. But angle DAC is right, so angle BAC is right.

## Critical Apparatus: Alleged Assumption and Reply

**Objection.** The supplied critique notes an apparently unstated rule: equal straight lines have equal squares, and conversely equality of the constructed squares permits equality of their sides.

**Reply within the reconstruction.** In the reconstruction, a square is not an independent numerical value but the figure constructed on a given side by I.46. Equal sides generate congruent square constructions through the same operations; conversely the SSS comparison in I.48 reduces the final step to equality of the constructed sides. What may be written in modern algebra as equality of squares is therefore derived through congruent construction and I.8, not introduced as a separate algebraic common notion.

### The construction, with its permissions named

*Aris's numbered actions above, restated so that every step carries the permission that authorises it. If a step cannot be tagged, it is not a step.*

1. Let $ABC$ have the square on $BC$ equal to the sum of the squares on $AB$, $AC$. [given]
2. Draw $AD$ at right angles to $AC$ at $A$, and make $AD = AB$. [I.11, I.3]
3. Join $DC$. [P1]
4. The square on $DC$ equals the sum of the squares on $AD$, $AC$. [I.47]
5. So the square on $DC$ equals the square on $BC$, and $DC = BC$. [C.N. 1]
6. Triangles $DAC$, $BAC$ have three sides equal; so $\angle BAC = \angle DAC$, a right angle. [I.8]

---

HOW DO YOU KNOW IT IS THERE?

**The point $D$.** Built by [I.11] and [I.3] — the perpendicular and the cut-off length. The converse is proved by constructing the right angle that is claimed and showing the given triangle must coincide with it.

---

Book I closes as it opened: with a construction carried out inside a repertoire whose every entry can be named.

---

ADDED TO THE REPERTOIRE BY THIS PROPOSITION

**Warrant I.48.** If the squares balance, the angle is right. No new action: nothing is drawable now that was not drawable before. The second ledger grows; the first does not.

---

Part Three

# The Closed Ledger

# What Book I Bought

Three commands were granted at the outset. Fourteen further actions were earned. Two constraints were held throughout, and one of them was spent exactly twice.

---

### The prior layer, held throughout (Book 0)

`B0` The seven primitive terms; the five Common Notions; and from Books VII and V the determination of part, multiple, equality of wholes and remainders, and the general theory of magnitude. A warrant, never an action: it licenses no new line, only the right to reason about parts, multiples, halves and doubles of what the commands have drawn.

Spent visibly at **I.34**, **I.37**, **I.38** and **I.41** — the four places where the manuscript tradition inserted an axiom of doubles and halves that Book I never needed.

---

### The complete ledger of actions

`P1` join — straight line from any point to any point (granted)

`P2` extend — produce a finite straight line continuously (granted)

`P3` sweep — circle on any centre with any given radius (granted)

`I.1` Erect an equilateral triangle on a given line

`I.2` Place at a given point a line equal to a given line

I.3 Cut off from the greater a line equal to the less

I.9 Bisect a given rectilineal angle

I.10 Bisect a given finite straight line

I.11 Erect a perpendicular from a point on a line

I.12 Drop a perpendicular from a point outside a line

I.22 Construct a triangle from three given straight lines

I.23 Copy a given rectilineal angle at a given point

I.31 Draw a parallel to a given line through a given point

I.42 Construct a parallelogram equal to a given triangle, in a given angle

I.44 Apply to a given line a parallelogram equal to a given triangle

I.45 Reduce any rectilineal figure to a parallelogram in a given angle

I.46 Describe a square on a given straight line

---

THE CONSTRAINTS

C4 All right angles are equal. Relied upon wherever an angle determined in one place is compared with an angle determined in another. Never executed, because it is not a command.

C5 The meeting condition for produced straight lines. Dormant from the first proposition to the twenty-eighth. Spent at **I.29**, in a reductio, to license the converse inference from parallelism to equal angles. Spent again at **I.44**, positively, to guarantee that two produced lines meet so the application of areas can close. Used nowhere else in Book I.

---

## The shape the ledger reveals

Fourteen constructions out of forty-eight, and read in the order earned they form a single line of supply:

- **I.1–I.3** make length portable. The equilateral triangle exists chiefly to build the transfer; the transfer exists to make the cut-off possible.
- **I.9–I.12** make symmetry available — bisect an angle, bisect a segment, raise a perpendicular, drop one — and none of it needs a parallel.
- **I.22–I.23** make figures portable. An angle is copied by copying three lengths, because Book I has no way to carry an angle directly.
- **I.31** makes direction available, and does so without Postulate 5.
- **I.42–I.46** make area constructible, ending with the square on a given line — the equipment Book II opens by assuming.

