

EUCLID'S ELEMENTS

BOOK VII

Arithmetic and the Theory of Numbers

22 Definitions and 39 Propositions

CONTENTS

BOOK VII

Elementary Number Theory

Definitions

Definition 1 — Unit

A unit is that according to which each existing thing is said to be one.

EUCLID, BOOK VII — DEFINITION 1: UNIT

"A unit is that according to which each existing thing is said to be one."

1



ARISTOTLE: Alexander, before we speak of numbers, we must understand the very idea of "one."

2



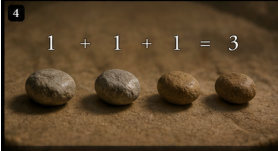
Behold this single stone.
It is **ONE**.

3



This "one" is what we call a **UNIT**—that according to which each existing thing is said to be one."

4



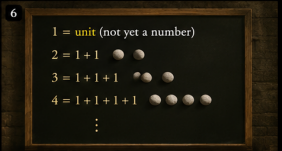
When we put units together, we can count:
three stones = $1 + 1 + 1$.

5



See how each stone is first taken as one, because of the **unit**.

6



$1 = \text{unit}$ (not yet a number)
 $2 = 1 + 1$
 $3 = 1 + 1 + 1$
 $4 = 1 + 1 + 1 + 1$
 $\vdots$

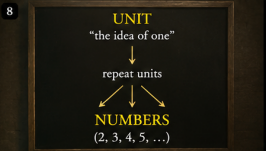
From repeated units, **numbers** arise:
2, 3, 4, and so on.

7



If we have apples, each apple is taken as one **unit**—one **unit**—relative to the collection.

8



UNIT
"the idea of one"
↓
repeat units
↙ ↘
NUMBERS
(2, 3, 4, 5, ...)

Thus the order is: first the **unit**, then, by repeating it, **numbers**.

9



Remember this well, Alexander: a **unit** gives the meaning of one, and **numbers** are made of units.

For Euclid, "number" means a positive integer greater than one. $1 = \text{unit}$ (not a number) $2 = 1 + 1$ $3 = 1 + 1 + 1$ $4 = 1 + 1 + 1 + 1$...

Definition 2 — Number

A number is a multitude composed of units.

EUCLID, BOOK VII — DEFINITION 2: NUMBER

"A number is a multitude composed of units."

1

Alexander, we begin with one.



This single stone is a **UNIT**. It is **ONE**.

2

Add another unit.



Now we have $1 + 1$. This multitude of units is called **2**.

3

Add another.



Now we have $1 + 1 + 1$. This multitude of units is called **3**.

4

Add one more.



$1 + 1 + 1 + 1 = 4$. A **number** is many units taken together.

5

And another.



$1 + 1 + 1 + 1 + 1 = 5$. And so on, without end...

6

$$\begin{aligned} 2 &= 1 + 1 \\ 3 &= 1 + 1 + 1 \\ 4 &= 1 + 1 + 1 + 1 \\ 5 &= 1 + 1 + 1 + 1 + 1 \\ &\dots \end{aligned}$$
$$n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$$

Every **number** is made by repeating the **unit**.

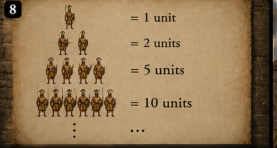
7

Look at our soldiers.



One soldier is a **unit**. Many soldiers together give us a **number**.

8

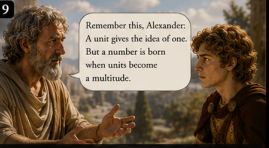


= 1 unit
= 2 units
= 5 units
= 10 units
...

Whatever we count—stones, soldiers, apples—each is counted as one unit.

9

Remember this, Alexander: A unit gives the idea of one. But a number is born when units become a multitude.



A NUMBER IS A MULTITUDE COMPOSED OF UNITS.

UNIT = ONE
(not yet a number)

→

NUMBER = MANY ONES TOGETHER
(a multitude composed of units)

5

Definition 3 — Part

A number is part of another number, the lesser of the greater, when it measures the greater.

EUCLID, BOOK VII — DEFINITION 3: PART

"A number is part of another number, the lesser of the greater, when it measures the greater."

1



Alexander, today we learn what it means for one number to be a **PART** of another.

Euclid says: "A number is **part** of another number, the lesser of the greater, when it **measures** the greater."

2

Suppose we have **3** and **12**.



Can the smaller number, **3**, fit into **12** exactly?

3

Yes!



$12 = 3 + 3 + 3 + 3$.
The number **3** **measures** **12** exactly four times.

4

Therefore, **3** is a **PART** of **12**.



More precisely, **3** is the fourth part of **12**.
 $3 = \frac{1}{4} 12$.

5

Another example.

$10 = 2 + 2 + 2 + 2 + 2$



2 **measures** **10** exactly five times.
So **2** is a **part** of **10**. ($2 = \frac{1}{5} 10$)

6

Another example.

$12 = 4 + 4 + 4$



4 **measures** **12** exactly three times.
So **4** is a **part** of **12**. ($4 = \frac{1}{3} 12$)

7

But look at **5** and **12**.

$12 = 5 + 5 = 10$

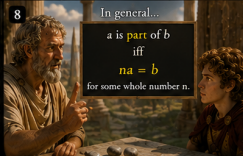


5 does **NOT** measure **12** exactly.
There are **2** left over.
So **5** is **NOT** a **part** of **12**.

8

In general...

a is **part** of b
iff
 $na = b$
for some whole number n .



The smaller number fits into the greater an **exact** whole number of times.

9

Think of soldiers, Alexander.



If a group occurs the same number of times in the whole army, then that group is a **PART** of the whole.



Remember:
A "part" fits into the whole **EXACTLY**.

Examples:
2 is a part of **6** • **3** is a part of **12** • **5** is a part of **20**

7

Definition 4 — Parts

The lesser is parts of the greater when it does not measure it.

EUCLID, BOOK VII — DEFINITION 4: PARTS

"But the lesser is PARTS of the greater when it does not measure it."

1



So far, Alexander, we learned what it means for a number to be a **PART** of another.

A number is a **PART** when it **MEASURES** the greater exactly.

2



Now we learn a different case.

Sometimes the smaller number does **NOT** measure the greater exactly.

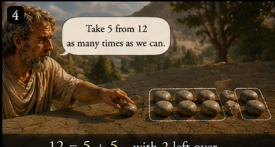
3



Consider 5 and 12.

Can 5 measure 12 exactly?

4



Take 5 from 12 as many times as we can.

$12 = 5 + 5$, with 2 left over. It does **NOT** measure exactly.

5



So 5 is not a **PART** of 12.

But it is contained in 12 as many times as possible, with something left over.

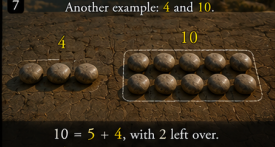
6



That is what we call **PARTS**.

The lesser is **PARTS** of the greater when it does not measure it.

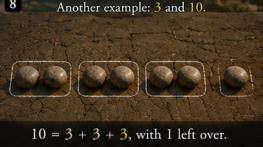
7



Another example: 4 and 10.

$10 = 5 + 4$, with 2 left over. So 4 is **PARTS** of 10, not a **PART**.

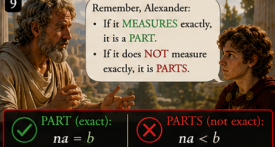
8



Another example: 3 and 10.

$10 = 3 + 3 + 3$, with 1 left over. So 3 is **PARTS** of 10.

9



Remember, Alexander:

- If it **MEASURES** exactly, it is a **PART**.
- If it does **NOT** measure exactly, it is **PARTS**.

✓ **PART** (exact):
 $na = b$
(no remainder)

✗ **PARTS** (not exact):
 $na < b$
(remainder left over)



DEFINITION 4:

The lesser is **PARTS** of the greater when it does not measure it.

EXAMPLES

5 is **PARTS** of 12
($12 = 5 + 5 + 2$)

4 is **PARTS** of 10
($10 = 4 + 4 + 2$)

3 is **PARTS** of 10
($10 = 3 + 3 + 3 + 1$)

Definition 5 — Multiple

The greater number is a multiple of the lesser when it is measured by the lesser.

"The greater number is a **multiple** of the lesser when it is measured by the lesser."

"The greater number is a **multiple** of the lesser when it is measured by the lesser."

1 Alexander, when one number measures another exactly, the greater number is called a **MULTIPLE** of the lesser.

We have learned: if 3 measures 12, then 3 is a **PART** of 12.

2 Let us see.
Here are 3. Here are 12.

Can 3 measure 12 exactly?
Yes!

3 $12 = 3 + 3 + 3 + 3$.

The number 3 **measures** 12 exactly four times.

4 Therefore...

3 is a **PART** of 12 (the fourth part).
But also...

5 The greater number, 12, is a **MULTIPLE** of the lesser, 3.

12 is a multiple of 3.

6 In general:
If the lesser number is repeated n times and makes the greater number, then the greater number is a multiple of the lesser.

If $b = n \times a$ (for some whole number n), then b is a **multiple** of a .

7 More examples.

2 measures 10.
 $10 = 2 + 2 + 2 + 2 + 2$

10 is a **multiple** of 2.

4 measures 16.
 $16 = 4 + 4 + 4 + 4$

16 is a **multiple** of 4.

8 Another example.

5 measures 20.
 $20 = 5 + 5 + 5 + 5$

20 is a **multiple** of 5.

9 But look here.

5 does **NOT** measure 12.

There is something left over.
So 12 is **NOT** a **multiple** of 5.

Remember:
The greater number is a **MULTIPLE** of the lesser when it is measured by the lesser.

Summary:

- If a measures b exactly $\rightarrow a$ is a **PART** of b .
- If a measures b exactly $\rightarrow b$ is a **MULTIPLE** of a .

$a \mid b$

(a divides b)

$\rightarrow a$ is a **PART** of b

$\rightarrow b$ is a **MULTIPLE** of a

Definition 6 — Even Number

An even number is one which can be divided in half.

EUCLID, BOOK VII — DEFINITION 6: EVEN NUMBER

"An even number is one which can be divided in half."

1

Now we learn what it means for a number to be **EVEN**.

Euclid says: "An **even** number is one which can be divided in half."

2

Let us try with 8.

We have 8 stones.

3

Can we divide these into two equal groups?

Yes! $8 = 4 + 4$.
So 8 can be divided in half.

4

Therefore, 8 is **EVEN**.

8 is an **even** number.

5

Let us try other numbers.

2

4

6

10

$2 = 1 + 1$ $4 = 2 + 2$ $6 = 3 + 3$ $10 = 5 + 5$

Each of these numbers can be divided in half into two equal groups. They are all **EVEN**.

6

What do we notice?

Every even number can be written as 2 times another whole number.

$2 = 2 \times 1$
 $4 = 2 \times 2$
 $6 = 2 \times 3$
 $8 = 2 \times 4$
 $10 = 2 \times 5$
 $12 = 2 \times 6$
...

So an **even** number $= 2 \times n$.

7

But what about 7?

$3 + 4$

We cannot divide 7 into two equal groups.
So 7 cannot be divided in half.

8

In other words...

An **EVEN** number can be split into **TWO** equal parts.
A number that cannot be split into two equal parts is **NOT** even.

9

Remember this, Alexander.

An **even** number is one which can be divided in half.



EVEN NUMBER = one which can be divided in half.

It can be written as $2 \times n$ for some whole number n .

EXAMPLES (EVEN)

$2 = 1 + 1$ $4 = 2 + 2$ $6 = 3 + 3$
 $8 = 4 + 4$ $10 = 5 + 5$ $12 = 6 + 6$

NOT EVEN (cannot be divided in half)

1, 3, 5, 7, 9, 11, ...

13

Definition 7 — Odd Number

An odd number is one which cannot be divided in half, or which differs from an even number by a unit.

EUCLID, BOOK VII — DEFINITION 7: ODD NUMBER

"An odd number is one which cannot be divided in half, or which differs from an even number by a unit."

1 Today we learn what it means for a number to be **ODD**.

Euclid says: "An **odd** number is one which **cannot be divided in half**, or which differs from an **even** number by a unit."

2 Let us take 7 stones.

Can we divide these into two equal groups?

3 If we try...

We get 3 and 4, not two equal groups. So 7 **cannot** be divided in half. Therefore, 7 is **ODD**.

4 Now look at another way.

An **odd** number differs from an **even** number by a **UNIT**.

5 For example:

$$7 = 6 + 1$$

6 is **even**. 1 is a **unit**.

So 7 is different from the **even** number 6 by **one unit**.

6 Other examples:

$$9 = 8 + 1$$

$$11 = 10 + 1$$

$$13 = 12 + 1$$

...

Each **odd** number is an **even** number + 1.

7 Think of pairing.

3 pairs, 1 left over.

Every stone has a partner, except one. That one leftover **unit** shows the number is **ODD**.

8 Let us try some numbers.

5: 1 left over
3: 1 left over
9: 1 left over

All leave one **unit** without a partner. So they are **ODD**.

9 Remember this, Alexander.

If a number cannot be divided into two equal whole-number groups, it is **ODD**.

In other words, it is one more than some **even** number.



ODD NUMBER

An **odd** number is one which cannot be divided in half, or which differs from an **even** number by a **unit**.

TWO WAYS TO RECOGNIZE AN ODD NUMBER

1. It **cannot** be divided in half.
(Two equal groups are impossible.)

2. It differs from an **even** number by a unit.
(**odd** = **even** + 1)

EXAMPLES (ODD)

1, 3, 5, 7, 9, 11, 13, ...
(All are **even** + 1)

Definition 8 — Even-times-even Number

An even-times-even number is one which is measured by an even number according to an even number.

EUCLID, BOOK VII — DEFINITION 8: EVEN-TIMES-EVEN NUMBER

"An even-times-even number is one which is measured by an even number according to an even number."

1

We have learned what it means for a number to be **EVEN**.

Now we learn a **special kind** of even number.

2

Let us take **12**.

12 is an **EVEN** number, for it can be divided in half:
 $12 = 6 + 6$.

3

Can we measure 12 by an **EVEN** number according to an **EVEN** number?

Yes! 12 is measured by **6** according to **2**. So 12 is built from an **even** number repeated an **even** number of times.

4

DEFINITION 8

An **even-times-even** number is one which is measured by an even number according to an **even** number.

In other words, it is the **product** of two even numbers.

5

More examples.

$16 = 4 \times 4$

4 rows (even)
4 in each row (even)
16 is even-times-even.

$24 = 4 \times 6$

4 rows (even)
6 in each row (even)
24 is even-times-even.

$20 = 2 \times 10$

2 rows (even)
10 in each row (even)
20 is even-times-even.

6

But not every even number is even-times-even.

Consider 6.

 $6 = 2 \times 3$

Here, 2 is even, but 3 is odd. Therefore, 6 is **NOT** even-times-even.

7

Another example.

 $10 = 2 \times 5$

2 is even, but 5 is odd. So 10 is **NOT** even-times-even.

8

Aristotle's way of seeing it.

An even-times-even number is like arranging soldiers in rows and columns where both the number of rows and the number in each row are even.

9

Remember this, Alexander.

EVEN $\times$ EVEN = EVEN-TIMES-EVEN.

Examples: 4, 8, 12, 16, 20, 24, ...

Not every even number belongs here! We will learn other kinds next.



DEFINITION 8:

An **even-times-even** number is one which is measured by an **even** number according to an **even** number.

WHAT IT MEANS

It is the **product** of two **even** numbers.

$2m \times 2n = 4mn$

(for some whole numbers m, n)

EXAMPLES (EVEN-TIMES-EVEN)

$4 = 2 \times 2$

$8 = 2 \times 4$

$12 = 2 \times 6$

$16 = 4 \times 4$

$20 = 2 \times 10$

$24 = 4 \times 6$

...

17

Definition 9 — Even-times-odd Number

An even-times-odd number is one which is measured by an even number according to an odd number.

EUCLID, BOOK VII — DEFINITION 9: EVEN-TIMES-ODD NUMBER

"An even-times-odd number is one which is measured by an even number according to an odd number."

1 We have learned even-times-even numbers. Now let us learn another kind. Listen carefully.

Today we learn what it means for a number to be **EVEN-TIMES-ODD**.

2 Let us take 6.

We have **six** stones.

3 Can we measure 6 by an **EVEN** number according to an **ODD** number?

Yes! **2** is even. **3** is odd.

So 6 is measured by an **even** number (**2**) according to an **odd** number (**3**).

4 **DEFINITION 9**
An **even-times-odd** number is one which is measured by an even number according to an odd number.

In other words, it is the **product** of an **even** number and an **odd** number.

5 More examples.

$10 = 2 \times 5$

 2 rows (even)
 5 in each row (odd)
 So 10 is even-times-odd.

$12 = 4 \times 3$

 4 rows (even)
 3 in each row (odd)
 So 12 is even-times-odd.

$18 = 6 \times 3$

 6 rows (even)
 3 in each row (odd)
 So 18 is even-times-odd.

6 Another way to see it.

Take 12 again.

We can see it as 4×3 .

Since one factor is even and the other is odd, 12 is **even-times-odd**.

7 Not every even number is even-times-odd.

Consider 8.

$8 = 2 \times 4$

Both factors are even.

So 8 is even-times-even, not even-times-odd.

8 More examples to remember.

Number	As a product	Even factor	Odd factor	Type
6	2×3	2	3	Even-times-odd
10	2×5	2	5	Even-times-odd
12	4×3	4	3	Even-times-odd
18	6×3	6	3	Even-times-odd
20	4×5	4	5	Even-times-odd

9 Remember this, Alexander.

When a number is built from an even number repeated an odd number of times, it is **EVEN-TIMES-ODD**.

WHAT IT MEANS

Even-times-odd numbers are the products of an even number and an odd number.

even $\times$ odd = even-times-odd

VISUAL PICTURE

Even (2 rows) by odd (3 in each row)

EXAMPLES (EVEN $\times$ ODD)

$2 \times 3 = 6$ $2 \times 5 = 10$ $4 \times 3 = 12$
 $6 \times 3 = 18$ $4 \times 5 = 20$...

Definition 10 — Odd-times-odd Number

An odd-times-odd number is one which is measured by an odd number according to an odd number.

1

We have learned three kinds of even products. Now let us learn another kind.

Today we learn what it means for a number to be **ODD-TIMES-ODD**.

2

Let us take 15.

We have **fifteen** stones.

3

Can we measure 15 by an **ODD** number according to an **ODD** number?

Yes! 3 is odd, 5 is odd.
So 15 is measured by an odd number (3) according to an odd number (5).

4

DEFINITION 10.
An odd-times-odd number is one which is measured by an odd number according to an odd number.

In other words, it is the **product** of two **odd** numbers.

5

More examples.

$9 = 3 \times 3$

3 rows (odd)
3 in each row (odd)
So 9 is odd-times-odd.

$21 = 3 \times 7$

3 rows (odd)
7 in each row (odd)
So 21 is odd-times-odd.

$25 = 5 \times 5$

5 rows (odd)
5 in each row (odd)
So 25 is odd-times-odd.

6

What do we notice?

ODD $\times$ ODD = ODD

The product of two odd numbers is always odd.

7

Let us check.

$3 \times 5 = 15$ (odd)

$3 \times 7 = 21$ (odd)

$5 \times 5 = 25$ (odd)

$7 \times 9 = 63$ (odd)

Each product is odd.
So each is **odd-times-odd**.

8

Remember this well, Alexander.

An **odd** number repeated an **odd** number of times makes an **odd-times-odd** number.

9

Examples of **odd-times-odd** numbers:

9, 15, 21, 25, 35, 45, 63, ...

All are odd.

We will now learn another kind of numbers.

21

Definition 11 — Prime Number

A prime number is one which is measured by a unit alone.

"A prime number is one which is measured by a unit alone."

"A prime number is one which is measured by a unit alone."

1 Today we learn another special kind of number.

DEFINITION 11
A prime number is one which is measured by a unit alone.

3 Let us try the number 7.

Can 2 measure 7 exactly? No
 Can 3 measure 7 exactly? No
 Can 4 measure 7 exactly? No
 Can 5 measure 7 exactly? No
 Can 6 measure 7 exactly? No
 Can the unit (1) measure 7 exactly? Yes

Only the unit measures 7 exactly.
Therefore, 7 is **PRIME**.

2 Now consider the number 12.

Can 2 measure 12 exactly? Yes
 Can 3 measure 12 exactly? Yes
 Can 4 measure 12 exactly? Yes
 Can 6 measure 12 exactly? Yes
 Can the unit (1) measure 12 exactly? Yes

Therefore, 12 is **NOT** prime.

5 Why is 12 not prime?

$12 = 2 \times 6$ $12 = 3 \times 4$

It can be measured by numbers other than the unit.

6 Aristotle explains to Alexander.

A prime number refuses to be built from smaller numbers.
No number smaller than it can measure it exactly, except the unit.

7 Visualizing 7 soldiers.

We can arrange them only as:

1 row of 7 or 7 single units

There is no other equal arrangement.
So 7 is **PRIME**.

8 But 12 soldiers can be arranged in many equal groups.

2 rows of 6 or 3 rows of 4

So 12 is not prime, because it has other measures.

9 Remember this, Alexander.

A prime number has no measure except the unit.

Examples of prime numbers:
2, 3, 5, 7, 11, 13, 17, ...

Definition 12 — Numbers Prime to One Another

Numbers prime to one another are those which are measured by a unit alone as a common measure.

EUCLID, BOOK VII — DEFINITION 12: NUMBERS PRIME TO ONE ANOTHER

“Numbers prime to one another are those which are measured by a unit alone as a common measure.”

1

Today we learn another kind of number relationship.

DEFINITION 12
Numbers prime to one another are those which are measured by a unit alone as a common measure.

2

Let us take the numbers 8 and 15.

Measures of 8:	Measures of 15:
1	1
2	3
4	5
8	15

What numbers measure both 8 and 15 exactly?

3

Only the unit (1) measures both 8 and 15 exactly.

Therefore, 8 and 15 are **PRIME TO ONE ANOTHER**.

Notice: 8 is not prime. 15 is not prime. Yet they are **prime to one another**!

4

Now let us try 12 and 18.

Measures of 12:	Measures of 18:
1	1
2	2
3	3
4	6
6	9
12	18

They have many common measures greater than the unit.

5

Since 12 and 18 are measured by numbers other than the unit in common, they are **NOT** prime to one another.

Their common measures are: 1, 2, 3, 6.

6

Let us see the idea in another way.

Imagine two companies of soldiers.

8 soldiers	15 soldiers

Can we divide both companies into equal groups of 2?
No, because 15 cannot.

Groups of 3? 4? 5?
None work for both.

Only groups of one soldier work for both.

7

In modern language, this means:

$\gcd(8, 15) = 1$

But for 12 and 18:

$\gcd(12, 18) = 6$

$\gcd$ = greatest common measure (or greatest common divisor)

8

More examples.

Prime to one another (only 1 is common)	Not prime to one another (common measure > 1)
8 and 15 $\gcd = 1$	9 and 10 $\gcd = 1$
14 and 25 $\gcd = 1$	14 and 21 $\gcd = 7$
	8 and 12 $\gcd = 4$

Look for the greatest number that measures both.

9

Remember this, Alexander.

Two numbers are prime to one another when their only common measure is the unit.

Prime number and prime to one another are **not** the same thing.

25

Definition 13 — Composite Number

A composite number is one which is measured by some number.

