

EUCLID'S ELEMENTS

BOOK VIII

Continued Proportion, Squares, Cubes, and Similar
Numbers

27 Propositions

A classical arithmetic edition with cinematic pedagogical illustrations

Contents

Book VIII

Continued Proportion

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the numbers are the least of those having the same ratio as them.

Proof.

Action 1. Let A, B, C, D be continuously proportional, so that

$$A : B = B : C = C : D,$$

and suppose that the outermost numbers A and D are prime to one another:

$$\gcd(A, D) = 1.$$

Action 2. Suppose, contrary to the required conclusion, that there is a smaller continuously proportional sequence E, F, G, H having the same corresponding ratios as A, B, C, D .

Action 3. Since the two sequences have the same number of terms and corresponding successive ratios, equality through the chains gives

$$A : D = E : H$$

(Prop. VII.14).

Action 4. Because A and D are prime to one another, they are the least numbers having their ratio (Prop. VII.21).

Action 5. The least numbers of a ratio measure any other numbers having that ratio an equal number of times, the leading measuring the leading and the following the following (Prop. VII.20). Hence A measures E and D measures H .

Action 6. But E was assumed smaller than A . A greater positive number cannot measure a smaller positive number. This is impossible.

Action 7. Therefore the supposed smaller continuously proportional sequence E, F, G, H cannot exist.

Action 8. Hence A, B, C, D are the least numbers having their continued ratio.

Therefore, if the extremes of a continued proportional sequence are prime to one another, the entire sequence is the least sequence having that ratio.

EUCLID – BOOK VIII, PROPOSITION 1 ARISTOTLE TEACHES YOUNG ALEXANDER

1 SET OUT THE CONTINUOUS PROPORTION

Let the numbers be continuously proportional.

$$\begin{array}{ccccccc} A & : & B & = & B & : & C \\ A & & B & & C & & D \end{array}$$

The outermost numbers A and D are prime to one another.
 $\gcd(A, D) = 1$

2 ASSUME SMALLER NUMBERS EXIST

Suppose there are smaller numbers having the same ratio.

$$\begin{array}{ccccccc} A & & B & & C & & D \\ E & & F & & G & & H \end{array}$$

$E < A, F < B, G < C, H < D$
and $E : F = F : G = G : H$

3 COMPARE THE TWO EXTREME RATIOS

Since the ratios correspond, the first to last ratios are equal.

$$A : D = E : H$$

4 USE THE FACT THAT A AND D ARE PRIME

From earlier (Book VII): Numbers prime to one another are the least numbers having their ratio.

A and D are prime
therefore A and D are the least numbers for the ratio
 $A : D$

5 THEREFORE A MEASURES E

Since $A : D = E : H$ and A, D are the least for this ratio, A must measure E (and D measures H).

$$A \mid E \text{ and } D \mid H$$

Thus $E = kA$
for some positive integer k.

6 BUT E WAS SUPPOSED TO BE SMALLER THAN A!

We assumed $E < A$, but we have just shown $A \mid E$.

$$A \mid E \text{ and } E < A$$

IMPOSSIBLE

7 REJECT THE SMALLER SEQUENCE

The assumed smaller numbers cannot exist.

$$\begin{array}{ccccccc} E & & F & & G & & H \end{array}$$

Therefore no smaller continuously proportional numbers have the same ratio.

8 THE CONCLUSION

Therefore A, B, C, D are the least numbers having this continued ratio.

$$\begin{array}{ccccccc} A & & B & & C & & D \end{array}$$

$A, D \text{ prime} \Rightarrow$
A, B, C, D are the least proportional sequence.

Thus, my young Alexander, truth in mathematics, as in all noble things, admits of no contrary.

Reason is the guide by which a king governs himself and the world.

Proposition 2

To find the least numbers, as many as may be prescribed, which are continuously proportional in a given ratio.

Proof.

Action 1. Let the given ratio be expressed by the least numbers $A : B$. By Prop. VII.22, A and B are prime to one another.

Action 2. Suppose, for example, that four continuously proportional numbers are required in the ratio $A : B$.

Action 3. Form the first products

$$C = A^2, \quad D = AB, \quad E = B^2.$$

Action 4. Form a second layer by multiplying C, D, E by A , and E by B :

$$F = A^3, \quad G = A^2B, \quad H = AB^2, \quad K = B^3.$$

Action 5. Since the same multiplier A produces C, D from A, B , and the same multiplier B produces D, E from A, B , Props. VII.17–18 give

$$C : D = D : E = A : B.$$

Action 6. Applying the same reasoning to F, G, H, K gives

$$F : G = G : H = H : K = A : B.$$

Action 7. Since A and B are prime to one another, equal powers of them are prime to one another (Prop. VII.27); hence C, E are prime to one another and likewise F, K are prime to one another.

Action 8. By Proposition VIII.1, the three-term chain C, D, E and the four-term chain F, G, H, K are therefore the least chains having the ratio $A : B$.

Action 9. The same construction continues: for any prescribed number n of terms, the sequence

$$A^{n-1}, A^{n-2}B, A^{n-3}B^2, \dots, AB^{n-2}, B^{n-1}$$

has each adjacent ratio equal to $A : B$.

Action 10. Thus one can construct the least continuously proportional numbers, in any prescribed multitude, in the given ratio.

Therefore, the required least continued proportional sequence exists for any prescribed number of terms.

EUCLID – BOOK VIII, PROPOSITION 2: TO FIND THE LEAST NUMBERS ARISTOTLE TEACHES YOUNG ALEXANDER

1 START WITH THE GIVEN RATIO
Let the given ratio be the least numbers $A : B$.
 A and B are prime to one another.

$A : B$
(A, B are in least terms)
 $\gcd(A, B) = 1$

2 DECIDE HOW MANY NUMBERS
Suppose four continuously proportional numbers are required.

We want:
 $X_1 : X_2 = X_2 : X_3 = X_3 : X_4 = A : B$

$X_1 \quad X_2 \quad X_3 \quad X_4$

3 BUILD THE FIRST POWERS
From A and B construct:
 $C = A^2, \quad D = AB, \quad E = B^2$

$C = A^2$
 $D = AB$
 $E = B^2$

4 EXTEND ONE LEVEL FURTHER
Multiply as follows:
 $F = AC = A^3, \quad G = AD = A^2B$
 $H = AE = AB^2, \quad K = BE = B^3$

$F \quad G \quad H \quad K$
 $A^3 \quad A^2B \quad AB^2 \quad B^3$

5 PROVE C, D, E ARE CONTINUOUSLY PROPORTIONAL
Since $A^2 : AB = A : B$ and $AB : B^2 = A : B$,
therefore $C : D = D : E = A : B$.

$C \quad D \quad E$
 $C : D = A : B$
 $D : E = A : B$

6 PROVE F, G, H, K ARE CONTINUOUSLY PROPORTIONAL
Likewise $F : G = A : B, G : H = A : B, H : K = A : B$.

$F \quad G \quad H \quad K$
 $F : G = A : B, G : H = A : B, H : K = A : B$

7 THE OUTERMOST NUMBERS ARE PRIME
Since A and B are prime, their powers are also prime:
 $\gcd(C, E) = 1$
 $\gcd(F, K) = 1$

$C = A^2$ prime to one another $E = B^2$
 $F = A^3$ prime to one another $K = B^3$

8 APPLY PROPOSITION 1
Since the outer numbers are prime and the numbers are continuously proportional, they are the least numbers having this ratio.

$C \quad D \quad E$ Least 3 numbers
 $F \quad G \quad H \quad K$ Least 4 numbers

9 THE PATTERN IS CLEAR

3 numbers: $A^2 \quad AB \quad B^2$
4 numbers: $A^3 \quad A^2B \quad AB^2 \quad B^3$
5 numbers: $A^4 \quad A^3B \quad A^2B^2 \quad AB^3 \quad B^4$

In general, for n numbers:
 $A^{n-1}, A^{n-2}B, A^{n-3}B^2, \dots, AB^{n-2}, B^{n-1}$
and every adjacent ratio is $A : B$.

10 THE CONCLUSION

Thus we have found the least numbers, as many as may be prescribed, continuously proportional in the given ratio $A : B$.

Knowledge, like numbers, grows by order and reason.
— Aristotle

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers which are the least of those having the same ratio as them, then the outermost of them are prime to one another.

Proof.

Action 1. Let A, B, C, D be the least numbers in their continued ratio:

$$A : B = B : C = C : D.$$

We must prove that A and D are prime to one another.

Action 2. Take the least pair E, F having the same basic ratio as $A : B$. By Prop. VII.22,

$$\gcd(E, F) = 1.$$

Action 3. By Proposition VIII.2 construct the least three-term sequence G, H, K in the ratio $E : F$.

Action 4. Continue the same construction, increasing the number of terms one at a time, until a least sequence L, M, N, O is obtained with the same number of terms as A, B, C, D .

Action 5. Since E, F are prime to one another, the successive extreme powers constructed from them are prime to one another (Prop. VII.27). Hence L and O are prime to one another.

Action 6. Both A, B, C, D and L, M, N, O are least sequences in the same continued ratio and have the same number of terms.

Action 7. Therefore corresponding terms are equal; in particular,

$$A = L, \quad D = O.$$

Action 8. Since L and O are prime to one another, A and D are also prime to one another.

Action 9. Thus the outermost terms of a least continued proportional sequence are prime to one another.

Therefore, $\gcd(A, D) = 1$.

EUCLID – BOOK VIII, PROPOSITION 3
IF THE NUMBERS ARE THE LEAST CONTINUOUSLY PROPORTIONAL,
THEN THE OUTERMOST ARE PRIME TO ONE ANOTHER.

1 SET OUT THE LEAST PROPORTIONAL SEQUENCE Take A, B, C, D continuously proportional, and suppose they are the least numbers having this ratio. $\begin{array}{ccccccc} A & B & B & - & C & = & D \\ A & B & C & & D & & \end{array}$ To prove: A and D are prime to one another.	2 TAKE THE LEAST PAIR HAVING THE SAME RATIO Let E and F be the least numbers having the ratio A : B. Then E and F are prime to one another. $E : F = A : B$ $\gcd(E, F) = 1$	3 BUILD THE LEAST THREE NUMBERS From E and F, construct the least three numbers continuously proportional. $G : H = H : K = E : F$ Since E and F are prime, G and K are prime to one another.
4 CONTINUE THE CONSTRUCTION Continue in the same way, increasing the number of terms each time, until we have as many terms as A, B, C, D. $\begin{array}{ccccccc} \text{2 terms} & & \text{3 terms} & & \text{4 terms} & & \\ \begin{array}{cc} E & F \end{array} & \begin{array}{ccc} G & H & K \end{array} & \begin{array}{cccc} L & M & N & O \end{array} \end{array}$ We now have four terms.	5 THE OUTERMOST OF THE NEW SEQUENCE ARE PRIME The sequence L, M, N, O is the least continuously proportional in this ratio. Their outermost numbers are prime to one another. $L : M = M : N = N : O = E : F$ $\gcd(L, O) = 1$	6 COMPARE THE TWO LEAST SEQUENCES Both A, B, C, D and L, M, N, O have the same number of terms, the same ratio, and are the least. $\begin{array}{ccccccc} A & B & C & D & & & \\ L & M & N & O & & & \end{array}$ Therefore corresponding terms are equal.
7 EQUAL CORRESPONDING TERMS Each of A, B, C, D is equal to its corresponding member of L, M, N, O. $\begin{array}{l} A = L \\ B = M \\ C = N \\ D = O \end{array}$	8 TRANSFER PRIMALITY Since L and O are prime to one another, and $A = L$, $D = O$, therefore A and D are prime to one another. $\gcd(L, O) = 1$ $\downarrow$ $A = L, D = O$ $\gcd(A, D) = 1$	9 THE CONCLUSION If the numbers are the least continuously proportional, then the outermost are prime to one another. Thus it is shown.

TOGETHER WITH PROPOSITION 1

If the outermost numbers are prime,
then the numbers are the least.

↔

If the numbers are the least,
then the outermost are prime.

*So truth in number is complete,
as beauty in the cosmos.*

– Aristotle

Proposition 4

For any multitude whatsoever of given ratios, expressed in the least numbers, to find the least numbers continuously proportional in these given ratios.

Proof.

Action 1. Let the given ratios, each already expressed in least numbers, be

$$A : B, \quad C : D, \quad E : F.$$

We seek four least numbers whose successive ratios are these three ratios.

Action 2. Let G be the least number measured by both B and C (Prop. VII.34).

Action 3. As many times as B measures G , let A measure H . Then

$$H : G = A : B.$$

Action 4. As many times as C measures G , let D measure K . Then

$$G : K = C : D.$$

Action 5. It remains to attach the ratio $E : F$ to K . There are two cases according as E measures K or does not measure it.

Action 6. If $E \mid K$, let $K = pE$ and set $L = pF$. Then

$$K : L = E : F,$$

so H, G, K, L have the three required successive ratios.

Action 7. Suppose instead that $E \nmid K$. Let M be the least number measured by both E and K . Scale H, G, K by the same factor which takes K to M , obtaining N, O, M ; then choose P so that

$$M : P = E : F.$$

Thus N, O, M, P have the required ratios.

Action 8. In either case, suppose a smaller chain existed. Its second term would be measured both by B and C , and therefore by their least common multiple G (Prop. VII.35), contradicting its being smaller than G ; in the second case the analogous argument uses the least common multiple of E and K .

Action 9. Hence the chain produced in the relevant case is the least chain continuously proportional in the prescribed ratios.

Therefore, the least numbers continuously proportional in any prescribed sequence of ratios can be constructed.

EUCLED – BOOK VIII. PROPOSITION 4

1 SET OUT THE GIVEN RATIOS
Let the given ratios (in least terms) be
 $A : B \quad C : D \quad E : F$

2 JOIN THE FIRST TWO RATIOS
Let G be the least number measured by both B and C .
(The least common multiple of B and C .)