The two most famous propositions, I.47 and I.48, add no action at all. They spend what the preceding forty-six earned.

## The question, answered

Every object introduced in Book I is the output of a command held at the moment of its introduction — one of the three granted, or one of the fourteen earned — executed in a plane the definitions and constraints have already determined, and judged by criteria the Common Notions supply. Where a critic finds a gap, what is missing is not a postulate but a tag: the reader has stopped asking which permission is in force, and has begun reading the diagram instead of the argument.

## The axiom that was never missing

Of the objections brought against Book I, the one with the longest life was not about circles meeting or triangles moving. It was about halves. At I.37 two triangles, each the half of one of two equal parallelograms, are declared equal, and no Common Notion granted anything about halves. The tradition

answered by writing one in, and it stayed in the manuscripts for two thousand years.

The ledger answers differently. `[B0]` was in force from the first page: part and multiple were determined before the geometry began, *half* and *double* are names of particular parts and multiples, and their equality properties follow from what those words already mean. Step 8 of I.37 has a tag, and the tag is not a sixth Common Notion.

Book I was not missing an axiom. It was missing a book, and the book was sitting four hundred pages further on.

---

### The rule, restated

If a step cannot be tagged, it is not a step. A line that appears with no permission beside it is a line that was seen rather than drawn — and nothing that was merely seen has ever belonged in a demonstration.

---

Part Four

# Critical Evaluation

# Critical Evaluation: Objections, Revisions, and Replies

## The strongest initial criticism

The most forceful critical reading of the reconstruction can be stated without caricature. It charges, first, that the relational interpretation of Definition 4 risks textual infidelity; second, that the analysis of Proposition I.1 risks circularity because the desired intersection appears in the analytic hypothesis; third, that historical claims about Postulate 5 can become anachronistic if they attribute nineteenth-century non-Euclidean concepts to Euclid; and fourth, that the later success of Hilbertian, Tarskian, and constructive formalizations shows the practical necessity of explicit axioms of order, continuity, and congruence. In this form the objection is serious because it concerns the distinction between interpretive reconstruction and formal derivability rather than merely the appearance of diagrams.

## First revision: Definition 4 is cumulative

The critical notes themselves subsequently revise the charge that Definition 4 is simply circular. The decisive observation is that the definition is not presented, on the reconstruction, as a closed one-sentence specification. It introduces “evenness,” while later definitions and postulates progressively determine what that evenness entails. Definition 17 supplies symmetry; Def-

inition 23 supplies uniform direction and non-intersection under indefinite production; Postulate 2 excludes finite termination; Postulate 1 universalizes straight connection; Postulate 4 makes right angularity invariant; and Postulate 5 completes the planar setting. Once this cumulative method is admitted, the criticism must move from "the definition is circular" to the more exact historical and philological question of whether this staged use of Euclid's words is justified.

## Second revision: spherical geometry changes the historical question

A further initial criticism asserted that Euclid could not have been distinguishing the plane from curved surfaces because such alternatives were conceptually unavailable. The later discussion corrects that claim. Greek spherical geometry and astronomy make the sphere an available mathematical object, and Euclid himself is associated with the *Phaenomena*. The appropriate question is therefore not whether Euclid knew that a sphere behaves differently from a plane, but how much interpretive weight should be assigned to that knowledge in reading Postulate 5. The reconstruction uses it as historical support for the possibility of a deliberately discriminating planar condition; it need not claim that Euclid possessed the later theory of hyperbolic models.

## The remaining formalist challenge to I.1

The most persistent objection concerns existence in I.1. Analysis and synthesis by themselves do not license conversion of the conditional "if $C$ exists, then the circles meet at $C$" into the existence statement "the circles meet." Modern proof systems therefore isolate a circle–circle continuity principle. The reply defended in this book is semantic and ontological rather than merely syllogistic: Postulate 3 is an existence-producing command for circles, the definitions determine what those circles are, and the prior architecture determines the plane in which they coexist. A modern formalization that removes this semantic content must restore it by an explicit axiom. The

two accounts therefore disagree about where the existence content resides, not about whether the construction requires existence content at all.