EUCLID, BOOK VII — DEFINITION 13: COMPOSITE NUMBER
"A composite number is one which is measured by some number."

1 Today we learn another kind of number.

DEFINITION 13
A composite number is one which is measured by some number.

2 Take the number 12.
We can arrange 12 stones in equal groups in more than one way.
2 groups of 6 3 groups of 4
So 12 is measured by 2, by 3, by 4, and by 6.
Therefore, 12 is **COMPOSITE**.

3 Take the number 15.
We can arrange it as:
3 groups of 5
So 15 is measured by 3 and by 5.
Therefore, 15 is **COMPOSITE**.

4 But look at the number 7.
Can 2 measure 7 exactly? **No.**
Can 3 measure 7 exactly? **No.**
Can 4 measure 7 exactly? **No.**
Can 5 measure 7 exactly? **No.**
Can 6 measure 7 exactly? **No.**
Only the unit (1) measures 7.
Therefore, 7 is **NOT** composite.
It is **PRIME**.

5 Aristotle explains to Alexander.
A prime number cannot be broken into equal groups using any smaller number.
A composite number can.

6 For example, 12 soldiers.
They can be arranged as 2 rows of 6.
They can also be arranged as 3 rows of 4.
Its ability to be measured by other numbers reveals its composite nature.

7 More examples of composite numbers.
 $8 = 2 \times 4$ $9 = 3 \times 3$ $10 = 2 \times 5$ $14 = 2 \times 7$
Each of these numbers is measured by some number greater than the unit.

8 The simple rule to remember.
A COMPOSITE NUMBER is measured by some number (other than the unit).
Examples:
4, 6, 8, 9, 10, 12, 14, 15, ...

9 The contrast is clear, Alexander.
PRIME NUMBER
Measured only by the unit.
Examples:
2, 3, 5, 7, 11, 13, ...
COMPOSITE NUMBER
Measured by some number.
Examples:
4, 6, 8, 9, 10, 12, ...
We will now learn yet another important idea.

Definition 14 — Numbers Composite to One Another

Numbers composite to one another are those which are measured by some number as a common measure.

“Numbers composite to one another are those which are measured by some number as a common measure.”

We will now learn yet another definition.

EUCLID, BOOK VII — DEFINITION 14: NUMBERS COMPOSITE TO ONE ANOTHER

“Numbers composite to one another are those which are measured by some number as a common measure.”

1

Today we learn the meaning of numbers composite to one another.

DEFINITION 14
Numbers composite to one another are those which are measured by some number as a common measure.

2

Let us take the numbers 12 and 18.

12 soldiers 18 soldiers

Can we divide both into groups of 2?

$12 = 2 \times 6$ $18 = 2 \times 9$

Yes! 2 measures both 12 and 18.

3

Can we divide both into groups of 3?

$12 = 3 \times 4$ $18 = 3 \times 6$

Yes! 3 also measures both 12 and 18.

4

Can we divide both into groups of 6?

$12 = 6 \times 2$ $18 = 6 \times 3$

Yes! 6 measures both 12 and 18.

5

Since 12 and 18 are measured by some number greater than the unit as a common measure, they are composite to one another.

6

Common measures of 12 and 18:
1, 2, 3, 6.
Because they have common measures greater than the unit, 12 and 18 are COMPOSITE TO ONE ANOTHER.

7

Now let us take 8 and 15.

8 soldiers 15 soldiers

Can we divide both into groups of 2? No. Groups of 3? No. Groups of 4? No.

Only the unit (1) measures both.

Therefore, 8 and 15 are NOT composite to one another (they are prime to one another).

8

So two numbers are composite to one another when they share a common measure greater than the unit!

Two numbers are composite to one another when their only common measure is some number greater than the unit.

9

To remember:

Prime to one another Common measure only 1. Example: 8 and 15 (gcd = 1)	Composite to one another Some common measure > 1. Example: 12 and 18 (gcd = 6)
--	---

Next we will learn what it means for one number to multiply another.

Definition 15 — Multiplication of Numbers

A number is said to multiply another number when the number being multiplied is added to itself as many times as there are units in the former number, and thereby some other number is produced.

EUCLID, BOOK VII — DEFINITION 15: MULTIPLICATION OF ONE NUMBER BY ANOTHER

"A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced."

1 Today we learn how one number multiplies another.

DEFINITION 15
A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.

2 Let the number 3 multiply the number 4.

The multiplying number is 3.
Therefore the number 4 is taken **three times**.

1st time 2nd time 3rd time

So we add 4 to itself **three times**.

$4 + 4 + 4 = 12$

Therefore, $3 \times 4 = 12$.

3 Another example.

Let the number 5 multiply the number 2.

The number 2 is taken **five times**.

$2 + 2 + 2 + 2 + 2 = 10$

Therefore, $5 \times 2 = 10$.

4 Yet another example.

Let the number 4 multiply the number 6.

The number 6 is taken **four times**.

$6 + 6 + 6 + 6 = 24$

Therefore, $4 \times 6 = 24$.

5 Aristotle explains to Alexander:

Imagine we have 4 soldiers in one squad.
If we make three identical squads, how many soldiers will we have in all?

$4 + 4 + 4 = 12$

Three copies of four produce twelve.

6 In general, this is what multiplication means.

multiplier (m) $\times$ number being multiplied (n) = result

That is,

$m \times n = \underbrace{n + n + \dots + n}_m$ times

7 More examples.

2 multiplies 7. $7 + 7 = 14$
 $2 \times 7 = 14$

3 multiplies 5. $5 + 5 + 5 = 15$
 $3 \times 5 = 15$

6 multiplies 3. $3 + 3 + 3 + 3 + 3 + 3 = 18$
 $6 \times 3 = 18$

8 Important things to remember.

Multiplication is repeated addition.

The order matters: 3×4 means three groups of four.

This idea will help us build **plane numbers** and **solid numbers**.

9 The simplest rule to hold in your mind, Alexander:

To multiply a number is to repeat it by addition a specified number of times.

This is the beginning of one of the most powerful ideas in all of mathematics.

Definition 16 — Plane Number

When two numbers multiplying one another make some other number, the number so created is called plane, and its sides are the numbers which multiply one another.

EUCLID, BOOK VII — DEFINITION 16: PLANE NUMBER

"When two numbers multiplying one another make some other number, then the number so created is called plane, and its sides are the numbers which multiply one another."

1 Today we learn how numbers can have shape.

DEFINITION 16
When two numbers multiplying one another make some other number, then the number so created is called **plane**, and its **sides** are the numbers which multiply one another.

2 Take the numbers 3 and 4.
Let 3 multiply 4.

$4 + 4 + 4 = 12$
(4 taken three times)



Therefore, $3 \times 4 = 12$.

3 According to Definition 16, the number 12 is called **PLANE**, because it is made by two numbers multiplying one another. Its **sides** are the numbers which multiply.

So 12 is a **plane number** with sides 3 and 4.

4 Why do we call it "plane"? Because we can arrange 12 objects in a rectangle on a flat surface.



It has length 4 and breadth 3. It is a **rectangle** — a figure in the plane.

5 Another example. Let 5 multiply 6.

$5 \times 6 = 30$



30 is a **plane number** with sides 5 and 6.

6 Yet another example. Let 2 multiply 7.

$2 \times 7 = 14$



14 is a **plane number** with sides 2 and 7.

7 In general, whenever $a \times b = c$,



c is a **PLANE NUMBER**, and its sides are a and b .

8 Many numbers can be plane numbers. For example:

$2 \times 5 = 10$ $3 \times 8 = 24$ $4 \times 9 = 36$



Each can be pictured as a **rectangle** with its two sides.

9 Remember, Alexander.

A **plane number** is a number produced by two numbers multiplying one another. Its **sides** are those two numbers.

This is the first step in seeing numbers not only as quantities, but also as **figures**.

Definition 17 — Solid Number

When three numbers multiplying one another make some other number, the number so created is called solid, and its sides are the numbers which multiply one another.

EUCLID, BOOK VII — DEFINITION 17: SOLID NUMBER

"When three numbers multiplying one another make some other number, then the number so created is called solid, and its sides are the numbers which multiply one another."

1 Today we learn about solid numbers.

DEFINITION 17
When three numbers multiplying one another make some other number, then the number so created is called **solid**, and its sides are the numbers which multiply one another.

2 Let us take the numbers 2, 3, and 4.

First, 2 multiplies 3:
 $2 \times 3 = 6$

Then 6 is multiplied by 4:
 $6 \times 4 = 24$

Therefore, $2 \times 3 \times 4 = 24$.

3 According to Definition 17, the number 24 is called **SOLID**, because it is made by three numbers multiplying one another.

Its **sides** are the numbers which multiply:
2, 3, and 4.

So 24 is a **SOLID NUMBER** with sides 2, 3, and 4.

4 Why do we call it "solid"? Because we can arrange 24 objects in a three-dimensional block.

2 layers (height)
3 rows (depth)
4 in each row (length)

It fills space. That is why it is called **solid**.

5 Imagine we have 24 soldiers. We form a solid block:

2 layers
3 rows
4 in each row

$2 \times 3 \times 4 = 24$ soldiers.

6 Another example.

Let 3, 4, and 5 multiply one another.

$3 \times 4 \times 5 = 60$

60 is a **SOLID NUMBER** with sides 3, 4, and 5.

7 In general, whenever $a \times b \times c = d$, the number d is called **SOLID**, and its sides are a , b , and c .

$a \times b \times c = d$

a : first side (height)
 b : second side (depth)
 c : third side (length)
 d : solid number (volume)

It is the product of three numbers.

8 More examples.

$1 \times 2 \times 7 = 14$
 $2 \times 2 \times 3 = 12$
 $3 \times 3 \times 3 = 27$

Each number has three sides, and the result is a solid number.

9 Remember, Alexander.

A **SOLID NUMBER** is a number produced by three numbers multiplying one another. Its sides are those three numbers.

Thus Euclid teaches us to see numbers not only as quantities, but also as shapes in space.

Definition 18 — Square Number

A square number is an equal times an equal, or a plane number contained by two equal numbers.

MATHEMATICS
DISCIPLINES
THE MIND

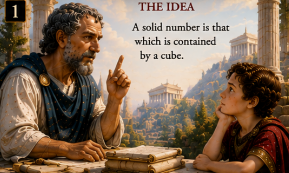
EUCLID, BOOK VII — DEFINITION 18

“A solid number is that which is contained by a cube.”

THE ELEMENTS
BOOK VII
DEFINITIONS

1 THE IDEA

A solid number is that which is contained by a cube.



2 A CUBE CONTAINS A NUMBER

A cube with 2 units on each edge contains $2 \times 2 \times 2 = 8$ units.

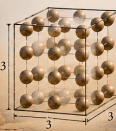


$2^3 = 8$

A solid number (8)

3 A LARGER CUBE

A cube with 3 units on each edge contains $3 \times 3 \times 3 = 27$ units.

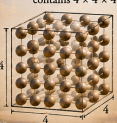


$3^3 = 27$

A solid number (27)

4 ANOTHER EXAMPLE

A cube with 4 units on each edge contains $4 \times 4 \times 4 = 64$ units.

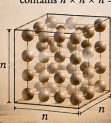


$4^3 = 64$

A solid number (64)

5 THE GENERAL FORM

A cube with n units on each edge contains $n \times n \times n = n^3$ units.



n^3

A solid number

6 NOT EVERY NUMBER IS SOLID

Numbers like 2, 3, 5, 6, 7, 10, 12 are not contained by a cube, and so are not solid numbers.



Only numbers of the form n^3 (1, 8, 27, 64, 125, ...) are solid numbers.



ARISTOTLE TO ALEXANDER:
“Just as a square number is contained by a square, so a solid number is contained by a cube — a number that fills space in all three dimensions.”

DEFINITION 18

A solid number is that which is contained by a cube.



ALEXANDER'S REFLECTION:
“So a solid number is a number that forms a perfect cube in space. It is the number of units that fill a cube completely.”

EUCLID — ORDER — REASON — IMMORTAL TRUTH

Definition 19 — Cube Number

A cube number is an equal times an equal times an equal, or a solid number contained by three equal numbers.

MATHEMATICS
DISCIPLINES
THE MIND

EUCLID, BOOK VII — DEFINITION 19

“The sides of a solid number are the three numbers which, when multiplied together, produce the solid number.”

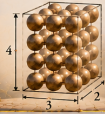
THE ELEMENTS
BOOK VII
DEFINITIONS

1 THE DEFINITION

The sides of a solid number are the three numbers which, when multiplied together, produce the solid number.

2 A SOLID NUMBER

A rectangular solid (a cube or rectangular parallelepiped) with 2 units in one direction, 3 in another, and 4 in the third contains:



$$2 \times 3 \times 4 = 24$$

Thus 24 is a solid number.

3 THE SIDES

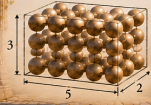
The three numbers 2, 3, and 4 are the sides of the solid number 24.



Sides:
2, 3, 4
Product:
 $2 \times 3 \times 4 = 24$

4 ANOTHER EXAMPLE

A solid number can have different sides.

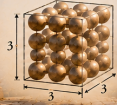


$$3 \times 5 \times 2 = 30$$

The sides of the solid number 30 are 3, 5, and 2.

5 A CUBE AS A SPECIAL CASE

When the three sides are equal, we have a cube.



$$3 \times 3 \times 3 = 27$$

The sides of the solid number 27 are 3, 3, and 3.

6 IN GENERAL

If the sides of a solid number are A , B , and C , then the solid number is

$A \times B \times C$

Conversely, if a number can be represented as the product of three numbers, those three numbers are the sides of that solid number.



ARISTOTLE TO ALEXANDER:
“A solid number has three sides, just as a geometric solid has three dimensions. The number arises from the multiplication of its three sides.”

DEFINITION 19

The sides of a solid number are the three numbers which, when multiplied together, produce the solid number.



ALEXANDER'S REFLECTION:
“The sides show how the number is built — length, breadth, and height in number, just as in a geometric solid. By multiplying the three sides, the whole is formed.”

EUCLID — ORDER — REASON — IMMORTAL TRUTH

Definition 20 — Proportional Numbers

Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

Definition 21 — Similar Plane and Solid Numbers

Similar plane and solid numbers are those having proportional sides.

EUCLID, BOOK VII — DEFINITION 21: SIMILAR PLANE AND SOLID NUMBERS

"Similar plane and solid numbers are those having proportional sides."

1 Today we learn about similar plane and solid numbers.

DEFINITION 21
Similar plane and solid numbers are those having **proportional sides**.

2 For plane numbers, each number is represented by two sides.

First plane number: $2 \times 3 = 6$

Second plane number: $4 \times 6 = 24$

Are these two plane numbers similar? Let us see.

3 Compare their corresponding sides.

First number sides: 2 and 3

Second number sides: 4 and 6

Do the ratios match?

$2 : 3 = 4 : 6$
 $\left(= \frac{2}{3} \right) \quad \left(= \frac{2}{3} \right)$

Yes! The sides are proportional. Therefore, the two plane numbers are **similar**.

4 Visually, they have the same shape, only different size.

The second is twice as large in each direction. Both sides are multiplied by the same number 2.

Same shape → **similar**. Different size → **not equal**.

5 Now let us see solid numbers.

First solid number with sides 2, 3, 4.

Second solid number with sides 4, 6, 8.

Do their corresponding sides have the same ratios?

6 Let us compare.

$2 : 4 = 3 : 6 = 4 : 8$
 $\left(= \frac{1}{2} \right) \quad \left(= \frac{1}{2} \right) \quad \left(= \frac{1}{2} \right)$

All three pairs of sides are proportional. Therefore, the two solid numbers are **similar**.

7 Similarity does not mean equality.

$2 \times 3 = 6$ $4 \times 6 = 24$

$6 \neq 24$

The numbers are different, but their shapes are similar.

8 In general:

Plane numbers: $a : b = c : d$
 $a \times b$ and $c \times d$ are similar plane numbers.

Solid numbers: $a : b : c = p : q : r$
 $a \times b \times c$ and $p \times q \times r$ are similar solid numbers.

"Similar numbers have proportional sides—the same shape, different size."

9 Remember, Alexander.

Similar plane and solid numbers are those having **proportional sides**.

Same proportion → similar shape.
Scale changes → size changes, but not the shape.

Thus Euclid teaches us that **proportional sides** create **similarity** in number—as in figures.

Definition 22 — Perfect Number

A perfect number is that which is equal to its own parts.

EUCLID, BOOK VII — DEFINITION 22: PERFECT NUMBER

"A perfect number is that which is equal to its own parts."

1 Today, Alexander, we learn the last definition of this Book.

DEFINITION 22
A perfect number is that which is equal to its own parts; that is, equal to the sum of its own factors (except itself).

2 Let us begin with the number 6. What are its parts (factors)?

The parts of 6, except itself, are 1, 2, and 3.

$1 + 2 + 3 = 6$

The sum of its parts is equal to the number itself.

Therefore, 6 is a perfect number.

3 Another example: the number 28. What are its parts (factors)?

These are the parts of 28, except itself:

$1 + 2 + 4 + 7 + 14 = 28$

The sum of its parts is equal to the number itself.

Therefore, 28 is a perfect number.

4 That is why Euclid calls such a number perfect:

Nothing is missing and nothing is excessive. Its own parts reconstruct the whole exactly.

As a kingdom whose provinces together have exactly the strength of the whole kingdom.

5 Let us see it clearly.

For 6: $1 + 2 + 3 = 6$

For 28: $1 + 2 + 4 + 7 + 14 = 28$

In both cases, the parts add up to the whole.

6 Not every number is perfect.

Take 8. Its parts are 1, 2, and 4. $1 + 2 + 4 = 7$. So 8 is not perfect.

Take 12. Its parts are 1, 2, 3, 4, and 6. $1 + 2 + 3 + 4 + 6 = 16$. So 12 is not perfect.

Only some numbers are perfect.

7 The first perfect numbers we know are:

6 = $1 + 2 + 3$

28 = $1 + 2 + 4 + 7 + 14$

There are others still greater, but these are the first and most famous.

8 In general,

A perfect number = the sum of its own proper divisors (those less than itself).

It is a number that is measured exactly by its own parts.

9 Remember, Alexander,

A perfect number is that which is equal to its own parts.

ΤΑ ΜΕΡΗ
ΙΣΑ ΤΟ ΟΛΟΝ ΠΟΙΕΙ.
The parts make the whole equal.

Thus Euclid completes the twenty-two definitions of Book VII. The study of numbers now prepares the mind for the propositions that follow.

Propositions

Proposition 1

Statement. Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.

Proof.

Action 1. For two (unequal) numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains.

Action 2. I say that AB and CD are prime to one another that is to say, that a unit alone measures (both) AB and CD.

Action 3. For if AB and CD are not prime to one another then some number will measure them.

Action 4. Let (some number) measure them, and let it be E.

Action 5. And let CD measuring BF leave F A less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring F H leave a unit, HA.

Action 6. In fact, since E measures CD, and CD measures BF, E thus also measures BF.

Action 7. And (E) also measures the whole of BA.

Action 8. Thus, (E) will also measure the remainder AF. And AF measures DG. Thus, E also measures DG.

Action 9. And (E) also measures the whole of DC.

Action 10. Thus, (E) will also measure the remainder CG. And CG measures F H. Thus, E also measures F H.

Action 11. And (E) also measures the whole of F A.

Action 12. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible.

Action 13. Thus, some number does not measure (both) the numbers AB and CD.

Action 14. Thus, AB and CD are prime to one another.

Action 15. (Which is) the very thing it was required to show. s b, and b measures c, then a also measures c, where all symbols denote , and a measures part of b, then a also measures the remainder of b, where

Therefore, Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.

EUCLID, BOOK VII — PROPOSITION 1

"If two unequal numbers, by being repeatedly diminished, leave a unit, and the remainders never in the process measure the last diminishing number, then the two numbers are prime to one another."

1

Begin with two unequal numbers, AB and CD.
Repeatedly subtract the smaller from the larger until the remainder is a unit.

107 and 36

$107 - 36 = 71$

$71 - 36 = 35$

$36 - 35 = 1$

At each earlier step, the remainder does not measure the preceding number.

2

We must prove:
AB and CD are prime to one another;
that is, the unit alone measures them both.

Definition 12:

Prime to one another are numbers which the unit alone measures.

3

Assume the opposite.
Suppose AB and CD are not prime to one another.
Then there exists a number E (> 1) that measures both.

$E > 1$

measures AB measures CD

4

Follow the subtraction process.
Let each remainder be the next number and continue until the final remainder is a unit.

AB → CD → AF → CG → AH

CD leaves remainder AF AF leaves remainder CG CG leaves remainder AH (unit)

$AB > CD > AF > CG > AH = 1$

5

Since E measures CD, it also measures every multiple of CD. And since E measures AB, it must also measure the remainder AF.

E measures CD
⇒ E measures any multiple of CD

E measures AB and also CD
⇒ E measures remainder AF

6

Since E measures AF, and AF has been subtracted from CD, while E also measures CD,
E must measure the next remainder CG.

E measures AF → E measures CD → Therefore E measures CG

7

The same reasoning again.
E measures CG and also the relevant whole, therefore E measures the final remainder AH.
But AH is a unit.

E measures CG Therefore E measures AH AH = 1 (unit)

8

Contradiction!
E was assumed to be a number greater than 1.
A number greater than the unit cannot measure a unit.

$E > 1$
but E measures AH = 1
Impossible!

9

Therefore no number greater than the unit measures both AB and CD.
Their only common measure is the unit.

AB and CD are prime to one another.
Q.E.D.

Euclid's first number-theoretic proof: the parts lead to a unit, so no common measure greater than the unit can exist.

The lesson: only the unit can measure both. Therefore the numbers are prime to one another.

Proposition 2

Statement. To find the greatest common measure of two given numbers (which are) not prime to one another.

Proof.

- Action 1.** Let AB and CD be the two given numbers (which are) not prime to one another.
- Action 2.** So it is required to find the greatest common measure of AB and CD.
- Action 3.** In fact, if CD measures AB, CD is thus a common measure of CD and AB, (since CD) also measures itself.
- Action 4.** And (it is) manifest that (it is) also the greatest (common measure).
- Action 5.** For nothing greater than CD can measure CD.
- Action 6.** But if CD does not measure AB then some number will remain from AB and CD, the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left.
- Action 7.** But if not, AB and CD will be prime to one another (Prop. VII.1).
- Action 8.** The very opposite thing was assumed.
- Action 9.** Thus, some number will remain which will measure the (number) preceding it.
- Action 10.** And let CD measuring BE leave EA less than itself, and let EA measuring DF leave F C less than itself, and let CF measure AE.
- Action 11.** Therefore, since CF measures AE, and AE measures DF, CF will thus also measure DF. And it also measures itself.
- Action 12.** Thus, it will also measure the whole of CD. And CD measures BE. Thus, CF also measures BE. And it also measures EA.

Action 13. Thus, it will also measure the whole of BA. And it also measures CD.

Action 14. Thus, CF measures (both) AB and CD.

Action 15. Thus, CF is a common measure of AB and CD.

Action 16. So I say that (it is) also the greatest (common measure).

Action 17. For if CF is not the greatest common measure of AB and CD then some number which is greater than CF will measure the numbers AB and CD.

Action 18. Let it (so) measure (AB and CD), and let it be G.

Action 19. And since G measures CD, and CD measures BE, G thus also measures BE.

Action 20. And it also measures the whole of BA.