3 SCALE A TO MEET G
As many times as B measures G , so many times let A measure H .
Then $H : G = A : B$.

4 SCALE D TO MEET G
As many times as C measures G , so many times let D measure K .
Then $G : K = C : D$.

5 DOES E MEASURE K?
We must now attach the third ratio $E : F$ to the last.
Does E divide K ?
Yes / No

6 DOES L DIVIDE K?
Let $L = K - E$.
Then $H : G :: K : L$.
Thus $H : G :: K : L$ are the numbers.

7 TO SHOW THEY ARE LEAST
Let any smaller number would give a number N measured to both H and L , $O = H - N$, containing the same common measure G .

8 LET $M = qG$
Multiply H and I by the same factor q .
 $N = qH$ and $O = qI$.
Then $N : O :: A : B$.

1 GIVEN RATIOS
(least terms)

$A : B$
 $C : D$
 $E : F$

2 JOIN THE FIRST TWO
TAKE $G = \text{lcm}(B, C)$

$B \quad \quad G \quad \quad C$
 $\quad | \quad G \quad \quad | \quad C$

3 FORM H AND K
 $H = mA$ where $m = \frac{G}{B}$
 $K = nD$ where $n = \frac{G}{C}$

$H \quad \quad G \quad \quad K$
 $\quad \quad \quad | \quad \quad \quad$

$H : G = A : B$

$G : K = C : D$

4 CHECK IF E | F | K
Two cases.

YES
(Case I)

NO
(Case II)

5 CASE I: E | K
 Let $K = pE$, $L = pF$.
 Then $K : L = E : F$.

$H \quad G \quad K \quad L$
 $\quad \quad \quad | \quad \quad \quad$

$H : G : K : L$
 are the least numbers.

6 CASE II: E | F | K
 Let $K = pE$, $L = pF$.
 $N = qH$, $O = qG$ where $M = qK$.
 Let $P = rF$ where $M = rE$.

$N \quad O \quad M \quad P$
 $\quad \quad \quad | \quad \quad \quad$

$N : O : M : P$
 are the least numbers.

7 BOTH CASES GIVE
 The numbers are continuously proportional in the given ratios and are the least.

WHY THIS WORKS

We always use the least common multiple to join ratios, and scale every adjoining number by the same factor.

Thus the ratios are preserved and the numbers remain least.

Thus, my young Alexander,

from many ratios we weave one chain of numbers, the smallest possible, bound together by reason and order.

— Aristotle

Proposition 5

Plane numbers have to one another the ratio compounded out of the ratios of their sides.

Proof.

Action 1. Let A and B be plane numbers. Let the sides of A be C, D and the sides of B be E, F :

$$A = CD, \quad B = EF.$$

Action 2. Isolate the two ratios of corresponding sides:

$$C : E \quad \text{and} \quad D : F.$$

Action 3. By Proposition VIII.4 take least numbers G, H, K continuously proportional in those two ratios, so that

$$G : H = C : E, \quad H : K = D : F.$$

Action 4. Let D multiply E and produce the bridge number

$$L = DE.$$

Action 5. Since D makes A from C and L from E , Prop. VII.17 gives

$$A : L = C : E = G : H.$$

Action 6. Since E makes L from D and B from F , Prop. VII.17 gives

$$L : B = D : F = H : K.$$

Action 7. Thus the two three-term chains A, L, B and G, H, K have corresponding equal successive ratios.

Action 8. Passing from first to last through equal ratios (Prop. VII.14) gives

$$A : B = G : K.$$

Action 9. But $G : K$ is, by construction, the ratio compounded out of $G : H$ and $H : K$, that is, out of $C : E$ and $D : F$.

Action 10. Hence the ratio of the plane numbers $A : B$ is compounded out of the ratios of their sides.

Therefore, $A : B$ is the ratio compounded from $C : E$ and $D : F$.

EUCLID - BOOK VIII, PROPOSITION 5
PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO
COMPOUNDED OUT OF THE RATIOS OF THEIR SIDES.

1 SET OUT THE TWO PLANE NUMBERS
 Let the plane numbers be A and B.
 Let the sides of A be C and D,
 and the sides of B be E and F.

$A = C \cdot D$

 $A = C \cdot D$

$B = E \cdot F$

 $B = E \cdot F$

2 ISOLATE THE SIDE-RATIOS
 The two ratios of corresponding sides are C : E and D : F.

$C : E \quad D : F$

3 BUILD A THREE-TERM PROPORTIONAL CHAIN
 By Proposition 4, take the least three numbers G, H, K such that

$G : H = C : E$
 $H : K = D : F$

$G \quad H \quad K$
 $\longleftarrow \quad \quad \longrightarrow$
 $G : H : K$
 encodes the two given side-ratios.

4 INTRODUCE A BRIDGE NUMBER
 Multiply D by E,
 and let

$L = D \cdot E$

This will serve as a bridge between A and B.

$A = C \cdot D$

 $A = C \cdot D$

$L = D \cdot E$

 $L = D \cdot E$

$B = E \cdot F$

 $B = E \cdot F$

5 COMPARE A WITH L
 Since D multiplies both C and E,
 and let

$A : L = C : E$

But $G : H = C : E$
 Therefore
 $A : L = G : H$

6 COMPARE L WITH B
 Since E multiplies both D and F,
 $L : B = D : F$
 But $H : K = D : F$
 Therefore
 $L : B = H : K$

7 LINE UP THE TWO CHAINS
 We now have two matching chains of three terms.

$A \quad L \quad B$
 $\longleftarrow \quad \quad \longrightarrow$
 $G \quad H \quad K$
 $A : L = G : H$
 $L : B = H : K$

8 PASS FROM FIRST-TO-LAST
 Since the ratios of corresponding parts are equal, the ratios of the first to the last are also equal.
 Therefore
 $A : B = G : K$

$A \quad L \quad B$
 $G \quad H \quad K$

9 INTERPRET G : K
 The ratio G : K is compounded from the two consecutive ratios C : E and D : F.

$G : K = (C : E) \text{ compounded with } (D : F)$
 $G : K = C : E \times D : F$

10 THE CONCLUSION
 Therefore the ratio of the plane numbers A and B is the ratio compounded out of the ratios of their sides.

$A : B = (C : E) \text{ compounded with } (D : F)$
 In modern terms,
 $\frac{A}{B} = \frac{C}{E} \times \frac{D}{F}$

WHAT WE HAVE LEARNED
 To compare areas, we do not compare the whole squares at once, but the ratios of their sides.
 Area ratios are built from side ratios.

$A = C \cdot D$

 $A = C \cdot D$

$\Rightarrow$

$B = E \cdot F$

 $B = E \cdot F$

$\Rightarrow$

$A : B = (C : E) \times (D : F)$

$\downarrow$
 ratio of first sides

$\downarrow$
 ratio of second sides

 $\downarrow$
 compounded ratio

Thus, my young Alexander, reason weaves together what seems apart, and from simple ratios the harmony of number and magnitude is revealed.
 — Aristotle

Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other number will measure any other number either.

Proof.

Action 1. Let A, B, C, D, E be continuously proportional and suppose

$$A \nmid B.$$

The successive terms therefore do not measure one another.

Action 2. We must show more generally that no earlier term measures any later term.

Action 3. Suppose, if possible, that A measures C .

Action 4. Take the least three numbers F, G, H having the same continued ratio as A, B, C (Prop. VII.33). Then

$$A : C = F : H$$

by equality through the chains (Prop. VII.14).

Action 5. Since $A : B = F : G$ and $A \nmid B$, Def. VII.20 gives $F \nmid G$. Hence F is not a unit, because the unit measures every number.

Action 6. The numbers F, G, H are least in their continued ratio; therefore their extremes F, H are prime to one another by Prop. VIII.3. Since F is not a unit, F cannot measure H .

Action 7. But $A : C = F : H$. If A measured C , then the leading term F would measure the following term H the same number of times (Def. VII.20), contradiction.

Action 8. Therefore $A \nmid C$. Repeating the same argument for any two nonadjacent members proves that no member of the continued proportional sequence measures any other.

Therefore, if the first does not measure the second, no term in the continued proportion measures another term.

EUCLID – BOOK VIII, PROPOSITION 6

IF THE SIDES OF TWO PLANE NUMBERS HAVE TO ONE ANOTHER THE SAME RATIO, THE PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO OF THEIR SIDES SQUARED.

1 SET OUT THE TWO PLANE NUMBERS
Let the plane numbers be A and B.
Let the sides of A be C and D.
Let the sides of B be E and F.

2 GIVEN: THE SIDES ARE IN THE SAME RATIO
Suppose the sides have to one another the same ratio.
 $C : E = D : F$
(That is, the first sides are proportional, and the second sides are proportional.)

3 CONSTRUCT SQUARES ON THE SIDES
On the sides of A construct the squares C² and D².
On the sides of B construct the squares E² and F².

4 APPLY PROPOSITION 5 TO THE SQUARES
By Proposition 5, the ratio of the squares on C and D has to the squares on E and F the compounded ratio of C : E and D : F.

Therefore
 $C^2 : D^2 :: E^2 : F^2$
 $= (C : E) \cdot (D : F)$

5 USE THE GIVEN RATIO
But $C : E = D : F$.
Therefore the ratio compounded from C : E and D : F is the same as the ratio C : E (or D : F).
Hence
 $C^2 : D^2 :: E^2 : F^2$
 $= C : E = D : F$

6 SUBSTITUTE THE PLANE NUMBERS
Since $A = C \cdot D$ and $B = E \cdot F$, then $C^2 : D^2$ is the square on A, and $E^2 : F^2$ is the square on B.
Therefore
(square on A) : (square on B) $= C : E = D : F$
That is, the squares on A and B have to one another the ratio of their sides.

AN ALTERNATIVE VIEW (THE SAME REASONING IN A MORE VISUAL FORM)

7 THE GIVEN PROPORTIONS
 $C : E = D : F$
Take any multiple ratio.
Let $C : E = m : n$
and $D : F = m : n$.
Then $C = m \cdot k$, $E = n \cdot k$,
 $D = m \cdot l$, $F = n \cdot l$
for some numbers k, l .

8 EXPRESS THE PLANE NUMBERS
 $A = C \cdot D = (m \cdot k)(m \cdot l) = m^2 \cdot k \cdot l$
 $B = E \cdot F = (n \cdot k)(n \cdot l) = n^2 \cdot k \cdot l$

9 TAKE THE RATIO OF A TO B
 $A : B = m^2 \cdot k \cdot l : n^2 \cdot k \cdot l$
 $= m^2 : n^2$
 $= (m : n)^2$
But $m : n = C : E = D : F$.
Therefore
 $A : B = (C : E)^2 = (D : F)^2$

10 CONCLUDE
Hence the squares on A and B have to one another the ratio of their sides squared.
 $A : B = (C : E)^2 = (D : F)^2$

11 FINAL STATEMENT
Therefore, if the sides of two plane numbers have to one another the same ratio, the plane numbers (or the squares on them) have to one another the ratio of their sides squared.
Thus, my young Alexander, in the harmony of ratios, areas reveal the power of squares.
— Aristotle

IN MODERN SYMBOLS

If $C : E = D : F$, then the ratio of the plane numbers $A = C \cdot D$ and $B = E \cdot F$ is

$$A : B = (C : E)^2 = (D : F)^2 \quad \text{or}$$

$$\frac{A}{B} = \left(\frac{C}{E}\right)^2 = \left(\frac{D}{F}\right)^2$$

Proposition 7

If there are any multitude whatsoever of continuously proportional numbers, and the first measures the last, then the first will also measure the second.

Proof.

Action 1. Let

$$A : B = B : C = C : D$$

and suppose that A measures the last term D :

$$A \mid D.$$

Action 2. We must prove that

$$A \mid B.$$

Action 3. Suppose, contrary to the desired conclusion, that

$$A \nmid B.$$

Action 4. By Proposition VIII.6, if the first term of a continued proportion does not measure the second, then no term measures any other term.

Action 5. In particular Proposition VIII.6 would imply

$$A \nmid D.$$

Action 6. But the hypothesis says

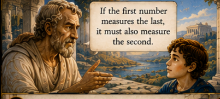
$$A \mid D.$$

This is a contradiction.

Action 7. Therefore the supposition $A \nmid B$ is false, and hence $A \mid B$.

Action 8. Thus the first term, if it measures the last term of a continued proportion, also measures the second.

Therefore, $A \mid B$.

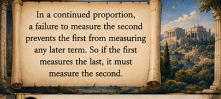


If the first number measures the last, it must also measure the second.

EUCLID'S ELEMENTS – BOOK VIII

PROPOSITION 7

If there are any multitude whatsoever of continuously proportional numbers, and the first measures the last, then the first will also measure the second.



In a continued proportion, a failure to measure the second prevents the first from measuring any later term. So if the first measures the last, it must measure the second.

1 Set out the continued proportion

Let A, B, C, D be continuously proportional numbers.

$A : B = B : C = C : D$



A B C D

Each number bears the same ratio to the next.

2 Assume the given divisibility

Suppose the first number A measures the last number D.

$A \mid D$



A D

D is a multiple of A.

3 Suppose the contrary

Suppose, for contradiction, that A does not measure B.

$A \nmid B$



A B

We assume that B is not a multiple of A.

4 Apply Proposition 6

By Proposition 6, if in a continued proportion the first does not measure the second, then the first does not measure any later number.

$A \nmid B \Rightarrow A \nmid D$



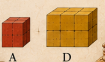
A D

Therefore, from $A \nmid B$ it follows that $A \nmid D$.

5 Compare with the hypothesis

But we were given that A measures D.

$A \mid D$



A D

So $A \mid D$.

6 Reach the contradiction

We thus have both $A \mid D$ and $A \nmid D$



A D

This is impossible.

7 Reject the supposition

Therefore the supposition that A does not measure B is false.

~~$A \nmid B$~~

We reject the assumption.

8 Conclude

Hence the first number A measures the second number B.