## Superposition after the reconstruction of I.4

The superposition controversy has a similar structure. If "apply" means physical displacement of a rigid body, Euclid would need a motion principle. If, however, application is read as construction of corresponding equal segments and angles followed by the coincidence criterion of Common Notion 4, then the result is obtained without a primitive dynamics. The reconstruction accordingly interprets the language of motion as a mode of exhibiting coincidence, while the logical force comes from constructions I.2–I.3, uniqueness of straight connection, and the identity of coincident figures.

## Order and betweenness

Hilbertian and Tarskian systems make the virtues of an explicit order relation unmistakable: one can state exactly when one point lies between two others and derive separation results with formal precision. The reconstruction does not deny that achievement. It contends that Euclid's text does not operate with an abstract order predicate detached from construction. "Beyond," "part," "whole," production of a straight line, cutting off an equal segment, and comparison of angles are operational notions whose content is fixed by the constructional language. Thus the modern axiom and the Euclidean operation need not occupy the same logical level.

## Figure equality and the Pythagorean theorem

The final major challenge concerns area. A modern treatment may introduce numerical area or an explicit polygon-additivity principle. Euclid instead builds a theory of equality of figures through parallelograms on the same or equal bases, triangles between the same parallels, diagonals, complements, and the common notions of addition and subtraction. The important distinction is therefore between *measure* and *equivalence*. Proposition I.47 needs

the latter. It shows that the square on the hypotenuse is decomposable into regions equal, through the established chain of propositions, to the two squares on the legs. The reconstruction thus reads I.47 as the culmination of a theory of geometrical equivalence rather than the application of an unstated numerical area function.

## A note on the “halves of equals” objection

Proposition I.37 deserves separate notice because the supplied catalogue explicitly calls “halves of equal things are equal” an additional common notion. This is the sharpest local challenge to the global claim that no hidden assumptions are needed. The answer adopted here treats halving as a proved geometrical relation, not as a primitive arithmetic operator: by I.34 a diagonal produces two equal triangles in each parallelogram; by I.35–I.36 the relevant wholes are equal; and Common Notions 1–3 govern substitution, composition, and decomposition. The statement about halves can therefore be presented as a derived corollary of those established equalities. Making that derivation explicit is preferable to leaving the maxim as an unexplained slogan.

## I.29–I.31 and the status of Postulate 5

The theory of parallels makes the methodological difference transparent. In a formal independence study, Postulate 5 is an axiom independent of the preceding postulates. In the ontological reading, it is also the condition that completes the determination of the plane. These descriptions are compatible so long as one does not infer from “independent” to “arbitrary” or “unmotivated.” I.29 unfolds the angular consequences of the plane so determined; I.30 establishes transitivity of parallelism through those consequences; and I.31 constructs the parallel through a given point by copying an angle and invoking I.27.

## I.46 as a synthesis of the planar structure

The construction of the square is likewise not treated as a fresh appeal to a cooperative diagram. The perpendicular comes from I.11, equality of sides from I.3, parallels from I.31, opposite-side equality from I.34, and the right-angle relations from I.29 together with Postulate 4. The square is thus a compact synthesis of properties already generated by the theory of the line, perpendicularity, equality, and parallels.

## Final methodological conclusion

The resulting position should be stated precisely. The claim is not that Hilbert, Tarski, or later constructive geometers were mistaken to introduce explicit axioms into their own formal systems. Their work shows what is required by those systems and clarifies distinctions that Euclid does not encode in modern predicate form. The stronger historical-philosophical claim defended here is that it does not follow that Euclid's demonstrations are therefore patched together by unconscious visual assumptions. On the cumulative reading, the allegedly missing content is distributed through the very meanings and operations that a modern formalization intentionally decomposes into independent primitives.

The disagreement is therefore ultimately between two architectures of rigor. One seeks completeness by isolating every primitive relation and stating an axiom for it. The other seeks completeness by progressively determining the being and permitted operations of its objects. The present paper argues that Book I is coherent on the second architecture. Its definitions are seeds, its postulates completion commands, its common notions rules of equality and order, and its propositions executable consequences in dependency order. Under that interpretation the alleged gaps concerning intersection, superposition, order, parallels, and figure equality are not additions from outside Euclid's system but explicit or derived consequences of the system as a whole.

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