Action 21. Thus, it will also measure the remainder AE. And AE measures DF . Thus, G will also measure DF .

Action 22. And it also measures the whole of DC.

Action 23. Thus, it will also measure the remainder CF , the greater (measuring) the lesser. The very thing is impossible.

Action 24. Thus, some number which is greater than CF cannot measure the numbers AB and CD.

Action 25. Thus, CF is the greatest common measure of AB and CD.

Action 26. ((Which is) the very thing it was required to show).

Corollary. So it is manifest, from this, that if a number measures two numbers then it will also measure their greatest common measure. (Which is) the very thing it was required to show.

Therefore, To find the greatest common measure of two given numbers (which are) not prime to one another.

MATHEMATICS
DISCIPLINES
THE MIND

EUCLID, BOOK VII — PROPOSITION 2

To find the greatest common measure of two given numbers which are not prime to one another.

THE ELEMENTS
BOOK VII
PROPOSITION 2

1 Given two numbers AB = 84 CD = 30 Let the two numbers be 84 and 30, which are not prime to one another.	2 Test divisibility Does 30 measure 84? $84 = 30 \times 2 + 24$ Since 30 does not measure 84, there is a remainder.	3 Begin repeated subtraction <table style="margin: auto;"> <tr><td>84</td></tr> <tr><td>− 30</td></tr> <tr><td>54</td></tr> </table> Subtract 30 from 84. Remainder 54.	84	− 30	54	4 Continue the process <table style="margin: auto;"> <tr><td>54</td></tr> <tr><td>− 30</td></tr> <tr><td>24</td></tr> </table> Subtract 30 from 54. Remainder 24.	54	− 30	24	5 Next stage <table style="margin: auto;"> <tr><td>30</td></tr> <tr><td>− 24</td></tr> <tr><td>6</td></tr> </table> Now subtract 24 from 30. Remainder 6.	30	− 24	6			
84																
− 30																
54																
54																
− 30																
24																
30																
− 24																
6																
6 Remainder measures previous <table style="margin: auto;"> <tr><td>24</td></tr> <tr><td>− 6</td></tr> <tr><td>18</td></tr> </table> Continue: $24 - 6 = 18$.	24	− 6	18	7 Next subtraction <table style="margin: auto;"> <tr><td>18</td></tr> <tr><td>− 6</td></tr> <tr><td>12</td></tr> </table> Continue: $18 - 6 = 12$.	18	− 6	12	8 Next subtraction <table style="margin: auto;"> <tr><td>12</td></tr> <tr><td>− 6</td></tr> <tr><td>6</td></tr> </table> Continue: $12 - 6 = 6$.	12	− 6	6	9 Termination condition 6 The remainder 6 measures the preceding remainder 12. The process terminates.	10 Common measure of both <table style="margin: auto;"> <tr><td>84</td></tr> <tr><td>30</td></tr> <tr><td>6</td></tr> </table> 6 measures both 84 and 30.	84	30	6
24																
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30																
6																
11 Divide the first number $84 = 6 \times 14$ Since 6 measures 84, we have $84 = 6 \times 14$.	12 Divide the second number $30 = 6 \times 5$ Since 6 measures 30, we have $30 = 6 \times 5$.	13 Suppose a greater measure Suppose $G > 6$ measures both. Assume there is a number G greater than 6 that measures both 84 and 30.	14 But then G measures 30 $G \mid 30$ and $30 \mid 84$ Since G measures 30, and 30 measures 84, G also measures 84.	15 Therefore G would measure 6 Any common measure of 84 and 30 must also measure the final remainder 6.												
16 But no number greater than 6 can measure 6 If $G > 6$, then G cannot measure 6. This is impossible.	17 Contradiction Therefore no greater common measure exists. Our assumption was false.	18 Conclusion 6 is the greatest common measure of 84 and 30.	19 General statement The same procedure finds the greatest common measure of any two numbers that are not prime to one another.	20 Corollary If a number measures two numbers, then it also measures their greatest common measure.												

ARISTOTLE TO ALEXANDER:

"By successive subtraction we reach the number that measures both and admits no greater measure."

ALEXANDER'S REFLECTION:

"So the greatest common measure is the last remainder — the measure shared by both, beyond which none can go."

EUCLID — ORDER — REASON — IMMORTAL TRUTH

Proposition 3

Statement. To find the greatest common measure of three given numbers (which are) not prime to one another.

Proof.

Action 1. Let A, B, and C be the three given numbers (which are) not prime to one another.

Action 2. So it is required to find the greatest common measure of A, B, and C.

Action 3. For let the greatest common measure, D, of the two (numbers) A and B have been taken (Prop. VII.2).

Action 4. So D either measures, or does not measure, C. First of all, let it measure (C). And it also measures A and B. Thus, D measures A, B, and C.

Action 5. Thus, D is a common measure of A, B, and C.

Action 6. So I say that (it is) also the greatest (common measure).

Action 7. For if D is not the greatest common measure of A, B, and C then some number greater than D will measure the numbers A, B, and C.

Action 8. Let it (so) measure (A, B, and C), and let it be E.

Action 9. Therefore, since E measures A, B, and C, it will thus also measure A and B.

Action 10. Thus, it will also measure the greatest common measure of A and B (Prop. VII.2 corr.).

Action 11. And D is the greatest common measure of A and B.

Action 12. Thus, E measures D, the greater (measuring) the lesser. The very thing is impossible.

Action 13. Thus, some number which is greater than D cannot measure the numbers A, B, and C.

Action 14. Thus, D is the greatest common measure of A, B, and C. So let D not measure C.

- Action 15.** I say, first of all, that C and D are not prime to one another.
- Action 16.** For since A, B, C are not prime to one another, some number will measure them.
- Action 17.** So the (number) measuring A, B, and C will also measure A and B, and it will also measure the greatest common measure, D, of A and B (Prop. VII.2 corr.). And it also measures C.
- Action 18.** Thus, some number will measure the numbers D and C.
- Action 19.** Thus, D and C are not prime to one another.
- Action 20.** Therefore, let their greatest common measure, E, have been taken (Prop. VII.2).
- Action 21.** And since E measures D, and D measures A and B, E thus also measures A and B. And it also measures C. Thus, E measures A, B, and C.
- Action 22.** Thus, E is a common measure of A, B, and C.
- Action 23.** So I say that (it is) also the greatest (common measure).
- Action 24.** For if E is not the greatest common measure of A, B, and C then some number greater than E will measure the numbers A, B, and C.
- Action 25.** Let it (so) measure (A, B, and C), and let it be F .
- Action 26.** And since F measures A, B, and C, it also measures A and B.
- Action 27.** Thus, it will also measure the greatest common measure of A and B (Prop. VII.2 corr.).
- Action 28.** And D is the greatest common measure of A and B. Thus, F measures D. And it also measures C. Thus, F measures D and C.
- Action 29.** Thus, it will also measure the greatest common measure of D and C (Prop. VII.2 corr.).
- Action 30.** And E is the greatest common measure of D and C.

Action 31. Thus, F measures E , the greater (measuring) the lesser.
The very thing is impossible.

Action 32. Thus, some number which is greater than E does not measure the numbers A , B , and C .

Action 33. Thus, E is the greatest common measure of A , B , and C .

Action 34. (Which is) the very thing it was required to show.

Therefore, To find the greatest common measure of three given numbers (which are) not prime to one another.

EUCLID, BOOK VII — PROPOSITION 3

"To find the greatest common measure of three given numbers which are not prime to one another."

1

Start with three numbers A, B, and C which are not prime to one another.

A = 84 B = 60 C = 36

We must find their greatest common measure.

2

First, find the greatest common measure of A and B (by Proposition 2).

$\gcd(A, B) = D$
 $\gcd(84, 60) = 12$
 So $D = 12$.

$84 - 60 = 24$
 $60 - 24 = 36$
 $36 - 24 = 12$
 $24 - 12 = 12$
 $12 - 12 = 0$

Thus 12 is the greatest common measure of 84 and 60.

3

Now ask: does $D = 12$ measure $C = 36$?

Yes. $36 \div 12 = 3$.

12 | 36

Since D measures A and B, and D also measures C, D is a common measure of A, B, and C.

4

Case 1: D measures C.
 Suppose some larger number E ($> D$) measures A, B, and C.

D = 12 E > 12

E | 84 E | 60 E | 36

Then E measures A and B,
 so by Prop. 2 (corollary) E also measures $D = 12$.

5

But a number greater than D cannot measure D.

$E > D$
 $E \nmid D$
 Impossible!
 Therefore no such E exists.
 So $D = 12$ is the greatest common measure of A, B, and C.

6

Case 2: D does not measure C.
 Then D and C cannot be prime to one another.
 Why? Because A, B, and C have a common measure.

A B C
 $\downarrow$ $\downarrow$ $\downarrow$
 Some common measure G ($> D$)
 G | A, G | B, G | C

So $G \mid D$ (since $D = \gcd(A, B)$).
 Thus some number > 1 measures both D and C.
 So D and C are not prime to one another.

7

Therefore, by Proposition 2, find the greatest common measure of D and C.

D = 12 and C = 36

$36 - 12 = 24$
 $24 - 12 = 12$
 $12 - 12 = 0$

$\gcd(D, C) = E$
 $\gcd(12, 36) = 12$
 So $E = 12$.

8

$E = 12$ measures D and C.
 Since D measures A and B, E also measures A and B.

E = 12

E | 84 E | 60 E | 36

So E is a common measure of A, B, and C.
 (Indeed, $84 \div 12 = 7$, $60 \div 12 = 5$, $36 \div 12 = 3$.)

9

Prove E is the greatest common measure.
 Suppose some larger number F ($> E$) measures A, B, and C.

F > 12

F | 84 F | 60 F | 36

Then F measures A and B so F measures $D = 12$.
 And F measures C.
 Thus F measures both D and C so F measures $E = 12$.
 But $F > E$, impossible.
 Therefore $E = 12$ is the greatest common measure of A, B, and C.

Summary:
 To find the greatest common measure of three numbers A, B, and C:
 1. Find $D = \gcd(A, B)$.
 2. If D measures C, then D is the answer.
 3. Otherwise find $E = \gcd(D, C)$.
 Then E is the greatest common measure of A, B, and C.

In symbols:
 $\gcd(A, B, C) = \gcd(\gcd(A, B), C)$

Corollary (from Proposition 2):
 If a number measures two numbers, then it also measures their greatest common measure.

Proposition 4

Statement. Any number is either part or parts of any (other) number, the lesser of the greater.

Proof.

Action 1. Let A and BC be two numbers, and let BC be the lesser.

Action 2. I say that BC is either part or parts of A.

Action 3. For A and BC are either prime to one another, or not.

Action 4. Let A and BC, first of all, be prime to one another.

Action 5. So separating BC into its constituent units, each of the units in BC will be some part of A. Hence, BC is parts of A.

Action 6. So let A and BC be not prime to one another.

Action 7. So BC either measures, or does not measure, A.

Action 8. Therefore, if BC measures A then BC is part of A.

Action 9. And if not, let the greatest common measure, D, of A and BC have been taken (Prop. VII.2), and let BC have been divided into BE, EF, and FC, equal to D.

Action 10. And since D measures A, D is a part of A.

Action 11. And D is equal to each of BE, EF, and FC.

Action 12. Thus, BE, EF, and FC are also each part of A. Hence, BC is parts of A.

Action 13. Thus, any number is either part or parts of any (other) number, the lesser of the greater.

Action 14. (Which is) the very thing it was required to show.

Therefore, Any number is either part or parts of any (other) number, the lesser of the greater.

EUCLID, BOOK VII — PROPOSITION 4

"Any number is either part or parts of any other number, the lesser of the greater."

1

Let A be any greater number,
and BC the lesser number.
A (greater)

BC (lesser)

We must show:
BC is either a PART of A,
or PARTS of A.

2

There are two cases.
Either A and BC are prime to one another,
or they are not.

Case 1:
A and BC
are prime to
one another.

Case 2:
A and BC
are not prime to
one another.

3

Case 1: Suppose A and BC are prime to one another.
Break BC into its constituent units.

A unit is some part
of A.

Therefore each unit
in BC is a part of A.

Therefore BC, being made of several units,
is PARTS of A.

4

Case 2: Suppose A and BC are not prime to
one another.

Ask: does BC measure A?

(i) If BC measures A,
then BC is a PARTS of A.

(ii) If BC does not
measure A,
go to the next step.

5

Case 2(i): If BC measures A,
Since BC is the lesser and it
measures A, by Definition 3,
BC is a PART of A.

A

BC

So in this case also,
BC is a PART of A.

6

Case 2(ii): If BC does not measure A,
By Proposition 2, take D as the greatest
common measure of A and BC.

A

BC

D measures both A and BC.

7

Divide BC into pieces each equal to D.
Let the pieces be BE, EF, FC.

B E BC F C

D D D

Since D measures A,
D is a PART of A.

Each piece (BE, EF, FC)
is equal to D,
so each piece is a PART of A.

8

Therefore BC is made from several parts of A.
Since each of the pieces (BE, EF, FC) is a part of A,
their whole together, BC, is PARTS of A.

A

BC

BE EF FC

D D D

Thus BC is PARTS of A
in this case.

9

In all cases, the lesser number (BC)
is either a PART of the greater (A)
or PARTS of the greater.

BC is a PART of A
(when BC measures A).

OR

BC is PARTS of A
(when BC does not
measure A or when
A and BC are prime).

A

BC

Proven!

Any number is either
part or parts of any other
number, the lesser of the greater.

ΠΑΣ ΑΡΙΘΜΟΣ ΕΙΤΕ ΜΕΡΟΣ ΕΣΤΙΝ
ΕΙΤΕ ΜΕΡΗ ΤΟΥ ΠΛΕΙΟΝΟΣ,
Ο ΟΝ ΕΛΑΤΤΟΝ.

Lesson for life:

Whatever is smaller,
it must fit into the greater
in one of two ways.

Proposition 5

Statement. If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.

Proof.

Action 1. For let a number A be part of a (number) BC, and another (number) D (be) the same part of another (number) EF that A (is) of BC.

Action 2. I say that the sum A, D is also the same part of the sum BC, EF that A (is) of BC.

Action 3. For since which(ever) part A is of BC, D is the same part of EF , thus as many numbers as are in BC equal to A, so many numbers are also in EF equal to D.

Action 4. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF , equal to D.

Action 5. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF .

Action 6. And since BG is equal to A, and EH to D, thus BG, EH (is) also equal to A, D.

Action 7. So, for the same (reasons), GC, HF (is) also (equal) to A, D.

Action 8. Thus, as many numbers as (are) in BC equal to A, so many are also in BC, EF equal to A, D.

Action 9. Thus, as many times as BC is (divisible) by A, so many times is the sum BC, EF also (divisible) by the sum A, D.

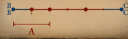
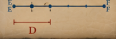
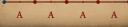
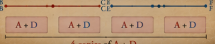
Action 10. Thus, which(ever) part A is of BC, the sum A, D is also the same part of the sum BC, EF .

Action 11. (Which is) the very thing it was required to show. = $(1/n)$ d then $(a + c) = (1/n) (b + d)$, where all symbols denote numbers.

Therefore, If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.

EUCLID, BOOK VII — PROPOSITION 5

“If one number is a part of another, and a second number is the same part of another, then the sum of the first pair is that same part of the sum of the second pair.”

1 Let A be a part of BC. So BC contains some exact number of copies of A. 	2 Let D be the same part of EF that A is of BC. So if A is one third of BC, D is also one third of EF. 	3 We must show that $A + D$ is the same part of $BC + EF$ that A is of BC. In symbols: $A + D$ is the same part of $BC + EF$ that A is of BC.
4 Divide BC into equal pieces, each equal to A. Suppose there are 4 copies of A. 	5 Divide EF into the same number of equal pieces, each equal to D. Since D is the same part of EF that A is of BC, there are also 4 copies of D. 	6 Pair corresponding pieces. Take one piece equal to A from BC and one piece equal to D from EF. Each pair has size $A + D$. 
7 Every paired block is equal to $A + D$. Since each piece from BC equals A, and each corresponding piece from EF equals D, each pair is $A + D$. And there are 4 such pairs.  4 pairs	8 Therefore the combined whole contains the same number of copies of $A + D$. $BC + EF = 4(A + D)$  4 copies of $A + D$	9 Thus $A + D$ is the same part of $BC + EF$ that A is of BC. In symbols: If A is $\frac{1}{3}$ of BC and D is $\frac{1}{3}$ of EF, then $A + D$ is $\frac{1}{3}$ of $BC + EF$.

ARISTOTLE:
Equal parts in equal wholes keep their proportion even when the wholes are added.

Example with numbers:
3 is one fourth of 12. 5 is one fourth of 20.
 $3 + 5 = 8$ and $12 + 20 = 32$ so indeed 8 is one fourth of 32.

LESSON FOR LIFE:
When relationships are the same, they remain the same under addition.

Proposition 6

Statement. If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.

Proof.

- Action 1.** For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another (number) F that AB (is) of C.
- Action 2.** I say that the sum AB, DE is also the same parts of the sum C, F that AB (is) of C.
- Action 3.** For since which(ever) parts AB is of C, DE (is) also the same parts of F , thus as many parts of C as are in AB, so many parts of F are also in DE.
- Action 4.** Let AB have been divided into the parts of C, AG and GB, and DE into the parts of F , DH and HE.
- Action 5.** So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE.
- Action 6.** And since which(ever) part AG is of C, DH is also the same part of F , thus which(ever) part AG is of C, the sum AG, DH is also the same part of the sum C, F (Prop. VII.5).
- Action 7.** And so, for the same (reasons), which(ever) part GB is of C, the sum GB, HE is also the same part of the sum C, F .
- Action 8.** Thus, which(ever) parts AB is of C, the sum AB, DE is also the same parts of the sum C, F .
- Action 9.** (Which is) the very thing it was required to show. $d \text{ c} = (m/n) d$ then $(a + c) = (m/n) (b + d)$, where all symbols denote

Therefore, If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.

EUCLID, BOOK VII — PROPOSITION 6

"If one number is parts of another, and a second number is the same parts of another, then the sum of the first pair is the same parts of the sum of the second pair."

1

Let AB be certain parts of C.
Here AB is two thirds of C.

So C is divided into three equal parts, and AB consists of two of them.

2

Let DE be the same parts of F that AB is of C.
Thus DE is also two thirds of F.

AB : C and DE : F are in the same ratio (two thirds).

3

We must show that AB + DE is the same parts of C + F that AB is of C.

4

Divide AB into its parts of C.
Since AB is two thirds of C, let each part be one third of C.

AG and GB are each one third of C.

5

Divide DE into the same parts of F. Each part is one third of F.

DH and HE are each one third of F.

6

Pair corresponding parts.
Pair AG with DH, and GB with HE.

7

By Proposition 5, each pair of corresponding parts added together is the same part of C + F, that each part is of C.

AG + DH → $\frac{C}{3}$ (one third of C + F)

GB + HE → $\frac{C}{3}$ (one third of C + F)

8

Add the paired parts together.
(AG + DH) + (GB + HE) = AB + DE

These two equal one-third parts of C + F together make two-thirds of C + F.

9

Therefore AB + DE is the same parts of C + F that AB is of C.

AB + DE is two thirds of C + F

In symbols:
If $a = \frac{m}{n} b$ and $c = \frac{m}{n} d$, then $a + c = \frac{m}{n} (b + d)$.

Example with numbers:
Let AB = 8 be two thirds of C = 12.
Let DE = 10 be two thirds of F = 15.

Indeed, 18 is two thirds of 27.
So the proposition is true.

Proposition 7

Statement. If a number is that part of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same part of the remainder that the whole (is) of the whole.

Proof.

Action 1. For let a number AB be that part of a number CD that a (part) taken away AE (is) of a part taken away CF .

Action 2. I say that the remainder EB is also the same part of the remainder F D that the whole AB (is) of the whole CD.

Action 3. For which(ever) part AE is of CF , let EB also be the same part of CG.

Action 4. And since which(ever) part AE is of CF , EB is also the same part of CG, thus which(ever) part AE is of CF , AB is also the same part of GF (Prop. VII.5).

Action 5. And which(ever) part AE is of CF , AB is also assumed (to be) the same part of CD.

Action 6. Thus, also, which(ever) part AB is of GF , (AB) is also the same part of CD. Thus, GF is equal to CD.

Action 7. Let CF have been subtracted from both.

Action 8. Thus, the remainder GC is equal to the remainder F D.

Action 9. And since which(ever) part AE is of CF , EB (is) also the same part of GC, and GC (is) equal to F D, thus which(ever) part AE is of CF , EB is also the same part of F D.

Action 10. But, which(ever) part AE is of CF , AB is also the same part of CD.

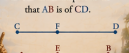
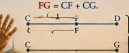
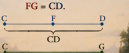
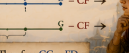
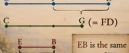
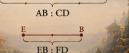
Action 11. Thus, the remainder EB is also the same part of the remainder F D that the whole AB (is) of the whole CD.

Action 12. (Which is) the very thing it was required to show. = $(1/n)$ d then $(a - c) = (1/n) (b - d)$, where all symbols denote numbers.

Therefore, If a number is that part of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same part of the remainder that the whole (is) of the whole.

EUCLID, BOOK VII — PROPOSITION 7

"If a number is that part of a number that a part taken away is of a part taken away, then the remainder will also be the same part of the remainder that the whole is of the whole."

1 Let AB be a part of CD.  AB has some fixed fractional relation to CD.	2 Take away matching parts. Take AE from AB, and CF from CD, with AE the same part of CF that AB is of CD.  $AE : CF = AB : CD$	3 We must prove that the remainder EB is the same part of the remainder FD that AB is of CD. 
4 Let EB be the same part of some number CG that AE is of CF.  $EB : CG = AE : CF$	5 Since AE is a part of CF, and EB is the same part of CG, by Proposition 5: $AB = AE + EB$ is the same part of $FG = CF + CG$.  $AB : FG = AE : CF$	6 But AB is already the same part of CD that AE is of CF. Therefore AB is the same part of both FG and CD. Hence: $FG = CD$. 
7 Subtract the same part CF from both FG and CD. Since the wholes are equal, the remainders are equal.  Therefore $CG = FD$.	8 We constructed EB to be the same part of CG that AE is of CF. Since $CG = FD$, it follows that EB is the same part of FD. EB is the same part of FD. 	9 Therefore the remainder EB is the same part of the remainder FD that AB is of CD. Thus the proposition is proved. 
In symbols: If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, then $a - c = \frac{1}{n} (b - d)$.	Example with numbers Let 6 be one half of 12, and 2 be one half of 4. $6 = \frac{1}{2} 12$ and $2 = \frac{1}{2} 4 \rightarrow 6 - 2 = 4$ and $12 - 4 = 8 \rightarrow 4 = \frac{1}{2} 8$	Lesson for life: When we remove equal portions from things that keep the same proportion, the balance of their relationship remains.

Proposition 8

Statement. If a number is those parts of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same parts of the remainder that the

Proof.

Action 1. whole (is) of the whole.

Action 2. G M K NH For let a number AB be those parts of a number CD that a (part) taken away AE (is) of a (part) taken away CF .

Action 3. I say that the remainder EB is also the same parts of the remainder F D that the whole AB (is) of the whole CD.

Action 4. For let GH be laid down equal to AB.

Action 5. Thus, which(ever) parts GH is of CD, AE is also the same parts of CF .

Action 6. Let GH have been divided into the parts of CD, GK and KH, and AE into the part of CF , AL and LE.