$A \mid B$



A B

Therefore, if the first of any multitude of continuously proportional numbers measures the last, it will also measure the second.

"Mathematics, my boy, is a chain of truths. Once a link fails, the whole chain is broken."
— Aristotle



*From the part to the whole,
from proportion to certainty.*

Proposition 8

If between two numbers there fall some numbers in continued proportion, then as many numbers as fall between them in continued proportion, so many numbers will also fall between any two numbers having the same ratio as them in continued proportion.

Proof.

Action 1. Let

$$A : C = C : D = D : B,$$

so that two mean proportionals C, D lie between A and B .

Action 2. Let another pair E, F satisfy

$$A : B = E : F.$$

We seek two means between E and F .

Action 3. Take the least four numbers G, H, K, L having the same continued ratio as A, C, D, B .

Action 4. Since G, H, K, L are least in their continued ratio, their extremes are prime to one another by Prop. VIII.3:

$$\gcd(G, L) = 1.$$

Action 5. Comparing first-to-last ratios gives

$$A : B = G : L.$$

Together with $A : B = E : F$, this gives $G : L = E : F$.

Action 6. Since G, L are prime to one another, they are the least numbers in their ratio (Prop. VII.21), and hence they measure E, F equally many times. Thus for some positive integer r ,

$$E = rG, \quad F = rL.$$

Action 7. Scale the interior terms by the same factor:

$$M = rH, \quad N = rK.$$

Action 8. Equal scaling preserves all adjacent ratios, so

$$E : M = M : N = N : F.$$

Action 9. Thus exactly two mean proportionals, M, N , lie between E and F , the same number as between A and B .

Action 10. The argument is unchanged for any number of intermediate proportional terms.

Therefore, pairs of numbers having the same outer ratio admit the same number of intervening mean proportionals.

EUCLID – BOOK VIII, PROPOSITION 8

IF BETWEEN TWO NUMBERS THERE FALL SOME NUMBERS IN CONTINUED PROPORTION, THEN, AS MANY NUMBERS AS FALL BETWEEN THEM, SO MANY NUMBERS WILL ALSO FALL BETWEEN ANY TWO NUMBERS HAVING THE SAME RATIO AS THEM IN CONTINUED PROPORTION.

1 SET OUT THE FIRST CONTINUED PROPORTION
Let A, C, D, B be in continued proportion.

$A : C = C : D = D : B$

There are two numbers, C and D, between A and B.

2 INTRODUCE ANOTHER PAIR WITH THE SAME RATIO
Take E and F such that $A : B = E : F$.

We must find two numbers, M and N, so that $E : M = M : N = N : F$.

3 REPLACE BY THE LEAST EQUIVALENT CHAIN
By Proposition 4, take the least numbers G, H, K, L, having the same ratio as A, C, D, B.

$G : H = H : K = K : L$

This chain has the same proportional structure as A, C, D, B.

4 THE EXTREMES ARE PRIME
Since G, H, K, L are the least such numbers, by Proposition 3 the extreme terms are prime to one another.

$\gcd(G, L) = 1$

G and L have no common measure.

5 COMPARE THE OUTERMOST RATIO
From the equality of the corresponding ratio of the two chains,

$A : B = G : L$

But we are given $A : B = E : F$. Therefore,

$G : L = E : F$

6 USE PRIMENESS TO SCALE
Since G and L are prime to one another, they are the least numbers for this ratio. Therefore they must measure E and F the same number of times.

$E = rG$ and $F = rL$ for some whole number r .

7 SCALE THE INTERIOR TERMS
As G measures E r times, let H measure M r times, and K measure N r times.

$M = rH$ and $N = rK$

We multiply the whole least chain by the same factor r .

$G : H = H : K = K : L$
 $E : M = M : N = N : F$

8 RATIOS ARE PRESERVED
Multiplying by the same factor preserves all ratios.

$G : H = E : M$
 $H : K = M : N$
 $K : L = N : F$

Since $G : H = H : K = K : L$, we get

$E : M = M : N = N : F$

9 COUNT THE NUMBERS
Between A and B there were two numbers, C and D.

$A : C = C : D = D : B$ (two numbers between)

Between E and F we have M and N.

$E : M = M : N = N : F$ (also two numbers between)

10 CONCLUSION
Therefore, if two numbers have some numbers between them in continued proportion, then any two numbers having the same ratio will also have the same number of such numbers between them.

The number of mean proportionals depends only on the ratio of the extremes, not on their magnitude.

THE IDEA
A continued proportion is determined, up to a common scale, by two things:

- the ratio of the extremes
- the number of mean proportionals.

$A : C = C : D = D : B$
(two means)
 $A : B = E : F$

⇒

$G : H = H : K = K : L$
(least chain)
 $G : L = E : F$

⇒

$E : M = M : N = N : F$
(two means)
Same ratio, same number

⇒

THE STRUCTURE OF HARMONY IS PRESERVED.

Proposition 9

If two numbers are prime to one another, and some numbers fall between them in continued proportion, then as many numbers as fall between them in continued proportion, so many numbers will also fall between each of them and a unit in continued proportion.

Proof.

Action 1. Let A and B be prime to one another and let

$$A : C = C : D = D : B.$$

Set out the unit $E = 1$.

Action 2. Take the least pair F, G having the same basic ratio as the continued chain.

Action 3. By Proposition VIII.2 construct successively the least three numbers H, K, L and the least four numbers M, N, O, P in that ratio.

Action 4. Because A, B are prime to one another, the original four-term chain is least (Prop. VIII.1), and therefore coincides term-by-term with the least four-term chain; in particular

$$M = A, \quad P = B.$$

Action 5. From the construction of Proposition VIII.2,

$$H = F^2, \quad M = F^3 = A,$$

and similarly

$$L = G^2, \quad P = G^3 = B.$$

Action 6. Since E is the unit,

$$E : F = F : H = H : A.$$

Thus F, H are two mean proportionals between the unit and A .

Action 7. Likewise,

$$E : G = G : L = L : B,$$

so G, L are two mean proportionals between the unit and B .

Action 8. Hence the same number of mean proportionals lies between 1 and each extreme as lay between A and B .

Action 9. The reasoning applies to any number of intervening terms.

Therefore, if prime extremes admit n mean proportionals, then each extreme admits n mean proportionals with the unit.

EUCLID - BOOK VIII, PROPOSITION 9

IF TWO NUMBERS ARE PRIME TO ONE ANOTHER, AND SOME NUMBERS FALL BETWEEN THEM IN CONTINUED PROPORTION, THEN AS MANY NUMBERS AS FALL BETWEEN THEM IN CONTINUED PROPORTION, SO MANY NUMBERS WILL ALSO FALL BETWEEN EACH OF THEM AND A UNIT IN CONTINUED PROPORTION.

1 SET OUT THE PRIME EXTREMES

Let A and B be prime to one another.

($\gcd(A, B) = 1$)

Between them fall two numbers in continued proportion, C and D, so that

$$A : C = C : D = D : B$$

2 SET OUT THE UNIT

Let E be the unit.

We will construct the same number of mean proportionals between E and A, and between E and B, so that

$$E : F = F : F^2 = F^2 : F^3 = F^3 : A$$

$$E : G = G : G^2 = G^2 : G^3 = G^3 : B$$

3 TAKE THE LEAST PAIR OF THE GIVEN RATIO

Take the least two numbers F and G having the same ratio as A and B. Then, by Proposition 2, construct successively three least numbers H, K, L, and four least numbers M, N, O, P.

$$F : G = H : K = L : M = N : O = P$$

4 IDENTIFY THE LEAST CHAIN WITH A, C, D, B

Since A and B are prime to one another, the chain A, C, D, B is the least chain of this ratio.

Therefore the least four numbers M, N, O, P are the same as A, C, D, B respectively.

$$M = A, N = C, O = D, P = B$$

The least chain represents the ratio between the prime extremes.

5 THE POWER STRUCTURE (PROOF 2)

From the construction:

- F multiplying itself makes H.
- F multiplying H makes M.
- G multiplying itself makes L.
- G multiplying L makes P.

$$H = F^2, L = G^2$$

$$M = F^3 = A, P = G^3 = B$$

Thus $A = F^3$ and $B = G^3$.

6 CONSTRUCT BETWEEN THE UNIT AND A

Since $E = 1$, and $H = F^2, M = F^3 = A$, we have

$$E : F = F : F^2 = F^2 : F^3 = F^3 : A$$

Hence $E : F = F : H = H : A$

So F and H fall between E and A in continued proportion.

7 CONSTRUCT BETWEEN THE UNIT AND B

Similarly, since $L = G^2, P = G^3 = B$, we have

$$E : G = G : G^2 = G^2 : G^3 = G^3 : B$$

Hence $E : G = G : L = L : B$

So G and L fall between E and B in continued proportion.

8 COUNT THE MEAN PROPORTIONALS

Between A and B we had two mean proportionals: C and D.

Between E and A we have two means: F and H.

Between E and B we have two means: G and L.

9 CONCLUSION

Therefore, as many numbers as fall between two prime numbers in continued proportion, so many numbers will also fall between each of them and a unit in continued proportion.

From $A : C = C : D = D : B$ (with A and B prime) we constructed

$$E : F = F : H = H : A$$

and

$$E : G = G : L = L : B$$

the same number of means.

THE WHOLE IDEA

If a continued proportion of length n lies between two prime numbers, then each of those numbers is an n+1 power of the same base.

$A : C = C : D = D : B$
(two means)
A and B prime

⇒

Take least ratio F : G
Build powers
 F^3, F^3 and G^3, G^3
So $A = F^3, B = G^3$

⇒

Between the unit and A
 $1 : F = F : F^2 = F^2 : F^3 = A$
(two means: F, F²)

⇒

Between the unit and B
 $1 : G = G : G^2 = G^2 : G^3 = B$
(two means: G, G²)

⇒

Same number of mean proportionals preserved.

Proposition 10

If some numbers fall between each of two numbers and a unit in continued proportion, then as many numbers as fall between each of the two numbers and the unit in continued proportion, so many numbers will also fall between the two numbers themselves in continued proportion.

Proof.

Action 1. Let $C = 1$ and suppose

$$C : D = D : E = E : A, \quad C : F = F : G = G : B.$$

Action 2. Since C is the unit, the first chain gives

$$E = D^2, \quad A = D^3.$$

Action 3. Similarly, the second chain gives

$$G = F^2, \quad B = F^3.$$

Action 4. Let D multiply F , producing

$$H = DF.$$

Action 5. Let

$$K = DH = D^2F, \quad L = FH = DF^2.$$

Action 6. From the products,

$$E : H = H : G = D : F.$$

Action 7. Since $A = D^3$ and $K = D^2F$,

$$A : K = D : F.$$

Action 8. Since $K = D^2F$ and $L = DF^2$,

$$K : L = D : F.$$

Action 9. Since $L = DF^2$ and $B = F^3$,

$$L : B = D : F.$$

Action 10. Therefore

$$A : K = K : L = L : B.$$

Action 11. Thus the same number of mean proportionals lies between A and B as lies between the unit and each of them.

Therefore, the two means K, L fall between A and B in continued proportion.

EUCLID - BOOK VIII, PROPOSITION 10

IF SOME NUMBERS FALL BETWEEN EACH OF TWO NUMBERS AND A UNIT IN CONTINUED PROPORTION, THEN, AS MANY NUMBERS AS FALL BETWEEN EACH OF THE TWO NUMBERS AND THE UNIT, SO MANY NUMBERS WILL ALSO FALL BETWEEN THE TWO NUMBERS THEMSELVES IN CONTINUED PROPORTION.

1 SET OUT THE TWO CHAINS
Let C be the unit.
Let two mean proportionals fall between C and A, and two between C and B.

$C = 1$
 $C : D :: D : E :: E : A$
 $C : F :: F : G :: G : B$

2 RECOGNIZE THE POWER STRUCTURE
From the first chain:
 $E = D^2$
 $A = D \cdot E = D^3$
From the second chain:
 $G = F^2$
 $B = F \cdot G = F^3$

3 CONSTRUCT THE BRIDGE
Let D multiply E.
Let
 $H = D \cdot E$

4 BUILD THE NEXT TWO NUMBERS
Let H be multiplied by D and by F.
Let
 $K = D \cdot H = D^2 E$
 $L = F \cdot H = D E^2$

5 SHOW $E : H = H : G$
Because $E = D^2$, $H = D E$, $G = F^2$, we have
 $E : H :: D^2 : D E :: D : F$
 $H : G :: D E : F^2 :: D : F$
Therefore $E : H = H : G$.

6 COMPARE A AND K
Since $A = D^3$ and $K = D^2 E$, we have
 $A : K = D^3 : D^2 E = D : E$

7 COMPARE K AND L
Since $K = D^2 E$ and $L = D E^2$, we have
 $K : L = D^2 E : D E^2 = D : E$

8 COMPARE L AND B
Since $L = D E^2$ and $B = F^3$, we have
 $L : B = D E^2 : F^3 = D : F$

9 CONCLUDE THE CONTINUED PROPORTION
Since $A : K = K : L = L : B = D : F$, Therefore
 $A : K :: K : L :: L : B$
K and L fall between A and B in continued proportion.

10 COUNT THE MEAN PROPORTIONALS
Between the unit and A there were two mean proportionals D and E.
Between the unit and B there were two F and G.
Between A and B we have two K and L.
The number of mean proportionals is the same in all three cases.
 $1 : D :: D : E :: E : A$ (two means)
 $1 : F :: F : G :: G : B$ (two means)
 $A : K :: K : L :: L : B$ (two means)

THE WHOLE IDEA
If each of two numbers has the same number of mean proportionals with the unit, then the two numbers have the same number of mean proportionals between them.

FROM THE UNIT TO A (two means)
 $1 : D :: D : E :: E : A$

FROM THE UNIT TO B (two means)
 $1 : F :: F : G :: G : B$

POWERS
 $A = D^3$
 $B = F^3$
 $E = D^2$, $G = F^2$

BRIDGE NUMBERS
 $H = D E$
 $K = D^2 E$
 $L = D E^2$

BETWEEN A AND B (two means)
 $A : K :: K : L :: L : B$

CONCLUSION
Same number of mean proportionals preserved.