Action 7. So the multitude of (divisions) GK, KH will be equal to the multitude of (divisions) AL, LE.

Action 8. And since which(ever) part GK is of CD, AL is also the same part of CF , and CD (is) greater than CF , GK (is) thus also greater than AL. Let GM be made equal to AL.

Action 9. Thus, which(ever) part GK is of CD, GM is also the same part of CF .

Action 10. Thus, the remainder M K is also the same part of the remainder F D that the whole GK (is) of the whole CD (Prop. VII.5).

Action 11. Again, since which(ever) part KH is of CD, EL is also the same part of CF , and CD (is) greater than CF , HK (is) thus also greater than EL. Let KN be made equal to EL.

Action 12. Thus, which(ever) part KH (is) of CD, KN is also the same part of CF .

Action 13. Thus, the remainder NH is also the same part of the remainder FD that the whole KH (is) of the whole CD (Prop. VII.5).

Action 14. And the remainder MK was also shown to be the same part of the remainder FD that the whole GK (is) of the whole CD .

Action 15. Thus, the sum MK, NH is the same parts of DF that the whole HG (is) of the whole CD .

Action 16. And the sum MK, NH (is) equal to EB , and HG to BA .

Action 17. Thus, the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD .

Action 18. (Which is) the very thing it was required to show. $d \mid c = (m/n) d$ then $(a - c) = (m/n) (b - d)$, where all symbols denote

Therefore, If a number is those parts of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same parts of the remainder that the

EUCLID, BOOK VII — PROPOSITION 8

"If a number is those parts of a number that a part taken away is of a part taken away, then the remainder will also be the same parts of the remainder that the whole is of the whole."

1

Let AB be the same parts of C that DE is of F.
(for example, two thirds)
So $AB = \frac{2}{3}C$ and $DE = \frac{2}{3}F$.

2

Take away the matching parts.
Remove DE from AB, and remove the corresponding part from C.
That is, $EB = \frac{2}{3}CG$.

3

We must prove that the remainder EB is the same parts of the remainder CG that AB is of C.
That is, $EB = \frac{2}{3}CG$.

4

Divide AB into the parts of C of which it is made.
Since AB is two thirds of C, divide it into two equal parts.
Thus each of AM and MB is one third of C.

5

Divide DE into the same parts of F.
Since DE is two thirds of F, divide it into two equal parts.
DN and NE are each one third of F.

6

Now apply Proposition 7 to each corresponding pair.
Since AM is the same part of C that DN is of F, the remainders after removing these parts are also in the same relation.
So MB is one third of EG.

7

Since MB is one third of EG, and MB is the same part of C that AM is of C, it follows that EG is also equal to CG. Because $AM + MB = AB = \frac{2}{3}C$.
Therefore $EG = CG$.

8

Now look at the whole remainders after removal.
We removed two thirds from both wholes.
Since $EG = CG$, the remainders stand in the same relation.
So the remainder EB is in the same relation to CG as AB is to C.

9

Therefore the remainder EB is the same parts of the remainder CG that AB is of C.
The proposition is proved.

In symbols:
If $a = \frac{m}{n}b$ and $c = \frac{m}{n}d$,
then $a - c = \frac{m}{n}(b - d)$.

EXAMPLE WITH NUMBERS

Let $AB = 8$ be two thirds of $C = 12$.
Let $DE = 4$ be two thirds of $F = 6$.

Before subtraction

$C = 12$
 $F = 6$

After subtracting matching parts

$12 - 4 = 8$
 $6 - 2 = 4$

Remainder = 8
Remainder = 4

LESSON FOR LIFE

Remove equal portions from things that keep the same proportion, and the balance of that proportion remains.

Proposition 9

Statement. If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or

Proof.

Action 1. the same parts, of the fourth.

Action 2. For let a number A be part of a number BC, and another (number) D (be) the same part of another EF that A (is) of BC.

Action 3. I say that, also, alternately, which(ever) part, or parts, A is of D, BC is also the same part, or parts, of EF .

Action 4. For since which(ever) part A is of BC, D is also the same part of EF , thus as many numbers as are in BC equal to A, so many are also in EF equal to D.

Action 5. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF , equal to D.

Action 6. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF .

Action 7. And since the numbers BG and GC are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) BG, GC is equal to the multitude of (divisions) EH, HC, thus which(ever) part, or parts, BG is of EH, GC is also the same part, or the same parts, of HF .

Action 8. And hence, which(ever) part, or parts, BG is of EH, the sum BC is also the same part, or the same parts, of the sum EF (Props. 7.5, 7.6).

Action 9. And BG (is) equal to A, and EH to D.

Action 10. Thus, which(ever) part, or parts, A is of D, BC is also the same part, or the same parts, of EF .

Action 11. (Which is) the very thing it was required to show. $= (1/n)$
 d then if $a = (k/l) c$ then $b = (k/l) d$, where all symbols
denote

Therefore, If a number is part of a number, and another (number) is
the same part of another, also, alternately, which(ever) part, or parts,
the first (number) is of the third, the second (number) will also be
the same part, or

EUCLID, BOOK VII — PROPOSITION 9

"If a number is part of a number, and another number is the same part of another, then alternately, whatever part or parts the first is of the third, the second will be the same part or parts of the fourth."

1

Let A be a part of BC.
Let D be the same part of EF.

So as A is of BC, so is D of EF.

2

We must prove alternately:
whatever part or parts
A is of D,
BC is the same part or parts
of EF.

In symbols:
If $A : D = k : l$,
then $BC : EF = k : l$.

3

Since A is a part of BC,
let us divide BC into equal
parts, each equal to A.

Suppose there are 4 such parts.

4

Since D is the same part of EF
that A is of BC, divide EF also
into the same number of
equal parts, each equal to D.

Also 4 parts.

5

Compare one part from each.
Here BG is equal to A,
and EH is equal to D.

Whatever part or
parts A is of D,
BG is the same part
or parts of EH.

6

Every other pair of corresponding
parts is equal to the first pair.
So GC is equal to BG (both A),
and HF is equal to EH (both D).

Therefore,
whatever part or parts
BG is of EH,
GC is of HF,
and so on.

7

Now add the corresponding parts.
By Proposition 5 (or 6),
if BG is a certain part or parts
of EH, then the sum of all the
BG, GC, ... is the same part or
parts of the sum of all the
EH, HF, ...

Thus BC is the
same part or parts
of EF.

8

But $BG = A$ and $EH = D$.
So whatever part or parts
A is of D,
BC is the same part or parts
of EF.

That is,
the relation is preserved
in the larger numbers.

9

Therefore, if A is k parts
of D out of l parts,
then BC is also k parts
of EF out of l parts.

In symbols:
If $A : D = k : l$,
then $BC : EF = k : l$.

ARISTOTLE:
See, Alexander, when two pairs
share the same scale, the relation
between the smaller numbers is
the same as between the wholes.

EXAMPLE WITH NUMBERS

Let $A = 3$ is one fourth of $BC = 12$.
Let $D = 5$ is one fourth of $EF = 20$.

Since A is three fifths of D (3 : 5),
then BC is three fifths of EF.

$3 : 5 = 12 : 20$

LESSON FOR LIFE:
Equal proportions run through
many things in nature and life.
Understand one, and you can
understand its greater whole.

Proposition 10

Statement. If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

Proof.

Action 1. For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another F .

Action 2. I say that, also, alternately, which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F .

Action 3. For since which(ever) parts AB is of C, DE is also the same parts of F , thus as many parts of C as are in AB, so many parts of F (are) also in DE.

Action 4. Let AB have been divided into the parts of C, AG and GB, and DE into the parts of F , DH and HE.

Action 5. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE.

Action 6. And since which(ever) part AG is of C, DH is also the same part of F , also, alternately, which(ever) part, or parts, AG is of DH, C is also the same part, or the same parts, of F (Prop. VII.9).

Action 7. And so, for the same (reasons), which(ever) part, or parts, GB is of HE, C is also the same part, or the same parts, of F (Prop. VII.9).

Action 8. And so (which(ever) part, or parts, AG is of DH, GB is also the same part, or the same parts, of HE.

Action 9. And thus, which(ever) part, or parts, AG is of DH, AB is also the same part, or the same parts, of DE (Props. 7.5, 7.6).

Action 10. But, which(ever) part, or parts, AG is of DH, C was also shown (to be) the same part, or the same parts, of F .

Action 11. And, thus) which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F .

Action 12. (Which is) the very thing it was required to show. = (m/n) d then if $a = (k/l) c$ then $b = (k/l) d$, where all symbols denote

Therefore, If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

EUCLID, BOOK VII — PROPOSITION 10

"If one number is parts of another, and a second number is the same parts of another, then alternately, whatever parts or part the first is of the third, the second will be the same parts or the same part of the fourth."

1 Let AB be certain parts of C, and let DE be the same parts of F.

So $AB = \frac{m}{n} C$, and $DE = \frac{m}{n} F$.

2 We must prove alternately: whatever parts or part AB is of DE, C is the same parts or part of F.

In symbols:
If $AB : DE = k : l$,
then $C : F = k : l$.

3 Divide AB into the individual parts of C of which it is made. Suppose AB consists of three equal parts.

$AB = AG + GB$

4 Divide DE into the matching individual parts of F.

$DE = DH + HE$

5 AG is the same single part of C that DH is of F.

By Proposition 9, alternately, whatever part or parts AG is of DH, C is the same parts or part of F.

$AG : DH = k : l \Rightarrow C : F = k : l$

6 In the same way, GB is to HE as C is to F.

$GB : HE = k : l \Rightarrow C : F = k : l$

7 Therefore every corresponding pair of parts has the same relation. Whatever part or parts AG is of DH, GB is of HE, and so on.

$AG : DH = k : l$
 $GB : HE = k : l$
 $\vdots$

8 Add the corresponding parts. By Proposition 5 and 6, whatever part or parts AG is of DH, AB is the same parts or parts of DE.

$AB : DE = k : l$

9 But the relation $AG : DH$ is the same as $C : F$ (from step 5). Therefore C is the same parts or part of F that AB is of DE.

Thus proved.
 $C : F = k : l$

ALTERNATE PROPORTION IN MANY PARTS

If two pairs contain the same collection of fractional parts, their crosswise ratio is preserved.

EXAMPLE WITH NUMBERS

Let $AB = 8$ be two thirds of $C = 12$.
 $AB = \frac{2}{3} C$

Let $DE = 10$ be two thirds of $F = 15$.
 $DE = \frac{2}{3} F$

Compare AB and DE.
 8 is four fifths of 10 .
 $AB : DE = 4 : 5$

Therefore by Proposition 10, C is four fifths of E.
 $C : F = 4 : 5$

Indeed, 12 is four fifths of 15.

Proposition 11

Statement. If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

Proof.

Action 1. Let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF .

Action 2. I say that the remainder EB is to the remainder F D as the whole AB (is) to the whole CD.

Action 3. (For) since as AB is to CD, so AE (is) to CF , thus which(ever) part, or parts, AB is of CD, AE is also the same part, or the same parts, of CF (Def. VII.20).

Action 4. Thus, the remainder EB is also the same part, or parts, of the remainder F D that AB (is) of CD (Props. 7.7, 7.8).

Action 5. Thus, as EB is to F D, so AB (is) to CD (Def. VII.20).

Action 6. (Which is) the very thing it was required to show. : $b :: a : c$: $b : d$, where all symbols denote numbers.

Therefore, If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

EUCLID, BOOK VII — PROPOSITION 11

"If the whole is to the whole as a part taken away is to a part taken away, then the remainder will also be to the remainder as the whole is to the whole."

1

Start with two wholes. Let the whole AB be to the whole CD in some fixed ratio.



So $AB : CD$ is some ratio.

2

Take away corresponding parts. Remove AE from AB, and remove CF from CD. Assume $AE : CF = AB : CD$.



The removed parts stand in the same ratio as the wholes.

3

Name the remainders. After subtraction, the remainders are:



$EB = AB - AE$
 $FD = CD - CF$

4

State what must be proved. We must show that the remainders have the same ratio as the wholes.

That is,
 $EB : FD = AB : CD$

5

Translate the given ratio into "part or parts". Since $AB : CD = AE : CF$, by Definition 20, whatever part or parts AB is of CD, AE is the same part or parts of CF.

6

Apply the subtraction results already proved. By Propositions 7 and 8, when corresponding quantities have the same part-or-parts relationship, subtracting them preserves that relationship in the remainders.

Therefore EB is the same part or parts of FD that AB is of CD.

7

Translate back into a ratio. By Definition 20 again, if EB is the same part or parts of FD that AB is of CD, then:
 $EB : FD = AB : CD$

8

Conclude. Therefore the remainder has the same ratio as the wholes.

So
 $EB : FD = AB : CD$

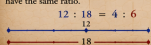
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In symbolic: If $a : b = c : d$, then
 $a : b = (a - c) : (b - d)$.

EXAMPLE WITH NUMBERS

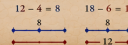
Suppose the wholes and the removed parts have the same ratio.

$12 : 18 = 4 : 6 \Rightarrow$



Subtract corresponding parts.

$12 - 4 = 8$ $18 - 6 = 12$



The remainders have the same ratio.

$8 : 12 = 12 : 18$
(that is, $2 : 3 = 2 : 3$)

LESSON FOR LIFE

When we remove matching parts from things that stand in the same proportion, the balance of their relationship remains unchanged.

79

Proposition 12

Statement. If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.

Proof.

Action 1. Let any multitude whatsoever of numbers, A, B, C, D, be proportional, (such that) as A (is) to B, so C (is) to D.

Action 2. I say that as A is to B, so A, C (is) to B, D.

Action 3. For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or parts, of D (Def. VII.20).

Action 4. Thus, the sum A, C is also the same part, or the same parts, of the sum B, D that A (is) of B (Props. 7.5, 7.6).

Action 5. Thus, as A is to B, so A, C (is) to B, D (Def. VII.20).

Action 6. (Which is) the very thing it was required to show. : $b :: a + c : b + d$, where all symbols denote numbers.

Therefore, If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.

EUCLID, BOOK VII — PROPOSITION 12

"If any multitude whatsoever of numbers are proportional, then as one of the leading numbers is to one of the following, so the sum of all the leading numbers will be to the sum of all the following."

1

Let A, B, C, D be numbers such that A is to B as C is to D.

$\frac{A}{2} \quad \frac{B}{3}$

$\frac{C}{4} \quad \frac{D}{6}$

So $A : B = C : D = 2 : 3$.

ARISTOTLE: Here, Alexander, two pairs of numbers share the very same proportion.

2

By Definition 20, whatever part or parts A is of B, C is the same part or parts of D.

$\frac{A}{2} \quad \frac{B}{3}$

$\frac{C}{4} \quad \frac{D}{6}$

A is $\frac{2}{3}$ of B.
C is $\frac{2}{3}$ of D.

ARISTOTLE: See how A is two thirds of B, so C is also two thirds of D.

3

Add the leading numbers together. Join A and C.

$\frac{A}{2} + \frac{C}{4}$

$\frac{A+C}{6}$

ARISTOTLE: Now we place the leading numbers end to end, A with C.

4

Add the following numbers together. Join B and D.

$\frac{B}{3} + \frac{D}{6}$

$\frac{B+D}{9}$

ARISTOTLE: And in the same way we join B with D.

5

Since A is the same part or parts of B that C is of D, by Propositions 5 and 6, A + C is the same part or parts of B + D that A is of B.

$\frac{A+C}{6}$

$\frac{B+D}{9}$

$A + C$ is $\frac{2}{3}$ of $B + D$.

ARISTOTLE: Their sums preserve exactly the same relationship.

6

By Definition 20 again, this means that the ratio of A to B is the same as the ratio of A + C to B + D.

$\frac{A : B = (A + C) : (B + D)}{\text{or}} \frac{A}{B} = \frac{A + C}{B + D}$

ARISTOTLE: Therefore the proportion remains unchanged when we add the corresponding numbers.

7

Let us see this with numbers.

A = 2, B = 3, C = 4, D = 6
A : B = 2 : 3 C : D = 4 : 6

$A + C = 2 + 4 = 6$

$B + D = 3 + 6 = 9$

So $A : B = 2 : 3 = 6 : 9 = (A + C) : (B + D)$.

ARISTOTLE: The same ratio appears again!

8

This is true not only for two pairs, but for any multitude whatsoever.

$\frac{A_1}{B_1} \quad \frac{A_2}{B_2} \quad \frac{A_3}{B_3} \quad \dots \quad \frac{A_n}{B_n}$

ARISTOTLE: However many proportional pairs we have, they all share one and the same proportion.

9

Join all the leading numbers together, and all the following numbers together.

$A_1 \quad A_2 \quad A_3 \quad \dots \quad A_n = \sum A_i$

$B_1 \quad B_2 \quad B_3 \quad \dots \quad B_n = \sum B_i$

Then $\frac{A_1}{B_1} = \frac{A_2}{B_2} = \dots = \frac{A_n}{B_n} = \frac{\sum A_i}{\sum B_i}$

That is, the sums are in the same proportion.

ARISTOTLE: Many proportions gathered together become one great proportion, unchanged.

LESSON FOR LIFE: When many things stand in the same proportion, their balance is preserved, even when we bring them together.

Proposition 13

Statement. If four numbers are proportional then they will also be proportional alternately.

Proof.

Action 1. Let the four numbers A, B, C, and D be proportional, (such that) as A (is) to B, so C (is) to D.

Action 2. I say that they will also be proportional alternately, (such that) as A (is) to C, so B (is) to D.

Action 3. For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or the same parts, of D (Def. VII.20).

Action 4. Thus, alterately, which(ever) part, or parts, A is of C, B is also the same part, or the same parts, of D (Props. 7.9, 7.10).

Action 5. Thus, as A is to C, so B (is) to D (Def. VII.20).

Action 6. (Which is) the very thing it was required to show. $a : c :: b : d$, where all symbols denote numbers.

Therefore, If four numbers are proportional then they will also be proportional alternately.

EUCLID, BOOK VII — PROPOSITION 13

"If four numbers are proportional, then they will also be proportional alternately."

1

Let four numbers A, B, C, D be such that A is to B as C is to D.

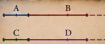
$$A : B = C : D$$


This is our starting proportion.

2

By Definition 20, whatever part or parts A is of B, C is the same part or parts of D.

If A is $\frac{k}{n}$ of B, then C is $\frac{k}{n}$ of D.



They share exactly the same part-of-whole relationships.

3

Now let us compare crosswise: A with C, and B with D.

Crosswise pairs



We change the pairing of the numbers.

4

By Propositions 9 and 10, since A is the same part or parts of B that C is of D, therefore whatever part or parts A is of C, B is the same part or parts of D.

The crosswise relation inherits the same equality of parts.

5

Whatever part or parts A is of C, B is the same part or parts of D.



So the crosswise proportion holds true.

6

By Definition 20 again, this means

$$A : C = B : D$$

So the crosswise proportion holds true.



7

Let us see with numbers. Take A = 2, B = 3, C = 4, D = 6.

$$A : B = 2 : 3 = 4 : 6 = C : D$$


The first proportion is true.

8

Alternate (compare crosswise):

$$A : C = 2 : 4$$
$$B : D = 3 : 6$$

And indeed,

$$2 : 4 = 3 : 6 = 1 : 2.$$


The crosswise proportion is also true.

9

Therefore,

If $A : B = C : D$, then $A : C = B : D$.

Four numbers proportional remain proportional when paired alternately.

TODAY'S LESSON:
A true proportion keeps its truth even when we look at it from the other diagonal.

THE RULE OF ALTERNATION

$$A : B = C : D \Rightarrow A : C = B : D$$

ARISTOTLE TO ALEXANDER:
In mathematics, as in life, relationships remain constant though the view may change.

83

Proposition 14

Statement. If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality. Let there be any multitude of numbers whatsoever, A, B, C, and (some) other (numbers), D, E, F, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that also, via equality, as A is to C, so D (is) to F.

Proof.

Action 1. For since as A is to B, so D (is) to E, thus, alternately, as A is to D, so B (is) to E (Prop. VII.13).

Action 2. Again, since as B is to C, so E (is) to F, thus, alternately, as B is to E, so C (is) to F (Prop. VII.13).

Action 3. And as B (is) to E, so A (is) to D.

Action 4. Thus, also, as A (is) to D, so C (is) to F.

Action 5. Thus, alternately, as A is to C, so D (is) to F (Prop. VII.13).

Action 6. (Which is) the very thing it was required to show. $c :: e : f$ then $a : c :: d : f$, where all symbols denote numbers.

Therefore, If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality. Let there be any multitude of numbers whatsoever, A, B, C, and (some) other (numbers), D, E, F, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that also, via equality, as A is to C, so D (is) to F.

EUCLID, BOOK VII — PROPOSITION 14

"If there are any number of numbers, and another set equal in multitude, and the corresponding pairs are in the same ratio, then the first and last numbers are also in the same ratio via equality."

1

Take two sets of numbers, equal in number.

Three in each set.

2

Assume the first pair of numbers are in the same ratio.

$A : B = D : E$

The ratio from A to B matches that from D to E.

3

Assume the second pair are also in the same ratio.

$B : C = E : F$

The ratio from B to C matches that from E to F.

4

By Proposition 13 (alternation), from $A : B = D : E$ we get:

$A : D = B : E$

We compared crosswise.

5

Again by Proposition 13, from $B : C = E : F$ we get:

$B : E = C : F$

We compared crosswise again.

6

Now look carefully. $A : D = B : E$ and $B : E = C : F$. Therefore both $A : D$ and $C : F$ are equal to the same ratio $B : E$. Hence, $A : D = C : F$.

$A : D = B : E = C : F$

(They are all equal.)

7

Once more, by Proposition 13 (alternate) on $A : D = C : F$, we get:

$A : C = D : F$

The first numbers compared to the last numbers have the same ratio.

8

Let us see with numbers. Take $A = 2, B = 4, C = 8$ and $D = 3, E = 6, F = 12$

	A	B	C
Top line	2	4	8
	D	E	F
Bottom line	3	6	12

Check:

$A : B = 2 : 4 = 1 : 2$
 $D : E = 3 : 6 = 1 : 2$
 $B : C = 4 : 8 = 1 : 2$
 $E : F = 6 : 12 = 1 : 2$

The successive ratios match.

9

Therefore, $A : C = 2 : 8 = 1 : 4$
 $D : F = 3 : 12 = 1 : 4$
 So $A : C = D : F$.

The first and last numbers are in the same ratio via equality.

THE MEANING

If each step in two chains has the same ratio, then the end-to-end ratio also matches.

Two chains with matching step ratios

Therefore $A : C = D : F$ (via equality)

ARISTOTLE TO ALEXANDER.

When relationships are consistent at every step, truth reaches from the beginning all the way to the end.

Proposition 15

Statement. If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.

Proof.

Action 1. For let a unit A measure some number BC, and let another number D measure some other number EF as many times.

Action 2. I say that, also, alternately, the unit A also measures the number D as many times as BC (measures) EF .

Action 3. For since the unit A measures the number BC as many times as D (measures) EF , thus as many units as are in BC, so many numbers are also in EF equal to D.

Action 4. Let BC have been divided into its constituent units, BG, GH, and HC, and EF into the (divisions) EK, KL, and LF , equal to D.

Action 5. So the multitude of (units) BG, GH, HC will be equal to the multitude of (divisions) EK, KL, LF .

Action 6. And since the units BG, GH, and HC are equal to one another, and the numbers EK, KL, and LF are also equal to one another, and the multitude of the (units) BG, GH, HC is equal to the multitude of the numbers EK, KL, LF , thus as the unit BG (is) to the number EK, so the unit GH will be to the number KL, and the unit HC to the number LF .

Action 7. And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following (Prop. VII.12).

Action 8. Thus, as the unit BG (is) to the number EK, so BC (is) to EF .

Action 9. And the unit BG (is) equal to the unit A, and the number EK to the number D.

Action 10. Thus, as the unit A is to the number D, so BC (is) to EF

.

Action 11. Thus, the unit A measures the number D as many times as BC (measures) EF (Def. VII.20).

Action 12. (Which is) the very thing it was required to show.

Therefore, If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.

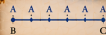
EUCLID, BOOK VII — PROPOSITION 15

“If a unit measures some number, and another number measures some other number the same number of times, then alternately, the unit will measure the third number as many times as the second number measures the fourth.”

1

Let a unit A measure a number BC .

Suppose the unit A fits into BC a certain number of times.

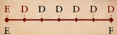


Example: If $A = 1$ unit, and $BC = 6$ units, then A fits into BC six times.

2

Let another number D measure another number EF the same number of times.

So D fits into EF exactly the same number of times as A fits into BC .



Example: If $D = 4$ units, and $EF = 24$ units, then D fits into EF six times.

3

We must show, alternately, that the unit A measures D as many times as BC measures EF .

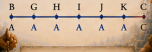
What we want to prove:

A measures D as many times as BC measures EF .

4

Divide BC into unit pieces.

Cut BC into parts equal to the unit A .

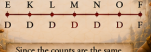


There are six unit pieces.

5

Divide EF into parts equal to D .

Cut EF into parts each equal to D .



Since the counts are the same, there are also six D -pieces.

6

Compare corresponding pieces.

Every unit piece is equal to every unit piece, and every D -piece is equal to every D -piece.

So each small pair has the same ratio:

$BG : EK = GH : KL = HI : LM$
 $IJ : MN = JK : NO = KC : OF$
(All are $A : D$.)

7

Apply Proposition 12.

If many pairs have the same ratio, then one of the pairs is to one of the pairs as the sum of the first terms is to the sum of the second terms.

$BC : EK = BC : EF$

The sum of the unit pieces is BC , and the sum of the D -pieces is EF .

8

Replace the small pieces by the original numbers.

$BG = A, EK = D$.

Therefore,

$A : D = BC : EF$

9

By Definition 20, this means that the unit A measures D as many times as BC measures EF .

That is what we wanted to prove.

SEE IT WITH NUMBERS

Unit $A = 1$	Number $D = 4$	Crosswise:	
1 fits into 6 six times.	4 fits into 24 six times.	$\Rightarrow$	1 fits into 4 four times. And 6 fits 24 four times.

THE IDEA

Equal repetition counts produce equal crosswise measurement counts.

ARISTOTLE TO ALEXANDER

Mathematics shows us that when the same measure is used equally, the balance of relationships holds true in every direction.

88

Proposition 16

Statement. If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.

Proof.

Action 1. Let A and B be two numbers.

Action 2. And let A make C (by) multiplying B, and let B make D (by) multiplying A. I say that C is equal to D.

Action 3. For since A has made C (by) multiplying B, B thus measures C according to the units in A (Def. VII.15).

Action 4. And the unit E also measures the number A according to the units in it.

Action 5. Thus, the unit E measures the number A as many times as B (measures) C.

Action 6. Thus, alternately, the unit E measures the number B as many times as A (measures) C (Prop. VII.15).

Action 7. Again, since B has made D (by) multiplying A, A thus measures D according to the units in B (Def. VII.15).

Action 8. And the unit E also measures B according to the units in it.

Action 9. Thus, the unit E measures the number B as many times as A (measures) D.

Action 10. And the unit E was measuring the number B as many times as A (measures) C.

Action 11. Thus, A measures each of C and D an equal number of times. Thus, C is equal to D.

Action 12. (Which is) the very thing it was required to show. symbols denote numbers.

Modern reading.

$$A \times B = B \times A.$$

Therefore, If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.

EUCLID, BOOK VII — PROPOSITION 16

"If two numbers multiplying one another make some numbers, then the numbers generated from them will be equal to one another."

1

Let A and B be two numbers.
Let A multiplying B make C.
Let B multiplying A make D.
 $A \times B = C$
 $B \times A = D$
I say that C and D are equal.
 $C = D$
This is what we must prove.

2

Since $A \times B = C$,
by Definition 15,
the number B measures C
as many times as there are
units in A.

B repeated the same number of times
(the number of units in A) produces C.

3

Let E be a unit.
The unit E measures A
as many times as there are
units in A.
 $E \ E \ E \ E \dots E$

Therefore, E measures A the same number of
times that B measures C.

4

By Proposition 15 (alternately),
E measures B as many times
as A measures C.
 $E \ E \ E \ E \dots E$

 $A \ A \ A \dots A$

We have changed the comparison.

5

Now, since $B \times A = D$,
by Definition 15,
A measures D according to
the number of units in B.
 $A \ A \ A \dots A$

A repeated the same number of times
(the number of units in B) produces D.

6

The unit E measures B
according to the number
of units in B.
 $E \ E \ E \ E \dots E$

 $A \ A \ A \dots A$

Therefore, E measures B the same number of
times that A measures D.

7

From (4) E measures B as many times
as A measures C.
From (6) E measures B as many times
as A measures D.
Therefore, A measures C
and D the same number of times.
 $A \ A \ A \dots A$

 $A \ A \ A \dots A$

They are measured equally by the same number.

8

But if the same number
measures two numbers
the same number of times,
the two numbers are equal.
 $C = D$

Thus the numbers produced are equal.

9

Therefore,
 $A \times B = B \times A$

The order of the factors does not change the product.

ARISTOTLE TO ALEXANDER:
In mathematics, truth is steady
and does not depend on order,
but on relation.

THE COMMUTATIVE LAW OF MULTIPLICATION
Two numbers, whatever their order,
produce the same number.

$AB = BA$

ALEXANDER:
So even when things change place,
their harmony remains the same.
How beautiful is reason!

Proposition 17

Statement. If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).

Proof.

Action 1. For let the number A make (the numbers) D and E (by) multiplying the two numbers B and C (respectively).

Action 2. I say that as B is to C, so D (is) to E.

Action 3. For since A has made D (by) multiplying B, B thus measures D according to the units in A (Def. VII.15).

Action 4. And the unit F also measures the number A according to the units in it.

Action 5. Thus, the unit F measures the number A as many times as B (measures) D.

Action 6. Thus, as the unit F is to the number A, so B (is) to D (Def. VII.20).

Action 7. And so, for the same (reasons), as the unit F (is) to the number A, so C (is) to E.

Action 8. And thus, as B (is) to D, so C (is) to E.

Action 9. Thus, alternately, as B is to C, so D (is) to E (Prop. VII.13).

Action 10. (Which is) the very thing it was required to show. then $d : e :: b : c$, where all symbols denote numbers.

Therefore, If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).

EUCLID, BOOK VII — PROPOSITION 17

"If a number multiplying two numbers makes some numbers, then the numbers generated from them will have the same ratio as the multiplied numbers."

1

Start with one multiplier.
Let A multiply two numbers B and C.
So,
 $A \times B = D$
 $A \times C = E$
We must prove that
 $B : C = D : E$

2

Look at the first product.
Since A multiplies B to make D,
B measures D as many times as there are units in A.

3

Introduce a unit F.
Let F be a unit.
The unit F measures A as many times as there are units in A.

4

Compare the measurements.
F measures A the same number of times that B measures D.
By Definition 20, this means
 $F : A = B : D$

5

Repeat for the second product.
Since A multiplies C to make E,
by the same reasoning,
 $F : A = C : E$

6

Compare the two equal ratios.
From (4) and (5),
 $F : A = B : D$
and
 $F : A = C : E$
Therefore,
 $B : D = C : E$

7

Alternate the proportion.
By Proposition 13 (alternation), from $B : D = C : E$ we get
 $B : C = D : E$

8

Interpretation.
Multiplying two numbers by the same number A does not change their ratio.
 $B \rightarrow A \times B = D$
 $C \rightarrow A \times C = E$

Visually:

9

See it with numbers.
Let $A = 4$, $B = 3$, $C = 5$.
Then $D = 4 \times 3 = 12$
and $E = 4 \times 5 = 20$
 $B : C = 3 : 5$
 $D : E = 12 : 20 = 3 : 5$
The ratio is preserved.

	Multiply by A = 4	Result
$B = 3$	3×4	12
$C = 5$	5×4	20
	$3 : 5 = 12 : 20$	

ARISTOTLE TO ALEXANDER:
When the same number is applied to two things, their relationship remains unchanged.

THE PRINCIPLE
Same multiplier $\rightarrow$ same proportion.
 $B : C = (A \times B) : (A \times C)$

"Thus, my boy," said Aristotle, "truth in number is like justice in rule—it treats equals equally."

Proposition 18

Statement. If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).

Proof.

Action 1. For let the two numbers A and B make (the numbers) D and E (respectively, by) multiplying some number C.

Action 2. I say that as A is to B, so D (is) to E.

Action 3. For since A has made D (by) multiplying C, C has thus also made D (by) multiplying A (Prop. VII.16).

Action 4. So, for the same (reasons), C has also made E (by) multiplying B.

Action 5. So the number C has made D and E (by) multiplying the two numbers A and B (respectively).

Action 6. Thus, as A is to B, so D (is) to E (Prop. VII.17).

Action 7. (Which is) the very thing it was required to show. e then $a : b :: d : e$, where all symbols denote numbers.

Therefore, If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).

EUCLID, BOOK VII — PROPOSITION 18

"If two numbers multiplying some number make some other numbers, then the numbers generated from them will have the same ratio as the multiplying numbers."

1

Let A and B be two multiplying numbers, and C a common number.

$A \times C = D$
 $B \times C = E$

We must prove that

$A : B = D : E$

This is what we want to prove.

2

Look at the first product. We are given

$A \times C = D$

$A \times C = D$

A multiplied by C makes D.

3

By Proposition 16 (commutativity), we may reverse the order:

$A \times C = C \times A$

Therefore,

$C \times A = D$

C multiplied by A also makes D.

4

Do the same for the second product. We are given

$B \times C = E$

By Proposition 16,

$B \times C = C \times B$

Therefore,

$C \times B = E$

C multiplied by B also makes E.

5

Now we see: The same number C multiplies both A and B.

$C \times A = D$
 $C \times B = E$

This is exactly the setting of Proposition 17.

6

By Proposition 17: When one number multiplies two numbers, the products have the same ratio as the multiplied numbers.

$A : B = D : E$

Thus the proposition is proved.

7

Interpretation. A and B are scaled by the same factor C. Therefore their ratio does not change.

$A : B$ $D : E$

Same factor on both sides — same balance.

8

See it with numbers. Let A = 3, B = 5, C = 4. Then D = A × C = 3 × 4 = 12 and E = B × C = 5 × 4 = 20

Original numbers	After multiplying by C = 4
A : B = 3 : 5	
D : E = 12 : 20	
$3 : 5 = 12 : 20$	

The ratio is preserved.

9

Conclusion. Multiplying both members of a ratio by the same number preserves the ratio.

$A : B = (A \times C) : (B \times C)$

This is Proposition 18.

ARISTOTLE TO ALEXANDER. Numbers are like lines in a harmony. Change their size together, and their harmony remains.

THE PRINCIPLE Same multiplier → same proportion
If $A \times C = D$ and $B \times C = E$, then $A : B = D : E$

ALEXANDER'S REFLECTION. When the same power acts on both, the relation between them remains forever the same.

95

Proposition 19

Statement. If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

Proof.

Action 1. Let A, B, C, and D be four proportional numbers, (such that) as A (is) to B, so C (is) to D.

Action 2. And let A make E (by) multiplying D, and let B make F (by) multiplying C. I say that E is equal to F .

Action 3. For let A make G (by) multiplying C.

Action 4. Therefore, since A has made G (by) multiplying C, and has made E (by) multiplying D, the number A has made G and E by multiplying the two numbers C and D (respectively).

Action 5. Thus, as C is to D, so G (is) to E (Prop. VII.17).

Action 6. But, as C (is) to D, so A (is) to B.

Action 7. Thus, also, as A (is) to B, so G (is) to E.

Action 8. Again, since A has made G (by) multiplying C, but, in fact, B has also made F (by) multiplying C, the two numbers A and B have made G and F (respectively, by) multiplying some number C.

Action 9. Thus, as A is to B, so G (is) to F (Prop. VII.18).

Action 10. But, also, as A (is) to B, so G (is) to E.

Action 11. And thus, as G (is) to E, so G (is) to F .

Action 12. Thus, G has the same ratio to each of E and F . Thus, E is equal to F (Prop. 5.9). So, again, let E be equal to F .

Action 13. I say that as A is to B, so C (is) to D.

Action 14. For, with the same construction, since E is equal to F , thus as G is to E, so G (is) to F (Prop. 5.7).

Action 15. But, as G (is) to E, so C (is) to D (Prop. VII.17).

Action 16. And as G (is) to F , so A (is) to B (Prop. VII.18).

Action 17. And, thus, as A (is) to B, so C (is) to D.

Action 18. (Which is) the very thing it was required to show.

Modern reading.

$$A : B = C : D \iff AD = BC.$$

Therefore, If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

EUCLID, BOOK VII — PROPOSITION 19
 "If four numbers are proportional, then the product of the first and fourth equals the product of the second and third; and conversely, if those products are equal, then the four numbers are proportional."

1 2 3

Let A multiply C.

We have:

EUCLID, BOOK VII — PROPOSITION 19

"If four numbers are proportional, the product of the first and fourth equals the product of the second and third; and conversely, if those products are equal, then the four numbers are proportional."

1

Let four numbers be proportional:
 $A : B = C : D$.
 We must prove that
 $A \times D = B \times C$.

2

Let A multiply C, producing a new number G.
 $A \times C = G$

3

Since $A \times C = G$ and $A \times D = E$, the same number A multiplies C and D. By Proposition 17,
 $C : D = G : E$.

4

But we assumed
 $A : B = C : D$.
 Therefore,
 $A : B = G : E$.

5

Now $A \times C = G$ and $B \times C = F$.
 The number C multiplies both A and B.
 By Proposition 18,
 $A : B = G : F$.

6

We have
 $A : B = G : E$
 and
 $A : B = G : F$.
 Thus $G : E = G : F$.
 The same number G has the same ratio to E and to F.

7

Therefore,
 $E = F$.
 That is,
 $A \times D = B \times C$.

8

For the converse, assume $A \times D = B \times C$.
 Using the same reasoning, we find
 $A : B = C : D$.

9

Hence we have the proportion
 $A : B = C : D$
 iff
 $A \times D = B \times C$.

ARISTOTLE TO ALEXANDER:
 "Proportion can be tested by multiplication. If the cross-products are equal, the ratios are equal."

THE PRINCIPLE
 $A : B = C : D \iff A \times D = B \times C$
 (Proportion $\iff$ Equality of cross-products)

ALEXANDER'S REFLECTION:
 "So the balance of numbers is kept—either by equal ratios, or by equal cross-multiplication."

99

Proposition 20

Statement. The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

Proof.

Action 1. For let CD and EF be the least numbers having the same ratio as A and B (respectively).

Action 2. I say that CD measures A the same number of times as EF (measures) B. For CD is not parts of A.

Action 3. For, if possible, let it be (parts of A).

Action 4. Thus, EF is also the same parts of B that CD (is) of A (Def. 7.20, Prop. 7.13).

Action 5. Thus, as many parts of A as are in CD, so many parts of B are also in EF .

Action 6. Let CD have been divided into the parts of A, CG and GD, and EF into the parts of B, EH and HF .

Action 7. So the multitude of (divisions) CG, GD will be equal to the multitude of (divisions) EH, HF .

Action 8. And since the numbers CG and GD are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) CG, GD is equal to the multitude of (divisions) EH, HF , thus as CG is to EH, so GD (is) to HF .

Action 9. Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following (Prop. VII.12).

Action 10. Thus, as CG is to EH, so CD (is) to EF .

Action 11. Thus, CG and EH are in the same ratio as CD and EF , being less than them. The very thing is impossible.

Action 12. For CD and EF were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, CD is not parts of A.

Action 13. Thus, (it is) a part (of A) (Prop. VII.4).

Action 14. And EF is the same part of B that CD (is) of A (Def. 7.20, Prop 7.13).

Action 15. Thus, CD measures A the same number of times that EF (measures) B.

Action 16. (Which is) the very thing it was required to show.

Modern reading. If C, D are the least numbers in the ratio of A, B , then

$$A = kC, \quad B = kD$$

for the same whole number k .

Therefore, The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

EUCLID, BOOK VII — PROPOSITION 20

“The least numbers of those having the same ratio measure those having the same ratio as them an equal number of times, the greater measuring the greater, and the lesser the lesser.”

1

Let A and B be two numbers.
Let CD and EF be the least numbers having the same ratio as A and B.

$CD : EF = A : B$

We must prove that CD measures A as many times as EF measures B.

This is what we want to prove.

2

By Proposition 4, the smaller number CD is either a part of A or parts of A.

Suppose, for contradiction, that CD is parts of A.

Then, because $CD : EF = A : B$, EF is the same parts of B that CD is of A.

We will show this leads to an impossibility.

3

Break CD into the parts of A. Break EF into the corresponding parts of B.

The numbers of parts are equal.

4

Compare one pair of corresponding parts. Since the divisions are equal in number and in size within each group.

$CG : EH = GD : HF = \dots$

All corresponding parts have the same ratio.

5

By Proposition 12, if the corresponding parts have the same ratio, then the sums have the same ratio.

$CG : EH = CD : EF$

Thus the smaller parts have the same ratio as the sums.

6

But $CG < CD$ and $EH < EF$. So CG and EH are smaller numbers having the same ratio as CD and EF.

$CG : EH = CD : EF$

This contradicts our assumption that CD and EF are the least numbers having that ratio.

Therefore our supposition was false.

7

Therefore CD is not parts of A. It must be a single part of A.

$CD \mid A$

And EF is the same part of B that CD is of A.

CD is a part of A, and EF is a part of B.

8

Hence CD measures A as many times as EF measures B.

Both measure an equal number of times.

$A = k \times CD$ and $B = k \times EF$ for the same whole number k.

9

Example:
 $A : B = 12 : 18 = 2 : 3$
The least numbers in this ratio are 2 and 3.

2 measures 12 exactly 6 times.
3 measures 18 exactly 6 times.

Thus every pair in the ratio $2 : 3$ is obtained by multiplying the least pair by the same number 6.

ARISTOTLE TO ALEXANDER:

The least pair is the pattern. All other pairs in the same ratio are built from it, by the same number.

THE PRINCIPLE

Least numbers $\rightarrow$ primitive ratio $\rightarrow$ all others by the same multiplier.

$CD : EF = A : B$ (least)
 $A = k \times CD$ and $B = k \times EF$

ALEXANDER'S REFLECTION:

So in harmony, the smallest pair is the seed, and all others are its fruits multiplied by the same number.

102

Proposition 21

Statement. Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Proof.

- Action 1.** Let A and B be numbers prime to one another.
- Action 2.** I say that A and B are the least of those (numbers) having the same ratio as them.
- Action 3.** For if not then there will be some numbers less than A and B which are in the same ratio as A and B. Let them be C and D.
- Action 4.** Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser that is to say, the leading (measuring) the leading, and the following the following C thus measures A the same number of times that D (measures) B (Prop. VII.20).
- Action 5.** So as many times as C measures A, so many units let there be in E.
- Action 6.** Thus, D also measures B according to the units in E.
- Action 7.** And since C measures A according to the units in E, E thus also measures A according to the units in C (Prop. VII.16).
- Action 8.** So, for the same (reasons), E also measures B according to the units in D (Prop. VII.16).
- Action 9.** Thus, E measures A and B, which are prime to one another. The very thing is impossible.
- Action 10.** Thus, there cannot be any numbers less than A and B which are in the same ratio as A and B.
- Action 11.** Thus, A and B are the least of those (numbers) having the same ratio as them.

Action 12. (Which is) the very thing it was required to show.

Modern reading.

$\gcd(A, B) = 1 \implies A, B$ are the least numbers in their ratio.

Therefore, Numbers prime to one another are the least of those (numbers) having the same ratio as them.

EUCLID, BOOK VII — PROPOSITION 21

“Numbers prime to one another are the least of those numbers having the same ratio as them.”

1

Let A and B be two numbers prime to one another.

We must prove that A and B are the least numbers having the ratio $A : B$.

That is our goal.

A and B are prime to one another.

2

Suppose the opposite. Suppose there are numbers C and D, both smaller than A and B, such that

$$C : D = A : B$$

We shall show this leads to a contradiction.

3

By Proposition 20, the least numbers C and D having the ratio $A : B$ measure A and B an equal number of times.

C measures A as many times as D measures B.

4

Let E contain as many units as the number of times C measures A.

Then

$$A = C \times E$$

And D measures B the same number of times.

So

$$B = D \times E$$

E is the common number of repetitions.

5

By Proposition 16 (commutativity), $A = C \times E$ becomes $A = E \times C$.

Therefore E measures A.

$B = D \times E$ becomes $B = E \times D$.

Therefore E measures B as well.

6

Thus E measures both A and B.

But A and B are prime to one another.

By Definition 12, no number greater than the unit can measure them both.

This is a contradiction.

7

Therefore our assumption was false.

There are no numbers C and D, both smaller than A and B, such that $C : D = A : B$.

So no smaller proportional pair exists.

8

Hence A and B are the least numbers having the ratio $A : B$.

Proved.

9

Examples.

$A = 8, B = 15$
 $\text{gcd}(8, 15) = 1 \rightarrow 8 : 15$ is already the least ratio.

$A = 12, B = 18$
 $\text{gcd}(12, 18) = 6 \rightarrow 12 : 18 = 2 : 3$ (not least)

ARISTOTLE TO ALEXANDER:
 When two numbers share no common measure, nothing greater than the unit can divide them both.

THE PRINCIPLE
 Numbers prime to one another are the seed of their ratio.
 All other pairs in the same ratio are obtained by multiplying both by the same whole number.

ALEXANDER'S REFLECTION:
 So the purest form of a ratio is written with numbers that share no common measure.

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Proposition 22

Statement. The least numbers of those (numbers) having the same ratio as them are prime to one another.

Proof.

Action 1. Let A and B be the least numbers of those (numbers) having the same ratio as them.

Action 2. I say that A and B are prime to one another.

Action 3. For if they are not prime to one another then some number will measure them.

Action 4. Let it (so measure them), and let it be C .

Action 5. And as many times as C measures A , so many units let there be in D .

Action 6. And as many times as C measures B , so many units let there be in E .

Action 7. Since C measures A according to the units in D , C has thus made A (by) multiplying D (Def. VII.15).

Action 8. So, for the same (reasons), C has also made B (by) multiplying E .

Action 9. So the number C has made A and B (by) multiplying the two numbers D and E (respectively).

Action 10. Thus, as D is to E , so A (is) to B (Prop. VII.17).

Action 11. Thus, D and E are in the same ratio as A and B , being less than them. The very thing is impossible.

Action 12. Thus, some number does not measure the numbers A and B .

Action 13. Thus, A and B are prime to one another.

Action 14. (Which is) the very thing it was required to show.