Proposition 11

There exists one number in mean proportion to two given square numbers; and one square number has to the other a squared ratio with respect to the ratio of their sides.

Proof.

Action 1. Let A, B be square numbers with respective sides C, D :

$$A = C^2, \quad B = D^2.$$

Action 2. Construct the mixed product

$$E = CD.$$

Action 3. Since $A = C \cdot C$ and $E = C \cdot D$, Prop. VII.17 gives

$$A : E = C : D.$$

Action 4. Since $E = C \cdot D$ and $B = D \cdot D$,

$$E : B = C : D.$$

Action 5. Hence

$$A : E = E : B,$$

so E is a mean proportional between A and B .

Action 6. Thus A, E, B are three continuously proportional numbers.

Action 7. Passing from the first term to the third takes the common ratio twice; hence $A : B$ is the squared ratio of $A : E$.

Action 8. But $A : E = C : D$.

Action 9. Therefore $A : B$ is the squared ratio of $C : D$.

Action 10. In modern notation,

$$\frac{A}{B} = \left(\frac{C}{D} \right)^2.$$

Therefore, one mean proportional exists between two squares, and the ratio of the squares is the squared ratio of their sides.

EUCLID – BOOK VIII, PROPOSITION 11

THERE EXISTS ONE NUMBER IN MEAN PROPORTION TO TWO GIVEN SQUARE NUMBERS, AND ONE SQUARE NUMBER HAS TO THE OTHER SQUARE NUMBER A SQUARED RATIO WITH RESPECT TO THE RATIO OF THEIR SIDES.

1 SET OUT THE TWO SQUARE NUMBERS

Let A and B be two square numbers.
Let C be the side of A, and D the side of B.

$A = C^2$ $B = D^2$

2 CONSTRUCT THE MIXED PRODUCT

Multiply the sides C and D together.
Let E = C x D.

$E = C \times D$

3 COMPARE A AND E

$A = C \times C$ and $E = C \times D$.
By Book VII, Prop. 17, if equals are multiplied by equals, the ratios are equal.
Therefore $A : E = C : D$

$A : E = C : C$

4 COMPARE E AND B

$E = C \times D$ and $B = D \times D$.
Again, by Book VII, Prop. 17, the ratios are equal.
Therefore $E : B = C : D$

$E : B = C : D$

5 RECOGNIZE THE MEAN PROPORTIONAL

Since $A : E = C : D$ and $E : B = C : D$, therefore
 $A : E = E : B$
E is a mean proportional between A and B.

$A : E = E : B$

6 SEE THE CONTINUED PROPORTION

The three numbers A, E, B are in continued proportion.
 $A : E = E : B$

$A : E = E : B$

7 MOVE FROM ONE STEP TO TWO STEPS

Since $A : E = E : B$, the ratio from the first term to the third term (A to B) contains two equal ratios.
So A : B is the squared ratio of A : E.

$A : B = (A : E)^2$

8 REPLACE A : E BY C : D

But we have already shown that $A : E = C : D$.
Therefore A : B is the squared ratio of C : D.
 $A : B = (C : D)^2$

$A : B = (C : D)^2$

9 TRANSLATE INTO MODERN NOTATION

Since $A = C^2$ and $B = D^2$, we get
 $\frac{A}{B} = \frac{C^2}{D^2} = \left(\frac{C}{D}\right)^2$
Thus the ratio of the squares is the square of the ratio of their sides.

$\frac{A}{B} = \frac{C^2}{D^2} = \left(\frac{C}{D}\right)^2$

10 CONCLUSION

Therefore:
• There exists one number in mean proportion to A and B – namely $E = C \times D$.
• One square number has to the other square number the squared ratio of the ratio of their sides.
 $A : B = (C : D)^2$
 $C^2 : D^2 = (C : D)^2$

$A : B = (C : D)^2$
 $C^2 : D^2 = (C : D)^2$

THE WHOLE IDEA

Take the sides C and D. Their squares are C^2 and D^2 . The mixed product, CD sits exactly between them. The ratio of the squares is the square of the ratio of the sides.

SIDES

C D

SQUARES

C^2 and D^2

MIXED PRODUCT (ON THE MIDDLE)

$E = C \times D$

CONTINUED PROPORTION

$A = C^2$ $E = CD$ $B = D^2$
 $A : E = E : B$

RATIO OF SQUARES

$A : B = (C : D)^2$
 $C^2 : D^2 = (C : D)^2$

29

Proposition 12

There exist two numbers in mean proportion to two given cube numbers; and one cube number has to the other a cubed ratio with respect to the ratio of their sides.

Proof.

Action 1. Let A, B be cube numbers with sides C, D :

$$A = C^3, \quad B = D^3.$$

Action 2. Construct

$$E = C^2, \quad F = CD, \quad G = D^2.$$

Action 3. Multiply the mixed product by the two sides and set

$$H = CF = C^2D, \quad K = DF = CD^2.$$

Action 4. Since A is the cube on C and B the cube on D ,

$$A = CE = C^3, \quad B = DG = D^3.$$

Action 5. From $E = C^2$ and $F = CD$,

$$E : F = C : D.$$

Action 6. From $F = CD$ and $G = D^2$,

$$F : G = C : D.$$

Action 7. Because $A = CE$ and $H = CF$,

$$A : H = E : F = C : D.$$

Action 8. Because $H = CF$ and $K = DF$,

$$H : K = C : D.$$

Action 9. Because $K = DF$ and $B = DG$,

$$K : B = F : G = C : D.$$

Action 10. Therefore

$$A : H = H : K = K : B,$$

so H, K are the two mean proportionals.

Action 11. The ratio from A to B consists of three successive copies of the side ratio.

Action 12. Hence

$$A : B = (C : D)^3.$$

Therefore, two mean proportionals lie between two cubes, and their ratio is the cubed ratio of their sides.

EUCLID – BOOK VIII, PROPOSITION 12

THERE EXIST TWO NUMBERS IN MEAN PROPORTION TO TWO GIVEN CUBE NUMBERS,
AND ONE CUBE NUMBER HAS TO THE OTHER CUBE NUMBER A CUBED RATIO
WITH RESPECT TO THE RATIO OF THEIR SIDES.

1 SET OUT THE TWO CUBE NUMBERS
Let A and B be two cube numbers.
Let C be the side of A, and D the side of B.

2 CONSTRUCT THE FIRST PRODUCTS
Make the square of each side and their product.
 $E = C^2$ ($C \times C$)
 $F = C \times D$
 $G = D^2$ ($D \times D$)

3 BUILD THE NEXT TWO NUMBERS
Multiply the middle product F by each of the original sides.
 $H = C \times F = C^2 \times D$
 $K = D \times F = C \times D^2$

4 RECOGNIZE THE CUBES
Now A and B are the cubes on C and D.
 $A = C \times E = C^3$
 $B = D \times G = D^3$

5 COMPARE E AND F, AND F AND G
By Book VII, Prop. 17,
 $E : F = C : D$
 $F : G = C : D$
So, $E : F = F : G = C : D$

6 COMPARE A AND H
Since $A = C \times E$ and $H = C \times F$, (the same multiplier C) therefore
 $A : H = E : F = C : D$

7 COMPARE H AND K
Since $H = C \times F$ and $K = D \times F$, (the same multiplier F) therefore
 $H : K = C : D$

8 COMPARE K AND B
Since $K = D \times F$ and $B = D \times G$, (the same multiplier D) therefore
 $K : B = F : G = C : D$

9 ASSEMBLE THE CONTINUED PROPORTION
We have
 $A : H = C : D$
 $H : K = C : D$
 $K : B = C : D$
Therefore
 $A : H = H : K = K : B$
So H and K are two mean proportionals between A and B.

10 THREE EQUAL STEPS
From A to B the same ratio C : D is taken three times.
 $A \times C : D \times C : D \times C : D \times C : D$
Thus A : B has the triple (cubed) ratio of C : D.

11 TRANSLATE TO MODERN NOTATION
Since $A = C^3$ and $B = D^3$
 $A : B = (C : D)^3$
or
 $\frac{A}{B} = \left(\frac{C}{D}\right)^3$

12 CONCLUSION
Therefore:
• There are two numbers, H and K, in mean proportion between the cube numbers A and B.
• And A to B has the cubed ratio of the ratio of their sides C to D.
 $A : H = H : K = K : B$
 $A : B = (C : D)^3$

THE WHOLE IDEA
If C and D are the sides of the cubes, A and B, then the numbers formed as shown fall in a continued proportion whose common ratio is C : D.

SIDES

SQUARES AND MIXED PRODUCT

NEXT TWO NUMBERS

CUBES

CONTINUED PROPORTION

RATIO OF CUBES

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some number by multiplying itself, then the numbers so produced will also be continuously proportional; and if the original numbers multiply the numbers so produced, the new numbers will likewise be continuously proportional.

Proof.

Action 1. Let

$$A : B = B : C.$$

Action 2. Let each original number multiply itself:

$$D = A^2, \quad E = B^2, \quad F = C^2.$$

Action 3. Let each original number multiply its own square:

$$G = A^3, \quad H = B^3, \quad K = C^3.$$

Action 4. Construct a bridge between A and B by setting

$$L = AB, \quad M = AL = A^2B, \quad N = BL = AB^2.$$

Action 5. Then

$$D : L = L : E,$$

$$\text{for } A^2 : AB = AB : B^2 = A : B.$$

Action 6. Likewise

$$G : M = M : N = N : H.$$

Action 7. Construct the corresponding bridge between B and C :

$$O = BC, \quad P = BO = B^2C, \quad Q = CO = BC^2.$$

Action 8. Then

$$E : O = O : F.$$

Action 9. Likewise

$$H : P = P : Q = Q : K.$$

Action 10. Since the original ratios satisfy $A : B = B : C$, the two square bridge-chains have equal corresponding ratios, and the two cube bridge-chains do also.

Action 11. Passing through the equal square chains gives

$$D : E = E : F.$$

Action 12. Passing through the equal cube chains gives

$$G : H = H : K.$$

Action 13. Thus equal powers of continuously proportional numbers are themselves continuously proportional; in modern notation,

$$A : B = B : C \implies A^n : B^n = B^n : C^n.$$

Therefore, squares, cubes, and successive equal powers preserve the continued proportion.

EUCLID — BOOK VIII, PROPOSITION 13

IF THERE ARE ANY MULTITUDE WHATSOEVER OF CONTINUOUSLY PROPORTIONAL NUMBERS, AND EACH MAKES SOME NUMBER BY MULTIPLYING ITSELF, THEN THE NUMBERS CREATED FROM THEM WILL ALSO BE CONTINUOUSLY PROPORTIONAL. AND IF THE ORIGINAL NUMBERS MULTIPLY THE NUMBERS SO CREATED, THE NEW NUMBERS WILL ALSO BE CONTINUOUSLY PROPORTIONAL.

1 SET OUT A CONTINUED PROPORTION Let A, B, C be three numbers in continued proportion. $A : B = B : C$ This is our starting proportion.	2 SQUARE EACH NUMBER (PRODUCE THE SECOND ROW) Let each number multiply itself. $D = A \times A = A^2$ $E = B \times B = B^2$ $F = C \times C = C^2$ $A : B = B : C$	3 CUBE EACH NUMBER (PRODUCE THE THIRD ROW) Now let each original number multiply the square made from it. $G = A \times D = A^3$ $H = B \times E = B^3$ $K = C \times F = C^3$ $A : B = B : C$	4 BUILD BRIDGES BETWEEN A AND B Let $L = A \times B$. Then multiply L by A and by B. $M = A \times L = A^2B$ $N = B \times L = AB^2$ $A : B = B : C$	5 BUILD BRIDGES BETWEEN B AND C Let $O = B \times C$. Then multiply O by B and by C. $P = B \times O = B^2C$ $Q = C \times O = BC^2$ $A : B = B : C$	6 OBTAIN THE SQUARE CHAINS From the bridges we get: $D : L = L : E$ (from A, B) $E : O = O : F$ (from B, C) So we have two equal ratios.	7 OBTAIN THE CUBE CHAINS From the bridges we get: $G : M = M : N = N : H$ (from A, B) $H : P = P : Q = Q : K$ (from B, C) So we have two equal ratios.
8 USE THE ORIGINAL RATIO Since $A : B = B : C$, the ratio of the first pair of chains (from A to B) is the same as the ratio of the second pair (from B to C). $A : B = B : C$	9 PASS BY EQUALITY (SQUARES) The chains D-L-E and E-O-F have the same ratio and the same length. Therefore, by equality from first to last, $D : E = E : F$ So the squares are in continued proportion. $A^2 : B^2 = B^2 : C^2$	10 PASS BY EQUALITY (CUBES) The chains G-M-N-H and H-P-Q-K have the same ratio and the same length. Therefore, by equality from first to last, $G : H = H : K$ So the cubes are in continued proportion. $A^3 : B^3 = B^3 : C^3$	11 THE PATTERN CONTINUES If we multiply each original number by itself again, and again, the pattern continues in the same way.	12 GENERAL RESULT For any power n, if $A : B = B : C$, then $A^n : B^n = B^n : C^n$ for $n = 2, 3, 4, \dots$ The ratio remains equal at every level.	13 VISUAL SUMMARY Each row is built from the one above by multiplying each term by itself. Original numbers: A, B, C Squares: A^2, B^2, C^2 Cubes: A^3, B^3, C^3 and so on...	14 CONCLUSION Hence, if numbers are in continued proportion, the numbers created from them by taking equal powers are also in continued proportion. $A^2 : B^2 = B^2 : C^2$ $A^3 : B^3 = B^3 : C^3$ $A^n : B^n = B^n : C^n$

THE WHOLE IDEA
Continuous proportion is stable under equal powers. Squares preserve it. Cubes preserve it. And the same for every higher power.