Modern reading.

$$A, B \text{ least in their ratio} \implies \gcd(A, B) = 1.$$

Therefore, The least numbers of those (numbers) having the same ratio as them are prime to one another.

"The least numbers of those having the same ratio as them are prime to one another."

"The least numbers of those having the same ratio as them are prime to one another."

1

Let C measure B.
Let A and B be the least numbers having their ratio.

We must prove that
A and B are
prime to one another.

Goal:
A and B are
prime to one another.

2

Suppose the opposite.
Suppose A and B are
not prime to one
another.

Then some number
greater than the unit
measures them both.
Let that number be C.

3

Let C measure A.
Let D contain as many
units as the number of
times C measures A.

C: A

D

Thus $A = C \times D$.

4

Let C measure B.
Let E contain as many
units as the number of
times C measure B.

C: B

E

Thus $B = C \times E$.

5

Now we have
 $A = C \times D$
and
 $B = C \times E$.

The same number C
multiplies D and E
to produce A and B.

6

By Proposition 17:
When one number multiplies
two numbers, the products
have the same ratio as the
multiplied numbers.

$D : E = A : B$

7

But C is greater than the unit.
Therefore $D < A$ and $E < B$.
So D and E are smaller
numbers having the
same ratio as A and B.

D < A
E < B

8

But A and B were the least
numbers having that ratio.
It is impossible for smaller
numbers D and E
to have the same ratio.
This is a contradiction.

9

Therefore our assumption
was false.
No number greater than
the unit measures both
A and B.
Hence, A and B are
prime to one another.

Proved.

ARISTOTLE TO ALEXANDER:
The least pair in a ratio has no common measure except the unit. That is the mark of harmony.

THE PRINCIPLE

Least numbers in a ratio → cannot be reduced further → must be coprime.

Least numbers in a ratio ⇔ Prime to one another.

ALEXANDER'S REFLECTION:
So purity of numbers is the foundation of all proportions.

Proposition 23

Statement. If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).

Proof.

Action 1. .

Action 2. Let A and B be two numbers (which are) prime to one another, and let some number C measure A.

Action 3. I say that C and B are also prime to one another.

Action 4. For if C and B are not prime to one another then (some) number will measure C and B.

Action 5. Let it (so) measure (them), and let it be D.

Action 6. Since D measures C, and C measures A, D thus also measures A. And (D) also measures B.

Action 7. Thus, D measures A and B, which are prime to one another. The very thing is impossible.

Action 8. Thus, some number does not measure the numbers C and B.

Action 9. Thus, C and B are prime to one another.

Action 10. (Which is) the very thing it was required to show.

Modern reading.

$$\gcd(A, B) = 1, C \mid A \implies \gcd(C, B) = 1.$$

Therefore, If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).

EUCLID, BOOK VII — PROPOSITION 23

"If two numbers are prime to one another, then a number measuring one of them will be prime to the remaining one."

1

Let A and B be two numbers prime to one another. Let C be a number measuring A.



We must prove that C and B are prime to one another.

2

Suppose the opposite. Suppose C and B are not prime to one another. Then some number D measures both C and B.



3

But C measures A.



4

Since D measures C, and C measures A, D also measures A.



5

And D also measures B.



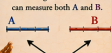
6

Therefore D measures both A and B. But A and B were assumed prime to one another.



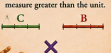
7

Hence this is impossible. No number greater than the unit can measure both A and B. This contradicts our assumption.



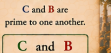
8

Therefore our supposition was false. C and B cannot have any common measure greater than the unit. So no number D > 1 measures both C and B.



9

Therefore C and B are prime to one another. C and B are coprime.



ARISTOTLE TO ALEXANDER:

Every divisor of one member of a coprime pair remains coprime to the other. This preserves the purity of the ratio.

THE PRINCIPLE

If A and B are coprime, and $C \mid A$, then C and B are coprime.

$\gcd(A, B) = 1$ and $C \mid A \implies \gcd(C, B) = 1$

ALEXANDER'S REFLECTION:

As harmony in music remains when a part is taken, so purity remains when a measure is taken.

Proposition 24

Statement. If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).

Proof.

- Action 1.** For let A and B be two numbers (which are both) prime to some number C.
- Action 2.** And let A make D (by) multiplying B.
- Action 3.** I say that C and D are prime to one another.
- Action 4.** For if C and D are not prime to one another then (some) number will measure C and D.
- Action 5.** Let it (so) measure them, and let it be E.
- Action 6.** And since C and A are prime to one another, and some number E measures C, A and E are thus prime to one another (Prop. VII.23).
- Action 7.** So as many times as E measures D, so many units let there be in F .
- Action 8.** Thus, F also measures D according to the units in E (Prop. VII.16).
- Action 9.** Thus, E has made D (by) multiplying F (Def. VII.15).
- Action 10.** But, in fact, A has also made D (by) multiplying B.
- Action 11.** Thus, the (number created) from (multiplying) E and F is equal to the (number created) from (multiplying) A and B.
- Action 12.** And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four numbers are proportional (Prop. 6.15).
- Action 13.** Thus, as E is to A, so B (is) to F .
- Action 14.** And A and E (are) prime (to one another).

Action 15. And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) (Prop. VII.21).

Action 16. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser that is to say, the leading (measuring) the leading, and the following the following (Prop. VII.20). Thus, E measures B. And it also measures C.

Action 17. Thus, E measures B and C, which are prime to one another. The very thing is impossible.

Action 18. Thus, some number cannot measure the numbers C and D.

Action 19. Thus, C and D are prime to one another.

Action 20. (Which is) the very thing it was required to show.

Modern reading.

$$\gcd(A, C) = \gcd(B, C) = 1 \implies \gcd(AB, C) = 1.$$

Therefore, If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).

EUCLID, BOOK VII — PROPOSITION 24

"If two numbers are prime to some number, then the number created from multiplying the former two numbers will also be prime to the latter number."

1

Let A and B be two numbers prime to some number C.

$\gcd(A, C) = 1$
 $\gcd(B, C) = 1$

Let A multiply B and produce D.

$D = A \times B$

We must prove:
 C and D are prime to one another.

2

Suppose the opposite.
 Suppose C and D are not prime to one another.
 Then some number E > 1 measures them both.

E measures C.

E measures D.

$E \mid C$ and $E \mid D$

3

Since A and C are prime to one another, and E measures C, by Proposition 23, A and E are prime to one another.

4

Let F contain as many units as the number of times E measures D.

E measures D.

$D = E \times F$

Therefore,
 $D = E \times F$

5

But D was also produced by A multiplying B.

$D = A \times B$

Therefore,
 $E \times F = A \times B$

6

From equality of products, we get the proportion:

$E : A = B : F$

$E \times F = A \times B$

7

But A and E are prime to one another.

By Proposition 21, E and A are the least numbers in the ratio E : A.

8

Since E : A = B : F, and E and A are the least numbers in this ratio, by Proposition 20, E measures B as many times as A measures F.

E measures B.

Therefore, E | B.

9

But we already have E | C.

Now E measures both B and C.

This contradicts our assumption that B and C are prime to one another.

ARISTOTLE TO ALEXANDER:

When purity (no common measure) holds separately, it holds together in the harmony of multiplication.

THE PRINCIPLE

If two numbers are each prime to a third number, then their product is also prime to that third number.

$\gcd(A, C) = 1$ and $\gcd(B, C) = 1 \Rightarrow \gcd(A \times B, C) = 1$

ALEXANDER'S REFLECTION:

As in the lyre, when each string is in harmony with a tuning note, the chord remains pure.

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Proposition 25

Statement. If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

Proof.

Action 1. Let A and B be two numbers (which are) prime to one another.

Action 2. And let A make C (by) multiplying itself.

Action 3. I say that B and C are prime to one another. For let D be made equal to A .

Action 4. Since A and B are prime to one another, and A (is) equal to D , D and B are thus also prime to one another. Thus, D and A are each prime to B .

Action 5. Thus, the (number) created from (multilying) D and A will also be prime to B (Prop. VII.24).

Action 6. And C is the number created from (multiplying) D and A .

Action 7. Thus, C and B are prime to one another.

Action 8. (Which is) the very thing it was required to show.

Modern reading.

$$\gcd(A, B) = 1 \implies \gcd(A^2, B) = 1.$$

Therefore, If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

EUCLID, BOOK VII — PROPOSITION 25

“If two numbers are prime to one another, then the number created from squaring one of them will be prime to the remaining number.”

1

Let A and B be two numbers prime to one another.

$\gcd(A, B) = 1$

A 

B 

We are given that A and B are prime to one another.

2

Let A multiply itself (square it) and produce C.

$C = A \times A = A^2$

A  

We must prove that C and B are prime to one another.

3

Let another number D be equal to A.

$D = A$

Then $C = D \times A$.

D  

So the square of A can be seen as the product of D and A.

4

Since $D = A$, and A is prime to B, D is also prime to B.

D  / 

(prime)

Whatever is prime to B, remains so if it is equal to A.

5

Thus both D and A are each prime to B.

$\gcd(D, B) = 1$

$\gcd(A, B) = 1$

D  

A  

B  

Each of the two factors is prime to B.

6

By Proposition 24:

If two numbers are each prime to a third number, then their product is also prime to that third number.

Therefore $D \times A$ is prime to B.

7

But $D = A$, so $D \times A = A \times A = C$.

C  

Hence the product in Prop. 24 is precisely the square C.

8

Therefore C and B are prime to one another.

$\gcd(C, B) = 1$

C  / 

(prime)

This is what we wished to prove.

9

Example.

Let $A = 4$, $B = 15$

$\gcd(4, 15) = 1$

$A^2 = 4^2 = 16$

$\gcd(16, 15) = 1$

The square remains prime to B.

ARISTOTLE TO ALEXANDER:

Squaring a number that is prime in its harmony with another number cannot introduce any common measure.

THE PRINCIPLE

If A and B are prime to one another, then the square of A is also prime to B.

$\gcd(A, B) = 1 \iff \gcd(A^2, B) = 1$

ALEXANDER'S REFLECTION:

As a chord in music, if a note is in harmony with another, the same harmony remains when it is repeated.

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Proposition 26

Statement. If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.

Proof.

Action 1. For let two numbers, A and B, both be prime to each of two numbers, C and D.

Action 2. And let A make E (by) multiplying B, and let C make F (by) multiplying D.

Action 3. I say that E and F are prime to one another.

Action 4. For since A and B are each prime to C, the (number) created from (multiplying) A and B will thus also be prime to C (Prop. VII.24).

Action 5. And E is the (number) created from (multiplying) A and B.

Action 6. Thus, E and C are prime to one another.

Action 7. So, for the same (reasons), E and D are also prime to one another. Thus, C and D are each prime to E.

Action 8. Thus, the (number) created from (multiplying) C and D will also be prime to E (Prop. VII.24).

Action 9. And F is the (number) created from (multiplying) C and D.

Action 10. Thus, E and F are prime to one another.

Action 11. (Which is) the very thing it was required to show.

Modern reading.

$$\begin{aligned} \gcd(A, C) = \gcd(A, D) = 1, \\ \gcd(B, C) = \gcd(B, D) = 1 \end{aligned} \implies \gcd(AB, CD) = 1.$$

Therefore, If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.

EUCLID, BOOK VII — PROPOSITION 26

"If two numbers are both prime to each of two numbers, then the numbers created from multiplying them will also be prime to one another."

1

Let four numbers A, B, C, D be such that every number in the first pair is prime to every number in the second pair.
 $\gcd(A, C) = 1$
 $\gcd(A, D) = 1$
 $\gcd(B, C) = 1$
 $\gcd(B, D) = 1$
This is our assumption.

2

Let A multiply B and produce E.
 $E = A \times B$

3

Let C multiply D and produce F.
 $F = C \times D$

4

We first show that E is prime to C.
We know
 $\gcd(A, C) = 1$
and
 $\gcd(B, C) = 1$
By Proposition 24,
 $\gcd(A \times B, C) = 1$

5

We next show that E is prime to D.
We know
 $\gcd(A, D) = 1$
and
 $\gcd(B, D) = 1$
By Proposition 24,
 $\gcd(A \times B, D) = 1$

6

So far we have proved:
 $\gcd(E, C) = 1$
 $\gcd(E, D) = 1$
Thus both C and D are each prime to E.

7

Now apply Proposition 24 again.
Since
 $\gcd(E, C) = 1$
and
 $\gcd(E, D) = 1$
By Proposition 24,
 $\gcd(C \times D, E) = 1$

8

But $C \times D = F$.
Hence
 $\gcd(F, E) = 1$,
that is,
 $\gcd(E, F) = 1$.
So E and F are prime to one another.

9

Example
Let $A = 4, B = 9,$
 $C = 5, D = 7.$
Check all cross-pairs:
✓ $\gcd(4, 5) = 1$
✓ $\gcd(4, 7) = 1$
✓ $\gcd(9, 5) = 1$
✓ $\gcd(9, 7) = 1$
 $E = A \times B = 4 \times 9 = 36$
 $F = C \times D = 5 \times 7 = 35$
 $\gcd(36, 35) = 1$
The products remain prime to one another.

ARISTOTLE TO ALEXANDER:
"See how harmony is preserved.
When each member of one pair has
no common measure with each member
of the other pair, their products
remain free of any common measure."

THE PRINCIPLE IN ONE LINE
If $\gcd(A, C) = \gcd(A, D) = \gcd(B, C) = \gcd(B, D) = 1$,
then $\gcd(A \times B, C \times D) = 1$.

ALEXANDER'S REFLECTION:
As in a melody, when every note
is in harmony with every other,
the chords created from them
remain in harmony as well."

118

Proposition 27

Statement. If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes.

Proof.

Action 1. Let A and B be two numbers prime to one another, and let A make C (by) multiplying itself, and let it make D (by) multiplying C.

Action 2. And let B make E (by) multiplying itself, and let it make F by multiplying E.

Action 3. I say that C and E, and D and F , are prime to one another.

Action 4. For since A and B are prime to one another, and A has made C (by) multiplying itself, C and B are thus prime to one another (Prop. VII.25).

Action 5. Therefore, since C and B are prime to one another, and B has made E (by) multiplying itself, C and E are thus prime to one another (Prop. VII.25).

Action 6. Again, since A and B are prime to one another, and B has made E (by) multiplying itself, A and E are thus prime to one another (Prop. VII.25).

Action 7. Therefore, since the two numbers A and C are both prime to each of the two numbers B and E, the (number) created from (multiplying) A and C is thus prime to the (number created) from (multiplying) B and E (Prop. VII.26).

Action 8. And D is the (number created) from (multiplying) A and C, and F the (number created) from (multiplying) B and E.

Action 9. Thus, D and F are prime to one another.

Action 10. (Which is) the very thing it was required to show. a^2 is also prime to b^2 , as well as a^3 to b^3 , etc., where all symbols denote

Modern reading.

$$\gcd(A, B) = 1 \implies \gcd(A^n, B^n) = 1.$$

Therefore, If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes.

EUCLID, BOOK VII — PROPOSITION 27

"If two numbers are prime to one another, and each multiplies itself to make a new number, then those new numbers will also be prime to one another; and if the original numbers multiply those squares to make further numbers, those too will be prime to one another — and so on."

1

Start with a coprime pair.
Let A and B be prime to one another.
 $\gcd(A, B) = 1$

We shall create their powers and show that they remain coprime.

2

Square A:
Let A multiply itself and produce C.
 $C = A^2$

Cube A:
Let A multiply C and produce D.
 $D = A \times C = A^3$

3

Square B:
Let B multiply itself and produce E.
 $E = B^2$

Cube B:
Let B multiply E and produce F.
 $F = B \times E = B^3$

4

First, A^2 remains coprime to B.
Since $\gcd(A, B) = 1$, and A squares to $C = A^2$, by Proposition 25,
 $\gcd(C, B) = 1$

So C and B are prime to one another.

5

Now square B.
Since $\gcd(C, B) = 1$, and B squares to $E = B^2$, by Proposition 25 again,
 $\gcd(C, E) = 1$

Then A^2 and B^3 are prime to one another.

6

We will now prepare for the cubes.
Since $\gcd(A, B) = 1$, and B squares to E, by Proposition 25,
 $\gcd(A, E) = 1$

Both A and C are each coprime to both B and E.

7

Apply Proposition 26.
Since A and C are each prime to B and E, the products of the pairs are prime to one another.
 $\gcd(A \times C, B \times E) = 1$

Therefore the products are coprime.

8

Identify the products.
 $A \times C = D = A^3$
 $B \times E = F = B^3$

Hence by Proposition 26,
 $\gcd(D, F) = 1$

So A^3 and B^3 are prime to one another.

9

And this continues for higher powers.
The same pattern goes on for fourth powers, fifth powers, and so on.

If $\gcd(A, B) = 1$, then for every $n \geq 1$, $\gcd(A^n, B^n) = 1$, and so on without end.

ARISTOTLE TO ALEXANDER:
"When numbers are in harmony—
free of any common measure—
that harmony is preserved
through all their equal powers."

THE PRINCIPLE IN ONE LINE

If $\gcd(A, B) = 1$, then for all $n \geq 1$, $\gcd(A^n, B^n) = 1$.

This is the endless harmony of numbers.

ALEXANDER'S REFLECTION:
"As in music, when the notes
are in harmony, every octave
sings the same concord."

121

Proposition 28

Statement. If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.

Proof.

Action 1. For let the two numbers, AB and BC, (which are) prime to one another, be laid down together.

Action 2. I say that their sum AC is also prime to each of AB and BC.

Action 3. For if CA and AB are not prime to one another then some number will measure CA and AB.

Action 4. Let it (so) measure (them), and let it be D.

Action 5. Therefore, since D measures CA and AB, it will thus also measure the remainder BC. And it also measures BA.

Action 6. Thus, D measures AB and BC, which are prime to one another. The very thing is impossible.

Action 7. Thus, some number cannot measure (both) the numbers CA and AB.

Action 8. Thus, CA and AB are prime to one another.

Action 9. So, for the same (reasons), AC and CB are also prime to one another.

Action 10. Thus, CA is prime to each of AB and BC.

Action 11. So, again, let CA and AB be prime to one another.

Action 12. I say that AB and BC are also prime to one another.

Action 13. For if AB and BC are not prime to one another then some number will measure AB and BC.

Action 14. Let it (so) measure (them), and let it be D.

Action 15. And since D measures each of AB and BC, it will thus also measure the whole of CA. And it also measures AB.

Action 16. Thus, D measures CA and AB, which are prime to one another. The very thing is impossible.

Action 17. Thus, some number cannot measure (both) the numbers AB and BC.

Action 18. Thus, AB and BC are prime to one another.

Action 19. (Which is) the very thing it was required to show.

Modern reading.

$$\gcd(A, B) = \gcd(A, A + B) = \gcd(B, A + B).$$

Therefore, If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.

EUCLID, BOOK VII — PROPOSITION 28

If two numbers are prime to one another, then their sum will also be prime to each of them.
And if the sum of two numbers is prime to either one of them, then the original numbers will also be prime to one another.

If two numbers are prime to one another, then their sum will also be prime to each of them. And if the sum of two numbers is prime to either one of them, then the original numbers will also be prime to one another.

1 Let two numbers be prime to one another. Let them be AB and BC. Let their sum be $AC = AB + BC$.

We first prove that AC is prime to each of AB and BC.

2 Suppose, for contradiction, that AC and AB are not prime to one another.

Then some number D > 1 measures both AC and AB.

We shall reach a contradiction.

3 But BC = AC - AB.

Since D measures both AC and AB, it must also measure their remainder.

Therefore D | BC.

Now D measures both AB and BC.

4 But AB and BC were assumed to be prime to one another.

No number greater than 1 can measure both.

Contradiction!

Therefore AC and AB are prime to one another.

5 By the same argument, if a number measures both AC and BC, it would also measure $AC - BC = AB$.

Again a contradiction.

Therefore AC is prime to each of AB and BC.

6 Now we prove the converse.

Suppose the sum AC is prime to one of the numbers, say AB.

$\gcd(AC, AB) = 1$.

We shall prove that AB and BC are prime to one another.

7 Suppose, for contradiction, that AB and BC are not prime to one another.

Then some number D > 1 measures both AB and BC.

D | AB and D | BC.

We shall again reach a contradiction.

8 Since D measures both AB and BC, it also measures their sum.

$AB + BC = AC$,
so D | AC.

But D already measures AB. So D measures both AC and AB.

This contradicts our assumption that $\gcd(AC, AB) = 1$.

THE PRINCIPLE IN ONE LINE

For any numbers A and B,

$\gcd(A, B) = \gcd(A + B) = \gcd(B, A + B)$.

Thus, A and B are coprime $\Leftrightarrow A + B$ is coprime to each of them.

ARISTOTLE TO ALEXANDER:

When you add or subtract one number from another, you do not change the common divisors. Only the harmony remains. This is the secret of Euclid's reasoning.

ALEXANDER'S REFLECTION:

So the harmony of numbers is not broken by their sum. It is like a melody – chord the note, the concord remains.

Proposition 29

Statement. Every prime number is prime to every number which it does not measure.

Proof.

Action 1. Let A be a prime number, and let it not measure B.

Action 2. I say that B and A are prime to one another.

Action 3. For if B and A are not prime to one another then some number will measure them. Let C measure (them).

Action 4. Since C measures B, and A does not measure B, C is thus not the same as A.

Action 5. And since C measures B and A, it thus also measures A, which is prime, (despite) not being the same as it. The very thing is impossible.

Action 6. Thus, some number cannot measure (both) B and A.

Action 7. Thus, A and B are prime to one another.

Action 8. (Which is) the very thing it was required to

Modern reading.

$$p \text{ prime, } p \nmid n \implies \gcd(p, n) = 1.$$

Therefore, Every prime number is prime to every number which it does not measure.

EUCLID, BOOK VII — PROPOSITION 29

"Every prime number is prime to every number which it does not measure."

1

Let A be a prime number.
Let B be another number
which A does not measure.

A is prime
 $A \nmid B$

We must prove that
 $\gcd(A, B) = 1$.

2

Suppose the opposite:
A and B are not
prime to one another.
Then some number $C > 1$
measures both of them.

$C > 1$
 $C \mid A$
 $C \mid B$

3

But C cannot be equal to A.
For if $C = A$, then since $C \mid B$,
we would have $A \mid B$.
This contradicts our
assumption that A does
not measure B.

Therefore $C \neq A$.

4

Yet C measures A,
and C is not equal to A.
So we have found a number
C, different from A,
that measures the
prime number A.

$C \mid A$ and $C \neq A$

5

But this is impossible.
A is prime, so it is measured
only by the unit (1) and
by itself.

Definition of Prime:
A prime number has no
measures other than
1 and itself.

$1 \rightarrow A \text{ (prime)} \rightarrow A$

6

Since $C > 1$ and $C \neq A$,
C cannot measure A.
This contradicts our
assumption that
 $C \mid A$.

Contradiction!

7

Therefore our supposition
was false.
No number greater than 1
can measure both
A and B.

8

Hence A and B have
no common measure
greater than 1.
Therefore
 $\gcd(A, B) = 1$

A and B are
prime to one another

9

Example
Let $A = 7$ (prime),
 $B = 20$.
Since 7 does not
measure 20,
by this proposition,
 $\gcd(7, 20) = 1$.

Divisors of 7:
1, 7
Divisors of 20:
1, 2, 4, 5, 10, 20
No common divisor
greater than 1.
So $\gcd(7, 20) = 1$.