STARTING PROPORTION

$A : B = B : C$

A — B — C

SQUARE EACH NUMBER

$A^2 : B^2 = B^2 : C^2$

A² — B² — C²

CUBE EACH NUMBER

$A^3 : B^3 = B^3 : C^3$

A³ — B³ — C³

GENERAL POWER n

$A^n : B^n = B^n : C^n$

Aⁿ — Bⁿ — Cⁿ

CONTINUED PROPORTION PRESERVED AT EVERY LEVEL

Proposition 14

If a square number measures another square number, then the side of the former will also measure the side of the latter; and if the side measures the side, the former square number will measure the latter square number.

Proof.

Action 1. Let $A = C^2$ and $B = D^2$ be square numbers, and first suppose

$$A \mid B.$$

Action 2. Construct the mean proportional $E = CD$. By Prop. VIII.11,

$$A : E = E : B = C : D.$$

Action 3. Since A, E, B are continuously proportional and $A \mid B$, Proposition VIII.7 gives

$$A \mid E.$$

Action 4. Since $A : E = C : D$, the equal numerical ratio implies

$$C \mid D.$$

Action 5. Thus

$$A \mid B \implies C \mid D.$$

Action 6. Conversely suppose

$$C \mid D.$$

Action 7. Again $A : E = C : D$, so the equality of ratios gives

$$A \mid E.$$

Action 8. Since $A : E = E : B$, the same continued proportional structure carries the divisibility onward, and $A \mid B$.

Action 9. Therefore

$$A \mid B \iff C \mid D.$$

Therefore, $C^2 \mid D^2$ if and only if $C \mid D$.

EUCLID – BOOK VIII, PROPOSITION 14

IF A SQUARE NUMBER MEASURES ANOTHER SQUARE NUMBER, THEN THE SIDE OF THE FORMER WILL ALSO MEASURE THE SIDE OF THE LATTER, AND IF THE SIDE OF A SQUARE NUMBER MEASURES THE SIDE OF ANOTHER SQUARE NUMBER, THEN THE FORMER SQUARE NUMBER WILL ALSO MEASURE THE LATTER.

FIRST DIRECTION If the square A measures the square B, then side C measures the side D.	1 SET OUT THE SQUARES Let A and B be square numbers. Let C be the side of A, and D the side of B. $A = C^2$ $B = D^2$ Assume: A measures B. ($A \mid B$)	2 CONSTRUCT THE MEAN PROPORTIONAL Multiply the sides C and D. Let $E = C \times D$. By Prop. 11, A, E, B are in continued proportion. $A : E = E : B$ and their common ratio is $C : D$.	3 USE THAT A MEASURES B Since A, E, B are in continued proportion, and A measures B ($A \mid B$), by Prop. 7, A also measures E. Therefore $A \mid E$.	4 TRANSFER TO THE SIDES But $A : E = C : D$ and we have $A \mid E$. Therefore the first side measures the second. Hence $C \mid D$. Therefore, from $A \mid B$ we have $C \mid D$.	SUMMARY OF FIRST DIRECTION $A \mid B$ $\downarrow$ $A : E = C : D$ (by Prop. 7) $\downarrow$ $A \mid E$ $\downarrow$ $C \mid D$
SECOND DIRECTION If the side C measures the side D, then the square A measures the square B.	5 ASSUME THE SIDES Assume C measures D. ($C \mid D$) $A = C^2$ $B = D^2$	6 USE THE SAME CONSTRUCTION Let $E = C \times D$. By Prop. 11, A, E, B are in continued proportion. $A : E = E : B$ and $A : E = C : D$.	7 TRANSFER FROM SIDES TO NUMBERS Since $C : D = A : E$ and C measures D ($C \mid D$), therefore A measures E. Thus $A \mid E$.	8 PASS ALONG THE CONTINUED PROPORTION Since A, E, B are in continued proportion, and A measures E ($A \mid E$), by Prop. 7, A also measures B. Therefore $A \mid B$.	SUMMARY OF SECOND DIRECTION $C \mid D$ $\downarrow$ $A : E = C : D$ $\downarrow$ $A \mid E$ $\downarrow$ $A \mid B$ (by Prop. 7)

THE WHOLE IDEA
Insert the mean proportional.
 $E = C \times D$ between the squares.
A, E, B are in continued proportion with common ratio $C : D$.

KEY TOOL
Property of Continued Proportion
If the first term measures the last, then it measures every term proportional between them.
(Prop. 7)

If $C \mid D$

$\Rightarrow A \mid B$

EQUALITY OF RATIOS
 $A : E = E : B = C : D$
is the bridge that connects squares and their sides.

Proposition 15

If a cube number measures another cube number, then the side of the former will also measure the side of the latter; and if the side measures the side, the former cube number will measure the latter cube number.

Proof.

Action 1. Let $A = C^3$ and $B = D^3$ be cube numbers, and first suppose

$$A \mid B.$$

Action 2. Construct the Proposition VIII.12 chain

$$A : H = H : K = K : B = C : D,$$

where $H = C^2D$ and $K = CD^2$.

Action 3. Since A, H, K, B are continuously proportional and $A \mid B$, Proposition VIII.7 gives

$$A \mid H.$$

Action 4. Because $A : H = C : D$, it follows that

$$C \mid D.$$

Action 5. Hence

$$A \mid B \implies C \mid D.$$

Action 6. Conversely suppose

$$C \mid D.$$

Action 7. From $C : D = A : H$ we obtain

$$A \mid H.$$

Action 8. Since $A : H = H : K = K : B$, the divisibility propagates through the chain and yields

$$A \mid B.$$

Action 9. Hence

$$C \mid D \implies A \mid B.$$

Action 10. Therefore

$$A \mid B \iff C \mid D.$$

Therefore, $C^3 \mid D^3$ if and only if $C \mid D$.

EUCLID – BOOK VIII, PROPOSITION 15

IF A CUBE NUMBER MEASURES ANOTHER CUBE NUMBER, THEN THE SIDE OF THE FORMER WILL ALSO MEASURE THE SIDE OF THE LATTER, AND IF THE SIDE OF A CUBE NUMBER MEASURES THE SIDE OF ANOTHER CUBE NUMBER, THEN THE FORMER CUBE NUMBER WILL ALSO MEASURE THE LATTER.

FIRST DIRECTION
If A measures B, then C measures D.

1 SET OUT THE CUBES
Let A and B be two cube numbers.
Let C be the side of A, and D the side of B.
 $A = C^3$ $B = D^3$

Assume: A measures B.
(A | B)

2 CONSTRUCT THE SQUARES AND PRODUCTS
 $E = C^2$ ($C \times C$)
 $F = C \times D$
 $G = D^2$ ($D \times D$)

Then multiply F by C and by D.
 $H = C \times F = C^3 D$
 $K = D \times F = CD^3$

3 BY PROPOSITION 12
A, H, K, B are in continued proportion and their common ratio is C : D.
Thus
 $A : H = H : K = K : B$
 $= C : D$

4 USE THAT A MEASURES B
Since A, H, K, B are in continued proportion and A measures B (A | B), by Proposition 7, A also measures H.

5 TRANSFER TO THE SIDES
But $A : H = C : D$, and we have $A | H$. Therefore the first side measures the second.
Hence $C | D$.

From A | B we have $C | D$.

SUMMARY OF FIRST DIRECTION

SECOND DIRECTION
If C measures D, then A measures B.

7 ASSUME THE SIDES
Assume C measures D.
(C | D)

$A = C^3$ $B = D^3$

8 USE THE SAME CONSTRUCTION
 $E = C^2$, $F = C \times D$, $G = D^2$
 $H = C^3 D$, $K = CD^3$

From Proposition 12,
 $A : H = H : K = K : B = C : D$

9 TRANSFER FROM SIDES TO NUMBERS
Since $C : D = A : H$ and C measures D (C | D), therefore A measures H.

10 PASS ALONG THE CHAIN OF PROPORTION
Since A, H, K, B are in continued proportion and A measures H (A | H), by Proposition 7, A also measures B.

SUMMARY OF SECOND DIRECTION

11 CONCLUSION
Therefore:
A measures B
iff
C measures D.
 $A | B \iff C | D$
or
 $C^3 | D^3 \iff C | D$

THE WHOLE IDEA
Between the cubes C^3 and D^3 insert the two mean proportionals $C^2 D$ and $C D^2$. Divisibility passes across this continued proportion.

POWERS BUILT FROM THE SIDES
Squares and Products
 $E = C^2$ $F = C \times D$ $G = D^2$

CUBES AND TWO MEANS
 $A = C^3$ $H = C^2 D$ $K = C D^2$ $B = D^3$

KEY TOOL
Proposition 7
If in a continued proportion the first term measures the last, it measures every intermediate term.

EQUIVALENCE
 $C^3 | D^3 \iff C | D$
Cube numbers divide iff their sides divide.

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Proposition 16

If a square number does not measure another square number, then the side of the former will not measure the side of the latter either; and if the side does not measure the side, the former square number will not measure the latter square number either.

Proof.

Action 1. Let $A = C^2$ and $B = D^2$ be square numbers and suppose

$$A \nmid B.$$

Action 2. Assume, contrary to the desired conclusion, that

$$C \mid D.$$

Action 3. By Proposition VIII.14, $C \mid D$ implies

$$A \mid B.$$

Action 4. This contradicts the hypothesis $A \nmid B$.

Action 5. Therefore

$$C \nmid D.$$

Action 6. Conversely suppose

$$C \nmid D.$$

and assume that $A \mid B$.

Action 7. Proposition VIII.14 would then imply $C \mid D$, contradiction.

Action 8. Hence

$$A \nmid B.$$

Action 9. Thus

$$A \nmid B \iff C \nmid D.$$

Therefore, $C^2 \nmid D^2$ if and only if $C \nmid D$.

EUCLID – BOOK VIII, PROPOSITION 16

IF A SQUARE NUMBER DOES NOT MEASURE ANOTHER SQUARE NUMBER, THEN THE SIDE OF THE FORMER WILL NOT MEASURE THE SIDE OF THE LATTER EITHER, AND IF THE SIDE OF A SQUARE NUMBER DOES NOT MEASURE THE SIDE OF ANOTHER SQUARE NUMBER, THEN THE FORMER SQUARE NUMBER WILL NOT MEASURE THE LATTER SQUARE NUMBER EITHER.

FIRST DIRECTION: If A does not measure B, then C does not measure D.

1 SET OUT THE SQUARES

Let A and B be square numbers.
Let C be the side of A,
and D the side of B.

$A = C^2$ $B = D^2$

Assume: A does not measure B. ($A \nmid B$)

2 ASSUME THE OPPOSITE

Suppose, on the contrary,
that C measures D.

$C \mid D$

3 APPLY PROPOSITION 14

But by Proposition 14,
if C measures D,
then A measures B.

$C \mid D$
 $\downarrow$ (Prop. 14)
 $A \mid B$

4 REACH A CONTRADICTION

But we assumed A does not measure B.
Thus we have both:

$A \mid B$ and $A \nmid B$

IMPOSSIBLE!

Therefore, our assumption was false.
Hence C does not measure D.

$C \nmid D$

SECOND DIRECTION: If C does not measure D, then A does not measure B.

5 SET OUT THE SIDES

Assume: C does not measure D.

$C \nmid D$

6 ASSUME THE OPPOSITE

Suppose, on the contrary,
that A measures B.

$A = C^2$ $B = D^2$

$A \mid B$

7 APPLY PROPOSITION 14 AGAIN

But by Proposition 14,
if A measures B,
then C measures D.

$A \mid B$
 $\downarrow$ (Prop. 14)
 $C \mid D$

8 REACH CONTRADICTION

But we assumed C does not measure D.
Thus we have both:

$C \mid D$ and $C \nmid D$

IMPOSSIBLE!

Therefore, our assumption was false.
Hence A does not measure B.

$A \nmid B$

THE WHOLE RESULT

A square number and another square number stand in the same relation of divisibility as their sides.

EQUIVALENCE

$A = C^2$ $B = D^2$ $A \nmid B \iff C \nmid D$

As in all true reasoning, what is impossible cannot be, and from the impossible, truth necessarily follows. — Aristotle

KEY PRINCIPLE USED

Proposition 14:
 $A \mid B \iff C \mid D$
Therefore, by contradiction,
 $A \nmid B \iff C \nmid D$

ARISTOTLE TEACHING ALEXANDER – THE BEAUTY OF REASONING LIES IN NECESSITY.

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Proposition 17

If a cube number does not measure another cube number, then the side of the former will not measure the side of the latter either; and if the side does not measure the side, the former cube number will not measure the latter cube number either.

Proof.

Action 1. Let $A = C^3$ and $B = D^3$ be cube numbers and suppose

$$A \nmid B.$$

Action 2. Assume, contrary to the desired conclusion, that

$$C \mid D.$$

Action 3. By Proposition VIII.15, $C \mid D$ implies

$$A \mid B.$$

Action 4. This contradicts $A \nmid B$.

Action 5. Therefore

$$C \nmid D.$$

Action 6. Conversely suppose $C \nmid D$ and assume $A \mid B$.

Action 7. Proposition VIII.15 would then imply $C \mid D$, contradiction.

Action 8. Hence

$$A \nmid B.$$

Action 9. Therefore

$$A \nmid B \iff C \nmid D.$$

Therefore, $C^3 \nmid D^3$ if and only if $C \nmid D$.

EUCLID – BOOK VIII, PROPOSITION 17

IF A CUBE NUMBER DOES NOT MEASURE ANOTHER CUBE NUMBER, THEN THE SIDE OF THE FORMER WILL NOT MEASURE THE SIDE OF THE LATTER EITHER, AND IF THE SIDE OF A CUBE NUMBER DOES NOT MEASURE THE SIDE OF ANOTHER CUBE NUMBER, THEN THE FORMER CUBE NUMBER WILL NOT MEASURE THE LATTER CUBE NUMBER EITHER.