ARISTOTLE TO ALEXANDER:
"A prime number is like a single
note in music. Either it divides
another number, or it is in perfect
harmony with it."

THE PRINCIPLE IN ONE LINE
If A is prime and $A \nmid B$, then $\gcd(A, B) = 1$.
A prime either divides a number, or it is coprime to it.

ALEXANDER'S REFLECTION:
"So a prime number is like a key
that either opens the number,
or fits no lock within it—
and so it stands in harmony."

126

Proposition 30

Statement. If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).

Proof.

- Action 1.** For let two numbers A and B make C (by) multiplying one another, and let some prime number D measure C.
- Action 2.** I say that D measures one of A and B. For let it not measure A.
- Action 3.** And since D is prime, A and D are thus prime to one another (Prop. VII.29).
- Action 4.** And as many times as D measures C, so many units let there be in E.
- Action 5.** Therefore, since D measures C according to the units E, D has thus made C (by) multiplying E (Def. VII.15).
- Action 6.** But, in fact, A has also made C (by) multiplying B.
- Action 7.** Thus, the (number created) from (multiplying) D and E is equal to the (number created) from (multiplying) A and B.
- Action 8.** Thus, as D is to A, so B (is) to E (Prop. VII.19).
- Action 9.** And D and A (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) (Prop. VII.21), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser that is to say, the leading (measuring) the leading, and the following the following (Prop. VII.20). Thus, D measures B.
- Action 10.** So, similarly, we can also show that if (D) does not measure B then it will measure A. Thus, D measures one of A and B.

Action 11. (Which is) the very thing it was required to show.

Modern reading.

$$p \text{ prime, } p \mid AB \implies p \mid A \text{ or } p \mid B.$$

Therefore, If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).

EUCLID, BOOK VII — PROPOSITION 30

"If two numbers make some number by multiplying one another, and some prime number measures the number so created, then it will also measure one of the original numbers."

1

Let A and B be two numbers that multiply to make C.

$$C = A \times B$$

Let D be a prime number that measures C.

D is prime, and $D \mid C$

We must prove that $D \mid A$ or $D \mid B$.

2

Suppose, for the sake of argument, that D does not measure A.

$$D \nmid A$$

By Proposition 29, a prime that does not measure a number is prime to it.

Therefore $\gcd(D, A) = 1$.

3

Since D measures C, let E be the number of times D measures C.

$$C = D \times E$$

4

But also $C = A \times B$.

So both $D \times E$ and $A \times B$ are equal to C.

$$D \times E = A \times B$$

5

By Proposition 19, from equality of products we get proportionality.

$$D : A = B : E$$

$$D \times E = A \times B$$

$$D : A = B : E$$

6

Since $\gcd(D, A) = 1$, by Proposition 21, D and A are the least numbers in the ratio $D : A$.

Numbers that are prime to one another are the least numbers having their ratio.

7

But $D : A = B : E$.

By Proposition 20, the least numbers in a ratio measure the corresponding numbers in that same ratio.

Least in $D : A$ measure in $B : E$

$$\begin{array}{cc} D : A & B : E \\ \downarrow & \downarrow \\ B & E \end{array}$$

8

Therefore D measures B.

$$D \mid B$$

So if a prime D does not measure A, it must measure B.

9

Similarly, if D does not measure B, then D must measure A.

Hence in every case, $D \mid A$ or $D \mid B$.

This is what we set out to prove.

ARISTOTLE TO ALEXANDER:

"A prime number is like a master key. If it opens the door of a product, it must open the door of at least one of the rooms inside."

THE PRINCIPLE IN ONE LINE

$$p \text{ prime, } p \nmid (a \times b) \Rightarrow p \nmid a \text{ or } p \nmid b.$$

A prime cannot enter a product without entering at least one of its factors.

ALEXANDER'S REFLECTION:

"So the prime is never lost. If it is in the product, it must be hidden in one of the parts."

129

Proposition 31

Statement. Every composite number is measured by some prime number.

Proof.

Action 1. Let A be a composite number.

Action 2. I say that A is measured by some prime number.

Action 3. For since A is composite, some number will measure it.

Action 4. Let it (so) measure (A), and let it be B .

Action 5. And if B is prime then that which was prescribed has happened.

Action 6. And if (B is) composite then some number will measure it.

Action 7. Let it (so) measure (B), and let it be C .

Action 8. And since C measures B , and B measures A , C thus also measures A .

Action 9. And if C is prime then that which was prescribed has happened.

Action 10. And if (C is) composite then some number will measure it.

Action 11. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure A).

Action 12. And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number A .

Action 13. The very thing is impossible for numbers.

Action 14. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure A .

Action 15. Thus, every composite number is measured by some prime number.

Action 16. (Which is) the very thing it was required to show.

Modern reading.

$$n \text{ composite} \implies \exists p \text{ prime with } p \mid n.$$

Therefore, Every composite number is measured by some prime number.

EUCLID, BOOK VII — PROPOSITION 31
 “Every composite number is measured by some prime number.”

- 1 Let A be a composite number.
- 2 Is B a prime number?
- 3 If not, suppose B is composite.

1 Since A is composite, some number $B > 1$ measures A .

B | A

We must show that A is measured by some prime number.

2 If it is prime, then it already measures A , and we are done.

If B is prime, then $B | A$ (as required).

3 Then some smaller number C divides A .

C | B

Since $B | A$ and $C | B$, therefore $C | A$.

4 Is C a prime number?

If it is prime, then $C | A$ and we are done.

5 If C is not prime, then some smaller number $D > 1$ measures C .

D | C

Again, since $C | A$, we have $D | A$.

6 We continue in the same way. From A we descend:

A > B > C > D > E > ...

Each number measures the one before it, and therefore each measures A .

A (composite)
 B
 C
 D
 E
 $\vdots$
 $D | C$

7 Could this descent go on forever? No, this is impossible in numbers. For we cannot have an infinite descending chain of positive numbers.

A > B > C > D > E > ...

The numbers would become smaller and smaller without end, which cannot be.

8 Therefore the descent must stop at some number P , which is no longer measured by any smaller number greater than 1. Such a number is prime.

P is prime and $P | A$.

9 Thus every composite number is measured by some prime number.

Let $A = 84$ (composite).
 $28 | 84$, but 28 is composite.
 $14 | 28$, but 14 is composite.
 $7 | 14$, and 7 is prime.
Therefore $7 | 84$.
So A is measured by the prime number 7 .

ARISTOTLE TO ALEXANDER:
 "In the realm of number, as in life, follow the path downward, and truth will reveal itself as something simple and indivisible."

THE PRINCIPLE IN ONE LINE

If A is composite, then $\exists$ prime p such that $p | A$.

Every composite number contains a prime within its chain of divisors.

ALEXANDER'S REFLECTION:
 "Even the greatest numbers hide a prime within them, as a tree hides its root."

Proposition 32

Statement. Every number is either prime or is measured by some prime number.

Proof.

Action 1. Let A be a number.

Action 2. I say that A is either prime or is measured by some prime number.

Action 3. In fact, if A is prime then that which was prescribed has happened.

Action 4. And if (it is) composite then some prime number will measure it (Prop. VII.31).

Action 5. Thus, every number is either prime or is measured by some prime number.

Action 6. (Which is) the very thing it was required to show.

Modern reading.

$$n > 1 \implies n \text{ is prime or has a prime divisor.}$$

Therefore, Every number is either prime or is measured by some prime number.

EUCLID, BOOK VII — PROPOSITION 32

"Every number is either prime or is measured by some prime number."

1

Let A be any number.
I shall show that
either A is prime,
or some prime number
measures A .

We must prove:
 A is prime
or
 $\exists$ prime P with $P \mid A$.

2

First, I ask:
Is A prime?

YES
Suppose
 A is prime.
Then we are
already done.

NO
Suppose
 A is not prime.
Then A is
composite.

There are only these two possibilities.

3

Since A is composite,
by Proposition 31,
 A is measured by
some prime number.

Therefore,
there exists
a prime number
 P such that
 $P \mid A$.

4

Proposition 31
Every composite number
is measured by
some prime number.
↓
Here A is composite,
so $\exists$ prime P
such that $P \mid A$.

5

Thus every number A
falls into one of these
two classes:

I
 A is
prime.

II
Some
prime P
measures A .

No third possibility remains.

6

Therefore,
every number is either
prime or is measured by
some prime number.

Which is what we set out to prove.

ARISTOTLE TO ALEXANDER:

"This is the great dividing of all numbers.
Either they stand on their own
as prime,
or a prime number stands within them
as within a fortress."

THE PRINCIPLE IN ONE LINE

For any number $A > 1$,

A is prime or $\exists$ prime P such that $P \mid A$.

Every number either is a prime itself, or
contains a prime factor.

ALEXANDER'S REFLECTION:

"So the world of numbers
is either made of prime atoms,
or built with them inside,
like a city whose hidden stones
are all of the finest kind."

134

Proposition 33

Statement. To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Proof.

- Action 1.** Let A, B, and C be any given multitude of numbers.
- Action 2.** So it is required to find the least of those (numbers) having the same ratio as A, B, and C.
- Action 3.** For A, B, and C are either prime to one another, or not.
- Action 4.** In fact, if A, B, and C are prime to one another then they are the least of those (numbers) having the same ratio as them (Prop. VII.22).
- Action 5.** And if not, let the greatest common measure, D, of A, B, and C have been taken (Prop. VII.3).
- Action 6.** And as many times as D measures A, B, C, so many units let there be in E, F, G, respectively.
- Action 7.** And thus E, F, G measure A, B, C, respectively, according to the units in D (Prop. VII.15).
- Action 8.** Thus, E, F, G measure A, B, C (respectively) an equal number of times.
- Action 9.** Thus, E, F, G are in the same ratio as A, B, C (respectively) (Def. VII.20).
- Action 10.** So I say that (they are) also the least (of those numbers having the same ratio as A, B, C).
- Action 11.** For if E, F, G are not the least of those (numbers) having the same ratio as A, B, C (respectively), then there will be (some) numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Let them be H, K, L.
- Action 12.** Thus, H measures A the same number of times that K, L also measure B, C, respectively.

- Action 13.** And as many times as H measures A, so many units let there be in M .
- Action 14.** Thus, K, L measure B, C, respectively, according to the units in M .
- Action 15.** And since H measures A according to the units in M , M thus also measures A according to the units in H (Prop. VII.15).
- Action 16.** So, for the same (reasons), M also measures B, C according to the units in K, L, respectively. Thus, M measures A, B, and C.
- Action 17.** And since H measures A according to the units in M , H has thus made A (by) multiplying M .
- Action 18.** So, for the same (reasons), E has also made A (by) multiplying D.
- Action 19.** Thus, the (number created) from (multiplying) E and D is equal to the (number created) from (multiplying) H and M .
- Action 20.** Thus, as E (is) to H, so M (is) to D (Prop. VII.19). And E (is) greater than H.
- Action 21.** Thus, M (is) also greater than D (Prop. 5.13). And (M) measures A, B, and C. The very thing is impossible.
- Action 22.** For D was assumed (to be) the greatest common measure of A, B, and C.
- Action 23.** Thus, there cannot be any numbers less than E, F , G which are in the same ratio as A, B, C (respectively).
- Action 24.** Thus, E, F , G are the least of (those numbers) having the same ratio as A, B, C (respectively).
- Action 25.** (Which is) the very thing it was required to show.

Modern reading.

$$d = \gcd(A, B, C), \quad (A/d) : (B/d) : (C/d) = A : B : C.$$

Therefore, To find the least of those (numbers) having the same ratio as any given multitude of numbers.

EUCLEID, BOOK VII – PROPOSITION 33

"To find the least of those numbers having the same ratio as any given multitude of numbers."

1

THE TASK

Let the numbers be
A, B, C.

We seek the least numbers
having the same ratio as
them.

Example:
A = 12, B = 18, C = 30.

Goal: Find the least numbers with the same ratio as 12 : 18 : 30.

2

FIRST QUESTION

Let A, B, C
prime to one another?

YES

Then A, B, C
are already the
least numbers.

NO

Then find their
greatest common
measure D.

If they are prime to one another, we are done.

3

FIND THE GREATEST COMMON MEASURE

Suppose they are not
prime to one another.

Let D be their greatest
common measure.

Example:
gcd(12, 18, 30) = 6.

By Proposition 3, such a greatest common measure exists.

4

DIVIDE BY D

Let
E = A/D, F = B/D, G = C/D.

The
A = D × E, B = D × F, C = D × G.

Example with D = 6:
E = 12/6 = 2, F = 18/6 = 3, G = 30/6 = 5

So E, F, G are smaller numbers.

5

SAME RATIO

Since each original number is obtained
by multiplying by the same number D,
E, F, G have the same ratio as A, B, C.

E : F : G = A : B : C

Example: 2 : 3 : 5 = 12 : 18 : 30

We have found a smaller proportional triple.

6

SUPPOSE EVEN SMALLER EXISTS

Assume, for contradiction, that there are smaller
numbers H, K, L such that

H : K : L = A : B : C

and
H < E, K < F, L < G.

H < 2, K < 3, L < 5

So H, K, L are even smaller but in the same ratio.

7

THE COMMON FACTOR M

Since H, K, L have the same ratio as A, B, C,
there is some number M such that

A = H × M, B = K × M, C = L × M.

Therefore, M measures A, B, and C.

M | A, M | B, M | C

So M is a common measure to A, B, C.

8

COMPARE M AND D

From A = D × E and A = H × M, we get
D × E = H × M.

By Proposition 19,
E : H = M : D.

E > H
↓
Hence M/D > 1, so M > D.

Thus M is greater than D.

9

CONTRADICTION AND CONCLUSION

We M measures A, B, C and M > D.
This contradicts that D is their greatest common measure.

Therefore no such H, K, L exist.

Hence E, F, G are the least numbers
having the same ratio as A, B, C.

RESULT (Example)
Least numbers in the
ratio of 12 : 18 : 30
are 2 : 3 : 5.

ARISTOTLE
"To find the least numbers in the same ratio,
first find the greatest common measure,
then divide each number by it."

THE PRINCIPLE IN ONE LINE

Let D = gcd(A, B, C). Then (A/D) : (B/D) : (C/D) = A : B : C
and the numbers A/D, B/D, C/D are the least possible.

ALEXANDER'S REFLECTION:
"By removing what is common to all,
we reveal the simplest numbers
that keep the same harmony."

Proposition 34

Statement. To find the least number which two given numbers (both) measure.

Proof.

- Action 1.** Let A and B be the two given numbers.
- Action 2.** So it is required to find the least number which they (both) measure.
- Action 3.** For A and B are either prime to one another, or not.
- Action 4.** Let them, first of all, be prime to one another.
- Action 5.** And let A make C (by) multiplying B.
- Action 6.** Thus, B has also made C (by) multiplying A (Prop. VII.16). Thus, A and B (both) measure C.
- Action 7.** So I say that (C) is also the least (number which they both measure).
- Action 8.** For if not, A and B will (both) measure some (other) number which is less than C.
- Action 9.** Let them (both) measure D (which is less than C).
- Action 10.** And as many times as A measures D, so many units let there be in E.
- Action 11.** And as many times as B measures D, so many units let there be in F .
- Action 12.** Thus, A has made D (by) multiplying E, and B has made D (by) multiplying F .
- Action 13.** Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F .
- Action 14.** Thus, as A (is) to B, so F (is) to E (Prop. VII.19).
- Action 15.** And A and B are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) (Prop. VII.21), and the least (numbers) measure

those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser (Prop. VII.20).

Action 16. Thus, B measures E, as the following (number measuring) the following.

Action 17. And since A has made C and D (by) multiplying B and E (respectively), thus as B is to E, so C (is) to D (Prop. VII.17). And B measures E.

Action 18. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible.

Action 19. Thus, A and B do not (both) measure some number which is less than C.

Action 20. Thus, C is the least (number) which is measured by (both) A and B.

Action 21. So let A and B be not prime to one another.

Action 22. And let the least numbers, F and E, have been taken having the same ratio as A and B (respectively) (Prop. VII.33).

Action 23. Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F (Prop. VII.19).

Action 24. And let A make C (by) multiplying E.

Action 25. Thus, B has also made C (by) multiplying F . Thus, A and B (both) measure C.

Action 26. So I say that (C) is also the least (number which they both measure).

Action 27. For if not, A and B will (both) measure some number which is less than C.

Action 28. Let them (both) measure D (which is less than C).

Action 29. And as many times as A measures D, so many units let there be in G.

- Action 30.** And as many times as B measures D, so many units let there be in H.
- Action 31.** Thus, A has made D (by) multiplying G, and B has made D (by) multiplying H.
- Action 32.** Thus, the (number created) from (multiplying) A and G is equal to the (number created) from (multiplying) B and H.
- Action 33.** Thus, as A is to B, so H (is) to G (Prop. VII.19).
- Action 34.** And as A (is) to B, so F (is) to E.
- Action 35.** Thus, also, as F (is) to E, so H (is) to G.
- Action 36.** And F and E are the least (numbers having the same ratio as A and B), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser (Prop. VII.20). Thus, E measures G.
- Action 37.** And since A has made C and D (by) multiplying E and G (respectively), thus as E is to G, so C (is) to D (Prop. VII.17). And E measures G.
- Action 38.** Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible.
- Action 39.** Thus, A and B do not (both) measure some (number) which is less than C.
- Action 40.** Thus, C (is) the least (number) which is measured by (both) A and B.
- Action 41.** (Which is) the very thing it was required to show.

Modern reading.

$$\text{lcm}(A, B) = \frac{AB}{\text{gcd}(A, B)}.$$

Therefore, To find the least number which two given numbers (both) measure.

"To find the least number which two given numbers both measure."

“To find the least number which two given numbers both measure.”

1 THE TASK

Let the two given numbers be A and B .

We seek the least number measured by both.

$A \mid C$ and $B \mid C$ and C is least.

First we ask: Are A and B prime to one another?

2 CASE I: A AND B ARE COPRIME

Suppose A and B are prime to one another.

Let $C = A \times B$.

$C = A \times B = B \times A$

By Prop. 16.

So $A \mid C$ and $B \mid C$.

So C is a common multiple of A and B .

3 IS C THE LEAST?

Suppose some smaller number D is also measured by both A and B .

$A \mid D$ and $B \mid D$ and $D < C$

Let $D = A \times E$ and also $D = B \times F$.

4 EQUAL PRODUCTS

Then $A \times E = B \times F$.

By Prop. 19.

$A \times F = B \times E$

Since A and B are coprime, by Prop. 21 they are the least numbers in this ratio.

5 THE TRANSDUCTION

But $B \mid E$ and $C = A \times B$.

$D = A \times E$.

Then by Prop. 17, $C \mid D$.

But $D < C$.

Impossible!

Therefore no such D exists.

Hence $C = A \times B$ is the least common multiple.

6 CASE II: A AND B NOT COPRIME

Now suppose A and B are not prime to one another.

By Prop. 33, let F and E be the least numbers having the same ratio as A and B and $F \mid E$.

$A : B = F : E$

(F, E) is the simplest pair in the ratio of A and B .

7 BUILD A COMMON MULTIPLE

By Prop. 19, from $A : B = F : E$.

$A \times E = B \times F = C$

$A \times C \mid B \times F$

Thus C is a common multiple of A and B .

By Prop. 19.

We claim C is the least such number.

8 SUPPOSE SMALLER EXISTS

Suppose some smaller number D is measured by both.

$A \mid D$, $B \mid D$, and $D < C$.

Let $D = A \times G$ and $D = B \times H$.

Then $A \times G \times B \mid B \times H$.

By Prop. 19.

$A \times B = H \times G$

9 COMPARE THE RATIOS

But $A : B = F : E$.

Therefore $F : E = H : G$.

Since F and E are the least numbers in this ratio, by Prop. 20, they measure the corresponding numbers.

$E \mid G$

So E divides G .

10 TRANSFER THE DIVISIBILITY

We have $C = A \times E$ and $D = A \times G$.

Since $E \mid G$, by Prop. 17,

$C \mid D$

Therefore on such D exists.

Hence C is the least number measured by both A and B .

11 THE TRANSDUCTION AGAIN

But $D < C$.

Impossible!

Therefore on such D exists.

12 CONCLUSION

In every case we have found that the number measured by both A and B .

That number is C .

ARISTOTLE TO ALEXANDER.

"When two numbers must together in harmony, seek the simplest rhythm that both can keep — and none smaller can keep as well."

THE PRINCIPLE IN ONE LINE

Let $d = \gcd(A, B)$. Then

$$\text{lcm}(A, B) = \frac{A \times B}{d}$$

If not coprime:

$$\text{lcm}(A, B) = \frac{A \times B}{\gcd(A, B)}$$

ALEXANDER'S REFLECTION:

"Thus the many paths of numbers lead to one least step that both can reach together."

Proposition 35

Statement. If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).

Proof.

Action 1. For let two numbers, A and B, (both) measure some number CD, and (let) E (be the) least (number measured by both A and B). I say that E also measures CD.

Action 2. For if E does not measure CD then let E leave CF less than itself (in) measuring DF .

Action 3. And since A and B (both) measure E, and E measures DF , A and B will thus also measure DF .

Action 4. And (A and B) also measure the whole of CD.

Action 5. Thus, they will also measure the remainder CF , which is less than E. The very thing is impossible. Thus, E cannot not measure CD. Thus, (E) measures (CD).

Action 6. (Which is) the very thing it was required to show.

Modern reading.

$$A \mid N, B \mid N \implies \text{lcm}(A, B) \mid N.$$

Therefore, If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).

EUCLID, BOOK VII — PROPOSITION 35

"If two numbers both measure some number, then the least number measured by them will also measure the same number."

1

Let A and B both measure some number CD.

Let E be the least number measured by both A and B.

$A \mid CD, B \mid CD$

$A \mid E, B \mid E$

We must prove that E also measures CD: $E \mid CD$.

2

Suppose, for contradiction, that E does not measure CD.

When E measures CD, there is a remainder.

$CD = DF + CF$

$\begin{matrix} \text{multiple} \\ \text{of E} \end{matrix}$ $\begin{matrix} \text{remainder} \\ \text{of E} \end{matrix}$

and $0 < CF < E$.

So CF is a remainder smaller than E.

3

Since DF is a multiple of E,

$E \mid DF$

And since A and B both measure E,

$A \mid E, B \mid E$

they also measure DF:

$A \mid DF, B \mid DF$

For what E measures, A and B also measure.

4

But A and B also measure the whole number CD.

$A \mid CD, B \mid CD$

Thus A and B measure both CD and DF.

5

Therefore A and B must also measure the remainder.

$A \mid CF, B \mid CF$

because $CF = CD - DF$.

Remove what both measure, what remains is still measured by both.

6

But CF is smaller than E.

So CF is a number measured by both A and B, smaller than their least common multiple E.

This contradicts that E is the least number measured by both A and B.

7

Our assumption was false. So E must measure CD.

$E \mid CD$

The least common multiple of A and B divides the number CD.

8

EXAMPLE

Let $A = 6, B = 8$.

Their least common multiple is $E = 24$.

Suppose a number N is measured by both.

$6 \mid N$ and $8 \mid N$

Then by Proposition 35,

$24 \mid N$

Whatever number they both measure, the least common multiple also measures.

9

PRINCIPLE

THE LEAST COMMON MULTIPLE DIVIDES EVERY COMMON MULTIPLE.

Thus the least common multiple is universal: it measures all that both measure.

ARISTOTLE TO ALEXANDER:

"As in a well-ordered army, the smallest unit that contains both powers commands every road where both can travel."