FIRST DIRECTION: IF A does not measure B, then C does not measure D.

1 SET OUT THE CUBES
Let A and B be cube numbers.
Let C be the side of A, and D the side of B.

$A = C^3$

 $C \mid A$

$B = D^3$

 $D \mid B$

Assume: A does not measure B. ($A \nmid B$)

2 ASSUME THE OPPOSITE
Suppose, on the contrary, that C measures D.

$C \mid D$


$C \mid D$

3 APPLY PROPOSITION 15
But by Proposition 15, if C measures D, then A measures B.

$C \mid D \Leftrightarrow A \mid B$
(Prop. 15)
 $A = C^3$ $B = D^3$
 $A \mid B$

4 REACH A CONTRADICTION
But we assumed A does not measure B. Thus we have both:

$A \mid B$ and $A \nmid B$
IMPOSSIBLE!

Therefore, our assumption was false. Hence C does not measure D.
 $C \nmid D$

SECOND DIRECTION: IF C does not measure D, then A does not measure B.

5 SET OUT THE SIDES
Assume: C does not measure D.

$C \nmid D$


$C \nmid D$

6 ASSUME THE OPPOSITE
Suppose, on the contrary, that A measures B.

$A \mid B$

 $A \mid B$

7 APPLY PROPOSITION 15 AGAIN
But by Proposition 15, if A measures B, then C measures D.

$A \mid B \Leftrightarrow C \mid D$
(Prop. 15)
 $C \nmid D$
 $C \mid D$

8 REACH CONTRADICTION
But we assumed C does not measure D. Thus we have both:

$C \mid D$ and $C \nmid D$
IMPOSSIBLE!

Therefore, our assumption was false. Hence A does not measure B.
 $A \nmid B$

THE WHOLE RESULT
For cube numbers and their sides, divisibility (or non-divisibility) is equivalent.

EQUIVALENCE

$A = C^3$


$B = D^3$


$A \nmid B \Leftrightarrow C \nmid D$


KEY PRINCIPLE USED
Proposition 15:
 $A \mid B \Leftrightarrow C \mid D$
From this, by contradiction, we obtain:
 $A \nmid B \Leftrightarrow C \nmid D$

"For the things that are impossible in themselves, we prove by showing that their opposites lead to what cannot be."
— Aristotle



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Proposition 18

There exists one number in mean proportion to two similar plane numbers; and one plane number has to the other a squared ratio with respect to that which a corresponding side has to a corresponding side.

Proof.

Action 1. Let A, B be similar plane numbers. Let the sides of A be C, D , and those of B be E, F :

$$A = CD, \quad B = EF.$$

Action 2. Since the plane numbers are similar,

$$C : D = E : F.$$

Action 3. Alternately (Prop. VII.13),

$$C : E = D : F.$$

Action 4. Let D multiply E and produce

$$G = DE.$$

Action 5. Since $A = CD$ and $G = DE$ share the multiplier D ,

$$A : G = C : E.$$

Action 6. Using $C : E = D : F$,

$$A : G = D : F.$$

Action 7. Since $G = DE$ and $B = EF$ share the multiplier E ,

$$G : B = D : F.$$

Action 8. Hence

$$A : G = G : B,$$

so G is one mean proportional between A and B .

Action 9. The chain is structurally

$$CD \longrightarrow DE \longrightarrow EF.$$

Action 10. Since $A : G = G : B$, the ratio $A : B$ is the squared ratio of $A : G$.

Action 11. But $A : G = C : E = D : F$.

Action 12. Therefore

$$A : B = (C : E)^2 = (D : F)^2.$$

Therefore, one mean proportional lies between similar plane numbers, and their ratio is the squared ratio of corresponding sides.

EUCLID – BOOK VIII, PROPOSITION 18

THERE EXISTS ONE NUMBER IN MEAN PROPORTION TO TWO SIMILAR PLANE NUMBERS,
AND ONE PLANE NUMBER HAS TO THE OTHER PLANE NUMBER A SQUARED RATIO
WITH RESPECT TO THAT WHICH A CORRESPONDING SIDE HAS TO A CORRESPONDING SIDE.

1 SET OUT THE TWO SIMILAR PLANE NUMBERS
Let A and B be two similar plane numbers.
Let the sides of A be C, D, and the sides of B be E, F.

$A = CD$
 $B = EF$

Thus $A = CD$, $B = EF$.

2 FROM SIMILARITY OF THE PLANE NUMBERS
Since A and B are similar, their sides are proportional.
 $C : D = E : F$

3 ALTERNATE THE PROPORTION
From $C : D = E : F$, we alternately obtain
 $C : E = D : F$

4 CONSTRUCT THE BRIDGE NUMBER
Let D multiply E.
Let $G = D \cdot E$.

This number G will lie between A and B.

5 COMPARE A WITH G
 $A = C \cdot D$ and $G = D \cdot E$.
Both contain D.
Therefore $A : G = C : E$.
But $C : E = D : F$.
Hence $A : G = D : F$.

6 COMPARE G WITH B
 $G = D \cdot E$ and $B = E \cdot F$.
Both contain E.
Therefore $G : B = D : F$.

7 THE MEAN PROPORTIONAL
From (5) and (6) we have
 $A : G = G : B = D : F$.
Therefore $A : G = G : B$.
So G is the one mean proportional between A and B.

8 FROM ONE STEP TO TWO STEPS
Since $A : G = G : B$, the same ratio $(D : F)$ is taken twice to go from A to B.
Hence $A : B$ is the square of $A : G$.

$A : B = (A : G)^2$
 $= (D : F)^2$

9 THE SQUARED RATIO OF CORRESPONDING SIDES
But $A : G = C : E$ (from step 5).
Therefore
 $A : B = (C : E)^2 = (D : F)^2$.

Thus one plane number has to the other plane number a squared ratio with respect to that which a corresponding side has to a corresponding side.

THE WHOLE IDEA
For two similar plane numbers $A = CD$ and $B = EF$ with corresponding sides proportional ($C : D = E : F$), there is one mean proportional $G = DE$ between them.

SUMMARY OF CONSTRUCTION

RESULT
 $A : G = G : B = D : F$
(One mean proportional)
 $CD \rightarrow DE \rightarrow EF$

EQUIVALENT RATIO
 $A : B = (C : E)^2 = (D : F)^2$
(Squared ratio of corresponding sides)
This generalizes Proposition 11 from squares to all similar plane numbers.

"In all things of nature there is something of the marvelous."
— Aristotle

Proposition 19

Two numbers fall between two similar solid numbers in mean proportion; and a solid number has to a similar solid number a cubed ratio with respect to that which a corresponding side has to a corresponding side.

Proof.

Action 1. Let A, B be similar solid numbers with sides

$$A = CDE, \quad B = FGH.$$

Similarity gives

$$C : D = F : G, \quad D : E = G : H.$$

Action 2. Form the plane numbers

$$K = CD, \quad L = FG.$$

Since $C : D = F : G$, K and L are similar plane numbers.

Action 3. By Proposition VIII.18 there is one mean M between K and L ; in the construction,

$$M = DF, \quad K : M = M : L.$$

Action 4. Because $K = CD$ and $M = DF$,

$$K : M = C : F.$$

Thus the plane chain has the ratio of corresponding first sides.

Action 5. Alternating the side proportions gives

$$C : F = D : G = E : H.$$

Action 6. Multiply M by E and H to form two intermediate solid numbers

$$N = EM, \quad O = HM.$$

Also $A = EK$ and $B = HL$.

Action 7. Since E multiplies K, M to give A, N ,

$$A : N = K : M = C : F.$$

Action 8. Since E, H multiply the same number M to give N, O ,

$$N : O = E : H = C : F.$$

Action 9. Since H multiplies M, L to give O, B ,

$$O : B = M : L = C : F.$$

Action 10. Therefore

$$A : N = N : O = O : B.$$

Thus N, O are two mean proportionals between A and B .

Action 11. The four-term chain contains three equal proportional steps, each equal to a corresponding-side ratio.

Action 12. Hence

$$A : B = (C : F)^3 = (D : G)^3 = (E : H)^3.$$

Therefore, two means lie between similar solid numbers, and their ratio is the cubed ratio of corresponding sides.

EUCLID – BOOK VIII, PROPOSITION 19

TWO NUMBERS FALL BETWEEN TWO SIMILAR SOLID NUMBERS IN MEAN PROPORTION, AND A SOLID NUMBER HAS TO A SIMILAR SOLID NUMBER A CUBED RATIO WITH RESPECT TO THAT WHICH A CORRESPONDING SIDE HAS TO A CORRESPONDING SIDE.

1 SET OUT THE TWO SIMILAR SOLID NUMBERS
Let A and B be similar solid numbers.
Let the sides of A be C, D, E,
and the sides of B be F, G, H.

A = CDE B = FGH

Since A and B are similar,
C : D = F : G and D : E = G : H.

2 FORM TWO SIMILAR PLANE NUMBERS
Take the first two sides of each solid.
Let K = CD and L = FG.
Since C : D = F : G, the plane numbers
K and L are similar.

K = CD L = FG

By Proposition 18, there is one mean
proportional between K and L.
Let it be M. Then K : M = M : L.
And M = DE.

M = DE

3 RELATE K : M TO CORRESPONDING SIDES
Since K = CD and M = DE,
both contain D.
Therefore K : M = C : E.

But K : M = M : L.

Hence
C : E = M : L.
C : E = D : G = E : H.

C : E = D : G = E : H

4 LIFT THE CONSTRUCTION TO THREE DIMENSIONS
Multiply the mean M successively
by the third sides to build
two intermediate solid numbers.

First, form N = FDE
(replace C by F).

Then form O = FGE
(replace D by G).

5 COMPARE THE SUCCESSIVE SOLIDS
A = CDE, N = FDE, O = FGE, B = FGH.

Compare step by step:

① A : N = C : F

② N : O = D : G

③ O : B = E : H

But, C : E = D : G = E : H.

6 CONCLUDE THE MEAN PROPORTIONALS
Therefore
A : N = N : O = O : B.
So two numbers,
N and O,
fall between A and B
in mean proportion.

Thus
A
↓
N
↓
O
↓
B

7 THREE EQUAL PROPORTIONAL STEPS
Since each step has the ratio of corresponding sides,
we have

A : N = N : O = O : B = C : F.

A = CDE N = FDE O = FGE B = FGH

Three equal proportional steps.

8 FROM THREE STEPS TO A CUBED RATIO
The ratio (C : F) is taken three times to go
from A to B.
Therefore A : B has the cubed ratio
of C : F.

A : B = (C : F)³
= (D : G)³
= (E : H)³

A : B = (C : F)³

9 THE WHOLE CHAIN
Two mean proportionals
N and O lie between
A and B.

Two mean proportionals
N and O lie between
A and B.

A : N = N : O = O : B
= C : F (= D : G = E : H)

SUMMARY OF CONSTRUCTION
K = CD, L = FG
M = DF with K : M = M : L
N = FDE (replace C by F)
O = FGE (replace D by G)
B = FGH (replace E by H)

THE RESULT
Two numbers fall between
two similar solid numbers
in mean proportion:
A, N, O, B.

And the ratio of the solids
is the cubed ratio of
corresponding sides.

A : B = (C : F)³

ARISTOTLE'S LESSON
"For Nature works
in threes in space,
as in all things
that are whole
and perfect."
— Aristotle

FROM TWO DIMENSIONS TO THREE: FROM SQUARED RATIOS TO CUBED RATIOS. THIS IS THE HARMONY OF PROPORTION.

Proposition 20

If one number falls between two numbers in mean proportion, then the numbers will be similar plane numbers.

Proof.

Action 1. Let

$$A : C = C : B.$$

Thus C is one mean proportional between A and B .

Action 2. Take the least numbers D, E having the same ratio as A, C .

Action 3. Since D, E are least, they measure A, C an equal number of times. Let that common multiplicity be F . Then

$$A = DF.$$

Thus A is a plane number with sides D, F .

Action 4. Because $A : C = C : B$, the same least ratio $D : E$ also corresponds to $C : B$.

Action 5. Let the corresponding multiplicity for B be G . Then

$$B = EG,$$

so B is a plane number with sides E, G .

Action 6. The proportional structure gives

$$D : E = A : C = C : B.$$

Action 7. Comparing the second pair of sides gives

$$F : G = C : B = D : E.$$

Action 8. Alternately,

$$D : F = E : G.$$

Thus the sides of A and B are proportional.

Action 9. Therefore A and B are similar plane numbers.

Therefore, A and B are similar plane numbers.

Scholarly note. The transmitted proof in some editions contains a defective intermediate step: one asserted product is not explicitly demonstrated. The intended proportional structure and the conclusion above are nevertheless the one transmitted by the proposition.

EUCLID – BOOK VIII, PROPOSITION 20

IF ONE NUMBER FALLS BETWEEN TWO NUMBERS IN MEAN PROPORTION,
THEN THE NUMBERS WILL BE SIMILAR PLANE NUMBERS.

1 SET OUT THE CONTINUED PROPORTION
Let the numbers A, C, B be such that
 $A : C = C : B$.
So C is the mean proportional between A and B.
To prove: A and B are similar plane numbers.

2 REDUCE TO THE LEAST RATIO
Let D, E be the least numbers having the same ratio as A and C.
 $D : E = A : C$
Since D, E are the least, they measure A, C an equal number of times.

3 CONSTRUCT A AS A PLANE NUMBER
Let F be that number of times D measures A.
Then $A = D \times F$.
Therefore A is a plane number with sides D and F.

4 USE THE SECOND EQUAL RATIO
Since $A : C = C : B$, and $D : E = A : C$, the same least numbers D, E have the same ratio with C and B.

5 CONSTRUCT B AS A PLANE NUMBER
Let G be that number of times E measures B.
Then $B = E \times G$.
Therefore B is a plane number with sides E and G.

6 COMPARE ONE PAIR OF SIDES
From the original proportion $A : C = C : B$,
 $D : E$ has the same ratio as $A : C$ (and as $C : B$).
 $D : E = A : C = C : B$

7 COMPARE THE OTHER PAIR
From the constructions we have $A = D \times F$ and $B = E \times G$.
It follows that $F : G = C : B$.

8 EQUAL RATIOS
But $C : B = D : E$ (from step 6).
Therefore $F : G = D : E$.
Hence $D : E = F : G$.
 $D : E = F : G$

9 ALTERNATE THE PROPORTION
From $D : E = F : G$, by alternation we obtain
 $D : F = E : G$
The sides of A are D and F.
The sides of B are E and G.

10 CONCLUDE
Since $D : F = E : G$, the sides of A are proportional to the sides of B.
Therefore A and B are similar plane numbers.
Q.E.D.

THE WHOLE IDEA
From one mean proportional between A and B, we build A and B as plane numbers and show their sides are proportional.

SUMMARY OF CONSTRUCTION
 $A : C = C : B$
 $D : E = A : C$
 $A = D \times F$
 $B = E \times G$

RESULT
 $A = D \times F$, $B = E \times G$
and
 $D : F = E : G$.
Therefore A and B are similar plane numbers.

KEY PRINCIPLE USED
The least numbers of a ratio measure any numbers in that ratio an equal number of times.
From two equal ratios, we obtain proportional sides.

From one mean, similarity is born.
—Aristotle

Proposition 21

If two numbers fall between two numbers in mean proportion, then the latter are similar solid numbers.

Proof.

Action 1. Let

$$A : C = C : D = D : B.$$

Thus C, D are the two mean proportionals between A and B .

Action 2. Take the least three numbers E, F, G having the same ratio as A, C, D :

$$E : F = F : G.$$

Action 3. The extremes E, G are prime to one another by Prop. VIII.3; and because one mean F lies between them, Prop. VIII.20 shows that E, G are similar plane numbers.

Action 4. Let the sides of E be H, K and those of G be L, M :

$$E = HK, \quad G = LM.$$

Then

$$H : L = K : M.$$

Action 5. From the preceding proposition, the continued ratio $E : F = F : G$ is also the ratio of corresponding sides:

$$E : F = F : G = H : L = K : M.$$

Action 6. Since E, F, G and A, C, D have the same ratio and the same number of terms,

$$E : G = A : D.$$

Action 7. Because E, G are prime to one another, they are the least numbers in their ratio and measure A, D equally many times. Let that multiplicity be N . Then

$$A = NE = N(HK) = HKN.$$

Thus A is a solid number with sides H, K, N .

Action 8. Applying the same least-number reasoning to the triple C, D, B , let the corresponding multiplicity be O . Then

$$B = OG = O(LM) = LMO.$$

Thus B is a solid number with sides L, M, O .

Action 9. We have therefore constructed two solid numbers

$$A = HKN, \quad B = LMO.$$

Action 10. Since N, O multiply the corresponding least number to produce the corresponding terms of the original continued proportion,

$$N : O = A : C.$$

Action 11. But

$$A : C = E : F = H : L = K : M.$$

Hence

$$N : O = H : L = K : M.$$

Action 12. Therefore all three pairs of corresponding sides are proportional.

Action 13. Hence A and B are similar solid numbers.

Therefore, A and B are similar solid numbers.

EUCLID – BOOK VIII, PROPOSITION 21

IF TWO NUMBERS FALL BETWEEN TWO NUMBERS IN MEAN PROPORTION,
THEN THE LATTER ARE SIMILAR SOLID NUMBERS.

1 SET OUT THE CONTINUED PROPORTION
Let A, C, D, B be numbers such that
 $A : C = C : D = D : B$
So C and D are the two mean proportionals between A and B.

2 TAKE THE LEAST EQUIVALENT TRIPLE
Let E, F, G be the least three numbers having the same ratio as A, C, D.
 $E : F = F : G$
Therefore E and G are prime to one another.

3 E AND G ARE SIMILAR PLANE NUMBERS
Since one mean proportional F lies between E and G,
 $E : F = F : G$
by Proposition 20, E and G are similar plane numbers.

4 GIVE SIDES TO E AND G
Let the sides of E be H, K and the sides of G be L, M.
 $E = H \times K$
 $G = L \times M$
Since E = G, their sides are proportional:
 $H : L = K : M$

5 THE RATIO OF THE THREE TERMS
From the continued proportion, we also have
 $E : F = F : G = H : L = K : M$

6 COMPARE THE EXTREMES
Since E, F, G have the same ratio as A, C, D, and both triples have the same number of terms,
 $E : G = A : D$
But E and G are prime to one another.

7 SCALE E TO OBTAIN A
Since $E : G = A : D$ and $E \perp G$, the least number E measures A as many times as G measures D.
Let N be that number of times.
 $A = N \times E$
But $E = H \times K$
Therefore
 $A = N \times H \times K$
So A is a solid number with sides H, K, N.

8 SCALE G TO OBTAIN B
From the continued proportion, E measures C as many times as G measures B.
Let O be that number of times.
 $B = O \times G$
But $G = L \times M$
Therefore
 $B = O \times L \times M$
So B is a solid number with sides L, M, O.

9 COMPARE THE THIRD SIDES
Since E measures A N times, and also measures C N times,
 $N : O = A : C$
But $A : C = E : F$ and $E : F = H : L = K : M$.
Therefore
 $N : O = H : L = K : M$
Thus the three pairs of sides are in the same ratio.

10 CONCLUDE
We have shown that the three corresponding sides of the solids A and B are proportional:
 $H : L = K : M = N : O$
Therefore A and B are similar solid numbers.
 $A = HKN$ $B = LMO$

THE WHOLE IDEA
From a continued proportion of four numbers
 $A : C = C : D = D : B$, we build the least triple $E : F = F : G$; then give sides to E and G, and scale them up to form the solids A and B.

SUMMARY OF CONSTRUCTION
 $A : C = C : D = D : B$
Least triple
 $E : F = F : G$
 $E = H \times K$
 $G = L \times M$
 $A = N \times E$
 $B = O \times G$
 $H : L = K : M = N : O$
so A and B are similar solid numbers.

THE RESULT
 $H : L = K : M = N : O$
so A and B are similar solid numbers.

"In proportion there is harmony; in harmony, the order of nature."
—Aristotle

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.

Proof.

Action 1. Let

$$A : B = B : C.$$

Action 2. Suppose that A is a square number.

Action 3. Because B is one mean proportional between A and C , Proposition VIII.20 shows that A and C are similar plane numbers.

Action 4. A square number is a plane number whose two sides are equal; hence the ratio of the two sides of A is $1 : 1$.

Action 5. Since similar plane numbers have proportional corresponding sides, the two sides of C must likewise be equal.

Action 6. Therefore C is a square number.

Therefore, C is square.

*In proportion
there is order,
and from one truth
another follows.*
— Aristotle

EUCLID'S ELEMENTS – BOOK VIII

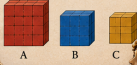
PROPOSITION 22

If three numbers are continuously proportional,
and the first is square, then the third will also be square.

So, my boy,
if the first number
is a square, and the
numbers are in
continued proportion,
the last is also a square.

1 Set out the continued proportion

Let A, B, C be three
continuously proportional numbers.

$A : B = B : C$


A B C

Each number bears the same ratio to the next.

2 Assume the first is square

Let the first number A be a square.
Then A can be expressed as
the product of two equal numbers.

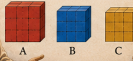
$A = a^2$


a

A is a square number
with equal sides.

3 Identify the mean proportional

Since $A : B = B : C$,
the number B is a mean proportional
between A and C.

$A : B = B : C$


A B C

B is a mean proportional
between A and C.

4 Apply Proposition 20

By Proposition 20, if one mean
proportional lies between two numbers,
then the two numbers are similar
plane numbers.



A and C are similar plane numbers.

Therefore A and C are similar plane numbers.

5 Use the fact that A is square

A square number is a plane number
whose two sides are equal.



a a

The two sides of A are equal.

6 Transfer equality of sides through similarity

Since A and C are similar plane numbers,
their corresponding sides are proportional.
But the two sides of A are equal,
so the two sides of C must also be equal.



a c

Hence the two sides of C are equal.

7 Conclude

Therefore C is a square number.

$C = c^2$


c c

Thus, if three numbers are continuously proportional,
and the first is square, the third will also be square.

"From proportion greater things are revealed."
— Aristotle



*Numbers, like forms, have harmony.
What begins in equality ends in equality.*

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.

Proof.

Action 1. Let

$$A : B = B : C = C : D.$$

Action 2. Suppose that A is a cube number.

Action 3. The numbers B, C are the two mean proportionals between A and D .

Action 4. By Proposition VIII.21, A and D are similar solid numbers.

Action 5. Since A is cube, its three sides are equal.

Action 6. Similarity of the solid numbers makes the three corresponding sides of D proportional to three equal sides.

Action 7. Therefore the three sides of D are equal.

Action 8. Hence D is a cube number.

Therefore, D is cube.

IF FOUR NUMBERS ARE CONTINUOUSLY PROPORTIONAL,
AND THE FIRST IS CUBE, THEN THE FOURTH WILL ALSO BE CUBE.

*"In proportion there is order;
in order there is harmony;
in harmony, there is beauty."
— Aristotle*

1. SET OUT THE CORRESPONDING REZANDER

Let the four numbers
A, B, C, D be such that

$$A : B :: B : C :: C : D$$

So B and C are the two
mean proportionals
between
A and D.

2. MARK THE FIRST NUMBER AS A CUBE

We are given that
the first number A is cube.
So there exists a number x
such that

$$A = x^3$$

3. RECOGNIZE THE TWO MEAN PROPORTIONALS AS A CUBE

Since $A : B :: B : C :: C : D$,
the numbers A and C
lie between A and D
in mean proportion.
B and C are
the two mean
proportionals.

4. APPLY PROPORTION 21

By Proposition 21,
if two mean numbers
lie between two numbers,
then those numbers
are similar solid numbers.

Therefore
A and D are
similar solid numbers.

5. WHAT IT MEANS THAT A IS A CUBE

A being a cube has three
equal sides.
Let the side of A be x.

So the three sides
of A are
 x, x, x .

6. USE SIMILARITY OF THE SOLID NUMBERS

Since A and D are similar
solid numbers,
their corresponding sides
are proportional.

$x : x$	$x : x$	$x : x$
$x : x$	$x : x$	$x : x$
$x : x$	$x : x$	$x : x$

where y, z, w are
the corresponding sides
of D.

7. CONCLUDE THE CORRESPONDING SIDES ARE EQUAL

Since $x : x = 1 : 1$,
we must have

$$\frac{y}{z} = \frac{z}{w}$$

Therefore
 $y = z = w$.

All three sides of D
are equal.

8. THEREFORE D IS A CUBE

Let the equal side of D
be y .

Then
 $D = y^3$

9. THE WHOLE ARGUMENT

From
 $A : B :: B : C :: C : D$
with $A = x^3$,
we have:

B and C are two means
between A and D
By Proposition 21,
A and D are similar
solid numbers
A is cube (x, x, x)
Therefore D has equal
corresponding sides
Therefore D is cube (y, y, y)
Q.E.D.

10. CONCLUSION

Since the fourth number D
is a cube,
there exists a number y
such that

$$D = y^3$$

Therefore
The result is also, cube.
Q.E.D.

THE BEAUTY OF PROPORTION

From a single cube,
through harmony of ratios,
another cube is born.
So does order give birth
to form.

THE PATTERN

3 terms continued proportion
= first square → third square

THE WHOLE ARGUMENT

4 terms continued proportion
= first cube → fourth cube

SUMMARY IN ONE LINE

If four numbers are in continued proportion and the first is cube,
then the last is cube.

ARISTOTLE'S LESSON

Always observe the order of the means,
and the nature of numbers will reveal itself.

Proposition 24

If two numbers have to one another the ratio which a square number has to another square number, and the first is square, then the second will also be square.

Proof.

Action 1. Let

$$A : B = C : D,$$

where C, D are square numbers, and suppose that A is square.

Action 2. Since C, D are squares, they are similar plane numbers.

Action 3. By Proposition VIII.18 one mean proportional lies between C and D ; let it be M :

$$C : M = M : D.$$

Action 4. Return to the given equality

$$A : B = C : D.$$

Action 5. By Proposition VIII.8, the same number of mean proportionals lies between any other pair having the same outer ratio. Thus one mean lies between A, B .

Action 6. Let that mean be N :

$$A : N = N : B.$$

Action 7. Since A is square and A, N, B are three continuously proportional numbers, Proposition VIII.22 applies.

Action 8. Therefore B is square.

Therefore, B is square.

ARISTOTLE:
Observe how properties travel through proportion, just as harmony travels through music.

EUCLID – BOOK VIII, PROPOSITION 24

IF TWO NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO ANOTHER SQUARE NUMBER, AND THE FIRST IS SQUARE, THEN THE SECOND WILL ALSO BE SQUARE.

In mathematics, as in life, order preserves itself through right relationships.

1 SET OUT THE EQUAL RATIOS

Let A, B be two numbers and C, D be two square numbers such that

$$A : B = C : D$$

A

 C

B

 D

Given: C and D are square. Also, A is square.

2 TWO SQUARES ARE SIMILAR PLANE NUMBERS

Let the sides of C be x, x and the sides of D be y, y.

$C = x^2$

 x

$D = y^2$

 y

That the ratios of their sides are $x : x = y : y = 1 : 1$.

Therefore C and D are similar plane numbers.

3 APPLY PROPOSITION 18

Since C and D are similar plane numbers, one mean proportional lies between them.

$$C : M = M : D$$

So there exists a number M such that

4 RETURN TO THE EQUAL RATIO

But we are given that

$$A : B = C : D$$

Thus A and B have the same ratio as C and D.

5 APPLY PROPOSITION 8

Since one mean proportional lies between C and D, the same number of mean proportionals will lie between any other two numbers having the same ratio.