THE PRINCIPLE IN ONE LINE

If $A \mid N$ and $B \mid N$, and $E = \text{lcm}(A, B)$, then $E \mid N$.

ALEXANDER'S REFLECTION:

"So the one least number stands at the gate of every path where both may go, never too high, never too low."

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Proposition 36

Statement. To find the least number which three given numbers (all) measure.

Proof.

- Action 1.** Let A, B, and C be the three given numbers.
- Action 2.** So it is required to find the least number which they (all) measure.
- Action 3.** For let the least (number), D, measured by the two (numbers) A and B have been taken (Prop. VII.34).
- Action 4.** So C either measures, or does not measure, D. Let it, first of all, measure (D). And A and B also measure D. Thus, A, B, and C (all) measure D.
- Action 5.** So I say that (D is) also the least (number measured by A, B, and C).
- Action 6.** For if not, A, B, and C will (all) measure (some) number which is less than D.
- Action 7.** Let them measure E (which is less than D).
- Action 8.** Since A, B, and C (all) measure E then A and B thus also measure E.
- Action 9.** Thus, the least (number) measured by A and B will also measure (E) (Prop. VII.35).
- Action 10.** And D is the least (number) measured by A and B.
- Action 11.** Thus, D will measure E, the greater (measuring) the lesser. The very thing is impossible.
- Action 12.** Thus, A, B, and C cannot (all) measure some number which is less than D.
- Action 13.** Thus, A, B, and C (all) measure the least (number) D. So, again, let C not measure D.
- Action 14.** And let the least number, E, measured by C and D have been taken (Prop. VII.34).

Action 15. Since A and B measure D, and D measures E, A and B thus also measure E. And C also measures (E).

Action 16. Thus, A, B, and C (also) measure E.

Action 17. So I say that (E is) also the least (number measured by A, B, and C).

Action 18. For if not, A, B, and C will (all) measure some (number) which is less than E.

Action 19. Let them measure F (which is less than E).

Action 20. Since A, B, and C (all) measure F , A and B thus also measure F .

Action 21. Thus, the least (number) measured by A and B will also measure F (Prop. VII.35).

Action 22. And D is the least (number) measured by A and B. Thus, D measures F . And C also measures F . Thus, D and C (both) measure F .

Action 23. Hence, the least (number) measured by D and C will also measure F (Prop. VII.35).

Action 24. And E is the least (number) measured by C and D.

Action 25. Thus, E measures F , the greater (measuring) the lesser. The very thing is impossible.

Action 26. Thus, A, B, and C cannot measure some number which is less than E.

Action 27. Thus, E (is) the least (number) which is measured by A, B, and C.

Action 28. (Which is) the very thing it was required to show.

Modern reading.

$$\text{lcm}(A, B, C) = \text{lcm}(\text{lcm}(A, B), C).$$

Therefore, To find the least number which three given numbers (all) measure.

"To find the least number which three given numbers all measure."

"To find the least number which three given numbers all measure."

1 THE TASK
Find the three given numbers be A, B, C .
We seek the least number which all three measure.

2 FIRST STEP
First find the least number measured by A and B .
Let $D = \text{lcm}(A, B)$.
So $A \mid D$ and $B \mid D$.
D is the least common multiple of A and B .

3 ASK THE QUESTION
Does C measure D ?
Yes? No?
 $C \mid D$ $(C \nmid D)$
We must show D is the least such number.

4 CASE I: C MEASURES D
Suppose $C \mid D$.
Then $A \mid D, B \mid D, C \mid D$.
So D is a common multiple of all three.
We must show D is the least such number.

5 CASE I - CONTRADICTION
Suppose a smaller number E exists with $E < D$ and $A \mid E, B \mid E, C \mid E$.
Since $A \mid E$ and $B \mid E$, by Proposition 35,
 $D \mid E$.
But $E < D$. Impossible!
Hence D is the least number.

6 CASE II: C DOES NOT MEASURE D
Suppose $C \nmid D$.
Let E be the least number measured by C and D .
So $C \mid E$ and $D \mid E$.
 $E = \text{lcm}(C, D)$.

7 A AND B ALSO MEASURE E
Since $A \mid D$ and $B \mid D$, and $D \mid E$.
 $A \mid E$ and $B \mid E$.
Therefore A, B , and C all measure E .

8 CASE II - CONTRADICTION
Suppose a smaller number F exists with $F < E$ and $A \mid F, B \mid F, C \mid F$.
From $A \mid F$ and $B \mid F$, by Proposition 35,
 $D \mid F$.

9 FORCE E TO DIVIDE F
We already have $C \mid F$ and $D \mid F$.
Since E is the least number measured by C and D , by Proposition 35,
 $E \mid F$.

10 CONTRADICTION AGAIN
But $F < E$.
So the larger number E would measure the smaller number F .
Impossible!

11 CONCLUSION
Therefore E is the least number measured by A, B , and C .
If $C \nmid D$, then the answer is $E = \text{lcm}(D, C)$.

12 THE RULE IN ONE LINE
Let $D = \text{lcm}(A, B)$.
If $C \mid D$, the answer is D .
If $C \nmid D$, the answer is $\text{lcm}(D, C)$.
That is the least number which A, B , and C all measure.

ARISTOTLE TO ALEXANDER:
"First bring two numbers to the same station, that is $\text{lcm}(A, B)$. Then bring the third. If he arrives at that station as well, we are done; if not, find the next station where he and the first two meet."

THE PRINCIPLE IN ONE LINE
 $\text{lcm}(A, B, C) = \text{lcm}(\text{lcm}(A, B), C)$
the first common multiple of all three.

ALEXANDER'S REFLECTION:
"So the path to the meeting of three is built by first uniting two, then joining the third... until all three travel together."

EUCLID, BOOK VII — PROPOSITION 36

"To find the least number which three given numbers all measure."

1

THE TASK

Let the three given numbers be A, B , and C . We seek the least number which all three measure.

A, B, C

Our goal: the smallest number divisible by A, B , and C .

2

STEP 1: FIND $\text{lcm}(A, B)$

By Proposition 34, let D be the least number measured by both A and B .

$D = \text{lcm}(A, B)$

So $A \mid D$ and $B \mid D$.

Now we ask: Does C measure D ?

3

CASE I: C MEASURES D

Suppose $C \mid D$.

Then $A \mid D, B \mid D$, and $C \mid D$.

So D is a common multiple of all three.

Is D the least?

Assume some smaller number E exists.

4

Suppose $E < D$ and $A \mid E, B \mid E$, and $C \mid E$.

Since A and B measure E , by Proposition 35 their least common multiple D must also measure E .

$D \mid E$

But $E < D$. Impossible!

Therefore D is the least.

5

CASE II: C DOES NOT MEASURE D

Suppose C does not measure D . Let E be the least number measured by both C and D .

$E = \text{lcm}(C, D)$

So $C \mid E$ and $D \mid E$.

We will show E is the least number for A, B , and C .

6

Since $A \mid D$ and $B \mid D$, and $D \mid E$,

$A \mid E$ and $B \mid E$.

Together with $C \mid E$, all three numbers measure E .

$A \mid E, B \mid E, C \mid E$

So E is a common multiple of A, B , and C .

7

ASSUME A SMALLER EXISTS

Suppose some number F exists such that $F < E$ and $A \mid F, B \mid F, C \mid F$.

We seek a contradiction.

8

Since $A \mid F$ and $B \mid F$, by Proposition 35, $D \mid F$.

Also $C \mid F$. Thus both C and D measure F .

Therefore the least number measured by C and D , which is E , must also measure F .

$E \mid F$

9

But $F < E$.

A larger number, E , measures a smaller number, F .

Impossible!

Our assumption is false.

10

Therefore no such F exists. So E is the least number measured by A, B , and C .

$E = \text{lcm}(A, B, C)$

This completes the construction.

11

EXAMPLES

Example 1 (Case I)

$A = 4, B = 6, C = 3$

$D = \text{lcm}(4, 6) = 12$

Since $3 \nmid 12$,

$\text{lcm}(4, 6, 3) = 12$

Answer: 12

Example 2 (Case II)

$A = 4, B = 6, C = 9$

$D = \text{lcm}(4, 6) = 12$

Since $9 \nmid 12$,

$E = \text{lcm}(12, 9) = 36$

Answer: 36

12

THE PRINCIPLE

Find lcm of the first two. Then take lcm of that result with the third.

$\text{lcm}(A, B, C) = \text{lcm}(\text{lcm}(A, B), C)$

Thus we obtain the least number measured by all three.

ARISTOTLE TO ALEXANDER:

"The wise builder sets the first stones well, and then adds the next. So in numbers, build the least common multiple step by step."

THE PRINCIPLE IN ONE LINE

Let $D = \text{lcm}(A, B)$.

If $C \mid D$, then $\text{lcm}(A, B, C) = D$. If $C \nmid D$, then $\text{lcm}(A, B, C) = \text{lcm}(D, C)$.

Either way, the result is the least number which A, B , and C all measure.

ALEXANDER'S REFLECTION:

"First two walk together, then the third joins the path. The least step that serves all is the step we seek."

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Proposition 37

Statement. If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).

Proof.

Action 1. For let the number A be measured by some number B.

Action 2. I say that A has a part called the same as B.

Action 3. For as many times as B measures A, so many units let there be in C.

Action 4. Since B measures A according to the units in C, and the unit D also measures C according to the units in it, the unit D thus measures the number C as many times as B (measures) A.

Action 5. Thus, alternately, the unit D measures the number B as many times as C (measures) A (Prop. VII.15).

Action 6. Thus, which(ever) part the unit D is of the number B, C is also the same part of A.

Action 7. And the unit D is a part of the number B called the same as it (i.e., a Bth part).

Action 8. Thus, C is also a part of A called the same as B (i.e., C is the Bth part of A).

Action 9. Hence, A has a part C which is called the same as B (i.e., A has a Bth part).

Action 10. (Which is) the very thing it was required to show.

Modern reading.

$$B \mid A \implies A \text{ has a } B\text{-th part.}$$

Therefore, If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).

EUCLID, BOOK VII — PROPOSITION 37

"If a number is measured by some number, then the measured number will have a part called the same as the measuring number."

1



Aristotle: "Today we learn a beautiful truth from Euclid. If a number measures another, then the measured number has a part called the same as the measuring number."

2

THE GIVEN

Let B measure A.

B | A



B measures A

Aristotle: "Here B divides A — that is, B measures A."

3

COUNT THE MEASURES

Suppose B measures A C times.

A



C times

Aristotle: "So A is made of C parts, each equal to B."

4

INTRODUCE THE UNIT

Let D be one unit. Then D measures C as many times as B measures A.

D



C times

Aristotle: "The unit D is such that it fits into C exactly as many times as B fits into A."

5

SWITCH THE VIEW

By the alternation of Proposition 15,

D measures B as many times as C measures A.

D → B



as many times as C → A

Aristotle: "So the same unit D, which is a part of B, measures C in the same way that it measures A."

6

IDENTIFY THE PART

Since D is one unit, it is the B-th part of B. Thus C is also the B-th part of A.

A



C is the B-th part of A

Aristotle: "Therefore the number C, which measures A so many times as B, is the B-th part of A."

7

CONCLUDE

From $A = B \times C$, we have $C = \frac{A}{B}$.

So $\frac{A}{B}$ is the B-th part of A.

Aristotle: "Thus, whenever B measures A, the quotient A by B is the very part of A named after B."

8

THE MEANING

B | A



A has a B-th part.

Alexander: "So every time a number is divisible by another, we can find its named part. This is a perfect measure."



ARISTOTLE'S LESSON:
"Divisibility tells us how a number is built — and how to find its parts."

IN MODERN TERMS:
If $B \mid A$, then $\frac{A}{B}$ is the B-th part of A.
Example: 20 is five times 4, so 4 is the fifth part of 20.

THE FIFTH PART OF 20:

20



(5 parts)



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EUCLID, BOOK VII — PROPOSITION 37

"If a number is measured by some number, then the measured number will have a part called the same as the measuring number."

1

THE TASK

Let A be a number measured by B.

B

A

We must prove that A has a part called the same as B.

Our starting point: B measures A.

2

COUNT HOW MANY TIMES

Suppose B measures A exactly C times.

A

B B ... B

C times

C is the number of times B contains in A.

3

INTRODUCE THE UNIT

Let D be one unit. D measures C exactly C times.

D

C

(1 unit) C times

D measures C as many times as B measures A.

4

APPLY PROP. 15 ALTERNATELY

Since D measures C as many times as B measures A, by Proposition 15 (alternately) D measures B as many times as C measures A.

D measures B as many times as C measures A.

We turn the comparison in the opposite way.

5

THE UNIT IS THE B-TH PART OF B

D is one unit. Therefore D is the part of B called the same as B — the B-th part of B.

B

D

D is the B-th part of B.

6

TRANSFER THE SAME PART TO A

Since D is to B as C is to A, C is the same part of A that D is of B.

Therefore C is the B-th part of A.

A

C

C is the B-th part of A.

7

IDENTIFY C

But $A = B \times C$. Therefore $C = A/B$.

C is the B-th part of A.

$A = B \times C$

$C = \frac{A}{B}$

8

CONCLUSION

If B measures A, then A has a part called the same as B.

B

A

$\Rightarrow$

A has a B-th part.

This completes the proposition.

ARISTOTLE TO ALEXANDER:

"When one number contains another a certain number of times, that count itself becomes the named part of the greater number."

EXAMPLE

Let A = 20 and B = 5.
Since 5 measures 20 exactly 4 times ($20 = 5 \times 4$), 4 is the fifth part of 20.

5

5

5

5

ALEXANDER'S REFLECTION:

"So when something is measured many times by a number, that count is the part called by that number's name!"

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Proposition 38

Statement. If a number has any part whatever then it will be measured by a number called the same as the part.

Proof.

Action 1. For let the number A have any part whatever, B.

Action 2. And let the (number) C be called the same as the part B (i.e., B is the Cth part of A). I say that C measures A.

Action 3. For since B is a part of A called the same as C, and the unit D is also a part of C called the same as it (i.e., D is the Cth part of C), thus which(ever) part the unit D is of the number C, B is also the same part of A.

Action 4. Thus, the unit D measures the number C as many times as B (measures) A.

Action 5. Thus, alternately, the unit D measures the number B as many times as C (measures) A (Prop. VII.15). Thus, C measures A.

Action 6. (Which is) the very thing it was required to show.

Modern reading.

$$A \text{ has a } B\text{-th part} \implies B \mid A.$$

Therefore, If a number has any part whatever then it will be measured by a number called the same as the part.

"If a number has any part whatever, then it will be measured by a number called the same as the part."

"If a number has any part whatever, then it will be measured by a number called the same as the part."

1 Now prove the converse of the previous proposition.

If a number has any part, then it will be measured by a number called the same as the part.

Let A be a number which has some part.

2 Let B be that part of A. Let C be the number called the same as that part. C.

$B = \frac{A}{C}$

$\frac{B}{B} = \frac{\frac{A}{C}}{\frac{A}{C}} = \frac{B}{B}$

C parts

So A has a C-th part equal to B.

3 Let D be one unit. D is the C-th part of C, because C contains C units.

$C = \frac{D}{D} = \frac{C \text{ units}}{1 \text{ unit}}$

Thus D is to C as B is to A — both are C-th parts.

4 COMPARE THE SAME PARTS

$D : C = B : A$

For the same named part holds in both pairs.

5 USE PROP. 15 (ALTERNATELY)

Since D measures C exactly C times (D being one unit), B will measure A as many times as D measures C.

B measures A exactly C times.

6 INTERPRET THE RESULT

If B measures A exactly C times, then A consists of C copies of B.

$A = \frac{A}{B} = \frac{C \times B}{B} = C \times B$

Hence $A = C \times B$.

7 CONCLUDE DIVISIBILITY

Thus C measures A.

$\frac{C}{C} \mid \frac{A}{A}$

Therefore the number called the same as the part, measures the number.

8 EXAMPLE

Let A = 20 and suppose its fifth part is 4.

$4 = \frac{20}{5}$

So C = 5.

$\frac{4}{4} \mid \frac{20}{4} = \frac{5}{1}$

5 parts

Hence $5 \mid 20$ since $20 = 5 \times 4$

9 THE PRINCIPLE IN ONE LINE

If A has a C-th part, then C measures A.

In modern words:

If A has an exact n-th part, then n divides A.

ARISTOTLE TO ALEXANDER:

"Propositions 37 and 38 are keys of arithmetic: one opens the door, the other walks back through it."

TOGETHER WITH PROPOSITION 37

$C \mid A \iff A \text{ has a } C\text{-th part.}$

Divisibility and named parts are the same truth seen from two sides.

ALEXANDER'S REFLECTION:

"So if I know a number has a certain part, I know another number that will measure it exactly."

Proposition 39

Statement. To find the least number that will have given parts.

Proof.

Action 1. Let A, B, and C be the given parts.

Action 2. So it is required to find the least number which will have the parts A, B, and C (i.e., an Ath part, a Bth part, and a Cth part).

Action 3. For let D, E, and F be numbers having the same names as the parts A, B, and C (respectively).

Action 4. And let the least number, G, measured by D, E, and F , have been taken (Prop. VII.36).

Action 5. Thus, G has parts called the same as D, E, and F (Prop. VII.37).

Action 6. And A, B, and C are parts called the same as D, E, and F (respectively). Thus, G has the parts A, B, and C.

Action 7. So I say that (G) is also the least (number having the parts A, B, and C).

Action 8. For if not, there will be some number less than G which will have the parts A, B, and C. Let it be H.

Action 9. Since H has the parts A, B, and C, H will thus be measured by numbers called the same as the parts A, B, and C (Prop. VII.38).

Action 10. And D, E, and F are numbers called the same as the parts A, B, and C (respectively).

Action 11. Thus, H is measured by D, E, and F . And (H) is less than G. The very thing is impossible.

Action 12. Thus, there cannot be some number less than G which will have the parts A, B, and C.

Action 13. (Which is) the very thing it was required to show.

Modern reading.

$$N_{\min} = \text{lcm}(d_1, d_2, \dots, d_n).$$

Therefore, To find the least number that will have given parts.

EUCLID, BOOK VII — PROPOSITION 39

"To find the least number that will have given parts."

1 THE PROBLEM We are given certain parts — for example a half, a third and a fourth. What is the smallest number that has all these parts exactly?	2 NAME THE PARTS Let the given parts be $\frac{1}{A} \quad \frac{1}{B} \quad \frac{1}{C}$ Let the numbers named after these parts be D, E, F	3 TAKE THE LEAST COMMON MULTIPLE By Proposition 36, find the least number measured by D, E and F: $G = \text{lcm}(D, E, F)$	4 OBTAIN THE PARTS Since D, E and F measure G, by Proposition 37, G has a D-th part, an E-th part and an F-th part. Therefore G has the given parts $\frac{1}{A} \quad \frac{1}{B} \quad \frac{1}{C}$	
5 SHOW IT IS THE LEAST Suppose there is a smaller number H that also has the parts $\frac{1}{A} \quad \frac{1}{B} \quad \frac{1}{C}$ and suppose $H < G$.	6 APPLY PROPOSITION 38 Since H has these parts, by Proposition 38 the numbers named after the parts must measure H: $D \mid H, E \mid H, F \mid H$	7 REACH A CONTRADICTION But $G = \text{lcm}(D, E, F)$ is the least number measured by D, E and F. So no smaller number than G can be measured by all three. Yet $H < G$ and D, E, F measure H. <i>Impossible!</i>	8 CONCLUDE Therefore G is the least number that has the given parts. $G = \text{lcm}(D, E, F)$	
9 NUMERICAL EXAMPLE Find the least number with parts $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. Take the LCM of 2, 3 and 4: $\text{lcm}(2, 3, 4) = 12$	10 CHECK THE PARTS The number 12 has: $12 \div 2 = 6 \quad (\text{a half})$ $12 \div 3 = 4 \quad (\text{a third})$ $12 \div 4 = 3 \quad (\text{a fourth})$ So 12 has all the required parts, and no smaller number does.	11 THE GENERAL RULE The least number that has given parts $\frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_n}$ is $\text{lcm}(d_1, d_2, \dots, d_n)$	12 THE WISDOM "Numbers obey harmony, my boy," said Aristotle. "The smallest number that contains many parts is born from the unity of their measures—their least common multiple."	
ARISTOTLE TO ALEXANDER: "Proposition 39 shows us how the many can live together in the one—the smallest number that contains all their parts, through the harmony of their measures."		THE ESSENCE To find the least number that will have given parts, take the least common multiple of the numbers named after those parts.		ALEXANDER'S REFLECTION: "So the half, the third and the fourth meet in one number—the smallest that welcomes them all. That is the power of harmony in numbers!"

EUCLID, BOOK VII — PROPOSITION 39

"To find the least number that will have given parts."

1

THE TASK

We must find the least number that will have the given parts.

Suppose the given parts are the A-th part, the B-th part, and the C-th part.

2

NAME THE PARTS

Let D, E, F be the numbers called the same as those parts.

A is the D-th part
B is the E-th part
C is the F-th part

3

FIND THE LEAST NUMBER MEASURED BY D, E, F

By Proposition 36, let G be the least number measured by D, E, F.

$G = \text{lcm}(D, E, F)$

$D \mid G \quad E \mid G \quad F \mid G$

So D, E, F all measure G.

4

USE PROPOSITION 37

Since $D \mid G$, G has a D-th part.
Since $E \mid G$, G has an E-th part.
Since $F \mid G$, G has an F-th part.

Therefore G has the parts A, B, and C.

5

SUPPOSE A SMALLER NUMBER EXISTS

Let H be a number smaller than G that also has the parts A, B, and C.

$H < G$

Could there be a smaller number?

6

APPLY PROPOSITION 38

Since H has the A-th part, the number D measures H.
Since H has the B-th part, the number E measures H.
Since H has the C-th part, the number F measures H.

$D \mid H$
 $E \mid H$
 $F \mid H$

7

BUT G WAS THE LEAST

But G was the least number measured by D, E, and F.

$G = \text{lcm}(D, E, F)$ is the least number that D, E, and F all measure.

Yet H is also measured by D, E, and F, and $H < G$.

This is impossible!

8

CONCLUDE

Therefore no number smaller than G can have all the given parts.

Thus G is the least number that has the parts A, B, and C.

9

EXAMPLE

Find the least number having the parts $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

The numbers named the same are 2, 3, and 4.

$G = \text{lcm}(2, 3, 4) = 12$

So 12 has a half, a third, and a fourth.

$12 \div 2 = 6$ (half)
 $12 \div 3 = 4$ (third)
 $12 \div 4 = 3$ (fourth)

ARISTOTLE TO ALEXANDER:

When we seek the least number with certain parts, we take the least number measured by the names of those parts.

THE PRINCIPLE

Least number with given parts = lcm of the numbers named after those parts.

$N_{\min} = \text{lcm}(D, E, F)$

ALEXANDER'S REFLECTION:

"So the harmony of numbers shows itself again—in both divisibility and in their parts."

157

Conclusion

Book VII develops a coherent arithmetic from the notions of unit and number through common measures, prime and composite numbers, numerical ratios, coprimality, prime divisors, least representatives of ratios, least common multiples, and named numerical parts. Its later propositions bind these themes together: Euclid's Lemma controls prime divisors of products, the descent argument ensures prime divisors of composite numbers, and the final constructions identify the least numbers satisfying simultaneous divisibility and part conditions.