6 THERE EXISTS N BETWEEN A AND B

So there exists a number N such that

$$A : N = N : B$$

A

 A

N

 N

B

 B

Hence A, N, B are three numbers continuously proportional.

7 RECALL: A IS SQUARE

We are given that A is square.

Let the side of A be x.

$$A = x^2$$

x

8 APPLY PROPOSITION 22

If three numbers are continuously proportional and the first is square, then the third is square.

$$A : N = N : B$$

A is square

Therefore B is square.

9 UNDERSTAND THE CHAIN

C and D are squares

C and D are similar plane numbers

By Prop. 18, one mean M lies between C and D.

By Prop. 8, one mean N lies between A and B.

By Prop. 22, since A is square, B is square.

10 CONCLUDE: B IS SQUARE

Therefore B has two equal sides.

Let the side of B be y.

$$B = y^2$$

y

11 THE WHOLE PICTURE

Given

$$A : B = C : D$$

C and D are squares

A is square

Then

C

 C

M

 M

D

 D

By Prop. 18

A

 A

N

 N

B

 B

By Prop. 8

By Prop. 22

B is square

12 THE BEAUTY OF PROPORTION

Thus from the ratio of squares and the square of the first, the square nature is carried to the second.

IN SYMBOLS

$$A : B = C : D, \quad C = x^2, D = y^2, \quad A = x^2 \Rightarrow B = y^2$$

WHAT WE USED

Prop. 18
Similar plane numbers $\Rightarrow$ one mean proportional.

Prop. 8
Equal ratios carry the same number of means.

Prop. 22
First square in three-term proportion $\Rightarrow$ third is square.

ARISTOTLE'S WORDS TO ALEXANDER

So, my boy, just as the lyre keeps the same harmony though the strings change, as proportion keeps the same nature though the numbers change.

Proposition 25

If two numbers have to one another the ratio which a cube number has to another cube number, and the first is cube, then the second will also be cube.

Proof.

Action 1. Let

$$A : B = C : D,$$

where C, D are cube numbers, and suppose that A is cube.

Action 2. Since C, D are cubes, they are similar solid numbers.

Action 3. By Proposition VIII.19 two mean proportionals lie between C, D ; let them be M, N :

$$C : M = M : N = N : D.$$

Action 4. Since $A : B = C : D$, Proposition VIII.8 transfers the same number of means to A, B .

Action 5. Let the two transferred means be E, F :

$$A : E = E : F = F : B.$$

Action 6. Thus A, E, F, B are four continuously proportional numbers and the first, A , is cube.

Action 7. By Proposition VIII.23, the fourth term B is therefore cube.

Therefore, B is cube.

EUCLID – BOOK VIII, PROPOSITION 25

IF TWO NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A CUBE NUMBER HAS TO ANOTHER CUBE NUMBER, AND THE FIRST IS CUBE, THEN THE SECOND WILL ALSO BE CUBE.

"In mathematics, as in life, harmony in ratios reveals hidden truths."
— Aristotle

"So it is, master."
— Alexander, aged 12

1 SET OUT THE EQUAL RATIOS

Let A, B be two numbers and C, D two cube numbers such that:

$$A : B = C : D$$

Also, A is cube.

2 TWO CUBES ARE SIMILAR SOLID NUMBERS

Let $C = x^3$ and $D = y^3$. Their three sides are equal:

Thus their corresponding sides are proportional:

$$x : y = x : y = x : y$$

Therefore C and D are similar solid numbers.

3 APPLY PROPOSITION 19

Since C and D are similar solid numbers, by Proposition 19, two mean proportionals lie between them.

So there exist numbers M and N such that:

$$C : M = M : N = N : D$$

4 RETURN TO THE EQUAL RATIO

But we are given that:

$$A : B = C : D$$

So A and B have the same ratio as C and D.

5 APPLY PROPOSITION 8

Since two mean proportionals lie between C and D, by Proposition 8, the same two mean proportionals will lie between any other two numbers having the same ratio.

Therefore there exist numbers E and F such that:

$$A : E = E : F = F : B$$

6 FOUR NUMBERS IN CONTINUED PROPORTION

Thus we have four numbers in continued proportion:

$$A : E = E : F = F : B$$

And we are given that A is cube.

7 RECALL PROPOSITION 23

Proposition 23 states: If four numbers are continually proportional, and the first is cube, then the fourth is cube.

$A : E = E : F = F : B$
A is cube
 $\Rightarrow$
B is cube.

8 APPLY PROPOSITION 23

Here we have exactly that situation:

$$A : E = E : F = F : B$$

and A is cube.

Therefore, by Proposition 23,

B is cube.

Thus the second number is also cube.

9 CONCLUSION

Therefore B is cube.

Let the side of B be b.

$$B = b^3$$

10 THE WHOLE ARGUMENT AT A GLANCE

C and D are cubes.

By Prop. 19, two means M, N lie between C and D.

By Prop. 8, the same two means E, F lie between A and B.

By Prop. 23, since A is cube, the fourth number B is cube.

11 THE CHAIN OF REASONING

Between cubes C and D:

Same ratio transferred to A and B:

Cube nature passes from first to fourth.

12 THE BEAUTY OF PROPORTION

As harmony in music is carried from note to note, so harmony in numbers carries the nature of squares and cubes from one to another.

IN SYMBOLS

Let $C = x^3$, $D = y^3$, $A = a^3$.

$$A : B = C : D = x^3 : y^3$$

$\Rightarrow B = b^3$

WHAT WE USED

Prop. 8

Similar solid numbers $\Rightarrow$ two mean proportionals exist.

WHAT WE USED

Prop. 8

Same number of mean proportionals transfers with the same ratio.

WHAT WE USED

Prop. 23

Four numbers in continued proportion, first cube $\Rightarrow$ fourth cube.

ARISTOTLE'S WORDS TO ALEXANDER

"Observe, my boy, how the order of means binds numbers together in a chain of harmony. From the first we may know the last."

Proposition 26

Similar plane numbers have to one another the ratio which some square number has to another square number.

Proof.

Action 1. Let A, B be similar plane numbers.

Action 2. By Proposition VIII.18 one mean proportional C lies between them:

$$A : C = C : B.$$

Action 3. Take the least three numbers D, E, F having the same continued ratio as A, C, B :

$$D : E = E : F.$$

Action 4. By the corollary to Proposition VIII.2, in a least three-term continued proportion the outermost numbers are squares. Hence D and F are square numbers.

Action 5. Because the least chain and the original chain have corresponding equal ratios and the same number of terms, their first-to-last ratios are equal:

$$A : B = D : F.$$

Action 6. Since D, F are squares, $A : B$ is the ratio of one square number to another square number.

Therefore, $A : B = \text{square} : \text{square}.$

EUCLID – BOOK VIII, PROPOSITION 26

SIMILAR PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH SOME SQUARE NUMBER HAS TO ANOTHER SQUARE NUMBER.

"Numbers are the language by which the universe speaks."
— Aristotle

1 SET OUT THE SIMILAR PLANE NUMBERS

Let A and B be similar plane numbers. Our goal is to show that A : B is the ratio of two square numbers.

~

A B

2 ONE MEAN PROPORTIONAL LIES BETWEEN THEM

By Proposition 18, since A and B are similar plane numbers, there is one mean proportional between them.

→→

A C B

$A : C = C : B$

3 REPLACE THE CHAIN BY ITS LEAST EQUIVALENT

Take the least three numbers D, E, F having the same continued ratio as A, C, B.

$D : E = E : F$

→→

D E F

4 THE EXTREMES OF A LEAST CHAIN ARE SQUARE NUMBERS

By Proposition 2, the outermost numbers of a least three-term continued proportion are squares.

Hence D is square and F is square.

$D = \square$ $F = \square$

5 COMPARE THE OUTER RATIOS

The original chain is $A : C = C : B : E$, and the least equivalent chain is $D : E = E : F$. Since the corresponding successive ratios are equal and the chains have the same number of terms, their first-to-last ratios are equal.

$A : B = D : F$

6 IDENTIFY THE RATIO AS A RATIO OF SQUARE NUMBERS

But D and F are square numbers. Therefore A : B is the same ratio as a square number to a square number.

$A : B = \text{square} : \text{square}$

7 THE CONCLUSION

Thus, the ratio of two similar plane numbers is a ratio of a square number to a square number.

$A : B = \text{square} : \text{square}$

8 THE BIG PICTURE

Similarity of plane numbers
→ one mean proportional
→ least three-term chain
→ square extremes
→ ratio of squares.

So similar plane numbers have to one another the ratio which some square number has to another square number.

IN SYMBOLS

$A \sim B$
 $\therefore A : B = D : F$,
with D and F squares.

WHAT WE USED

Prop. 18: one mean proportional between similar plane numbers.
Prop. 2: extremes of a least three-term chain are squares.

LOGIC IN A NUTSHELL

Similarity preserves the structure of ratios. A least chain with square ends reveals a square ratio.

ARISTOTLE'S LESSON TO ALEXANDER

"When two shapes are alike, their proportions are not arbitrary — they follow the rule of squares. Thus, from similarity, the language of number reveals the harmony of the universe."

66

Proposition 27

Similar solid numbers have to one another the ratio which some cube number has to another cube number.

Proof.

Action 1. Let A, B be similar solid numbers.

Action 2. By Proposition VIII.19 two mean proportionals C, D lie between them:

$$A : C = C : D = D : B.$$

Action 3. Take the least four numbers E, F, G, H having the same continued ratio as A, C, D, B :

$$E : F = F : G = G : H.$$

Action 4. By the corollary to Proposition VIII.2, the outermost numbers of a least four-term continued proportion are cubes. Hence E and H are cube numbers.

Action 5. The two four-term chains have corresponding equal ratios, so their first-to-last ratios are equal:

$$A : B = E : H.$$

Action 6. Since E, H are cubes, $A : B$ is the ratio of one cube number to another cube number.

Therefore, $A : B = \text{cube} : \text{cube}.$

EUCLID – BOOK VIII, PROPOSITION 27

SIMILAR SOLID NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH SOME CUBE NUMBER HAS TO ANOTHER CUBE NUMBER.

So it is, my boy, harmony in solid figures sings in the language of cubes.

From similarity of solids we learn that their volumes stand in the very ratio of cube to cube.

1 SET OUT THE GIVEN

Let A and B be similar solid numbers. We must show that the ratio of A to B is the ratio of two cube numbers.

2 APPLY PROPOSITION 19

Since A and B are similar solid numbers, by Proposition 19, two mean proportionals lie between them.

$$A : C = C : D = D : B$$

So there exist numbers C and D such that A, C, D, B are in continued proportion.

3 TAKE THE LEAST EQUIVALENT CHAIN

Take the least four numbers E, F, G, H having the same continued ratio as A, C, D, B.

$$E : F = F : G = G : H$$

This is the least four-term continued proportion with that ratio.

4 WHAT PROPOSITION 2 SAYS

By the corollary to Proposition 2, in the least four-term continued proportion, the first and the last are cube numbers.

Therefore E is cube and H is cube.

5 COMPARE THE OUTER RATIOS

Since the two chains (A, C, D, B) and (E, F, G, H) have the same continued ratio, their first to last ratios are equal.

$$A : B = E : H$$

Hence $A : B = E : H$.

6 CONCLUDE THE RATIO

But E and H are cube numbers. Therefore the ratio of A to B is the same as the ratio of two cube numbers.

$$A : B = E : H$$

where E and H are cube numbers.

Thus A : B is a ratio which some cube number has to another cube number.

7 THE WHOLE ARGUMENT AT A GLANCE

A and B are similar solid numbers.

By Prop. 19, two means C, D lie between A and B.

Take least chain E, F, G, H with same ratio.

By Prop. 2, E and H are cube numbers.

Therefore $A : B = E : H$, a ratio of cubes.

8 IN SYMBOLS

Similar solid numbers A and B

→

By Prop. 19, $A : C = C : D = D : B$

→

Least chain $E : F = F : G = G : H$ (Prop. 2 ⇒ E, H cubes)

→

Same ratio overall $A : B = E : H$ (as cube : cube)

Q.E.D.

9 A NUMERICAL EXAMPLE

Let the value of A be in ratio 2 : 3 : 4 and of B be in ratio 4 : 6 : 8.

Volumes: $A = 2^3 = 8$, $B = 4^3 = 64$

Take least chain in ratio 2 : 3 : 4 : 6

Outer terms 8 and 216 are cubes. And $8 : 64 = 8 : 216 / 3 = 1 : 8$ (which is $1^3 : 2^3$).

10 ANOTHER VIEW

If all corresponding sides of A are k times those of B (same scale factor k), then

$$A = k^3 B$$

Hence $A : B = k^3 : 1 = k^3 : 1$

$= k^3 : 1^3$

a ratio of cubes.

11 WHY IT WORKS

Similarity preserves the ratios of corresponding edges. In three dimensions, volume grows with the cube of the edge length. Therefore the ratio of volumes of similar solids is always a ratio of cubes.

12 THE BEAUTY OF ORDER

From shapes to solids, from squares to cubes—proportion weaves the hidden harmony of the universe.

Proposition 19

Similar solid numbers ⇒ two mean proportionals.

WHAT WE USED

Proposition 2 (Corollary)

In the least four-term chain, first and last are cubes.

Equality of Ratios

Equal continued ratios with same number of terms have equal outer ratios.

ARISTOTLE'S WORDS TO ALEXANDER

"When figures change, the order remains. What is true of one scale is true of every scale. Thus does reason lead us to the eternal."

Closing Note

Book VIII begins with the structure of continued numerical proportion and the least representatives of such proportions. It then develops the arithmetic of mean proportionals, square and cube numbers, divisibility and non-divisibility of powers, and finally the numerical characterization of similar plane and solid numbers. The closing propositions show that similar plane numbers stand in a ratio of squares, while similar solid numbers stand in a ratio of cubes. In this way the book binds numerical proportion, dimensional form, and powers into a single arithmetic theory.