

EUCLID'S ELEMENTS

BOOK VI

SIMILAR FIGURES

Similarity, Proportion, Geometric Construction, and the Golden
Section

Illustrated Edition

Contents

Book VI

Introduction

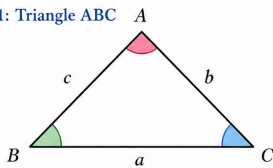
Book VI develops the geometry of similar rectilinear figures and proportional sides. It establishes criteria for triangle similarity, proportional constructions, geometric means, reciprocal proportionality, areas of similar figures, applications of areas, the extreme and mean ratio, the generalized Pythagorean theorem, and proportional relations between arcs and angles in equal circles.

Definition 1

Similar rectilinear figures are those which have their corresponding angles equal and the sides about the equal angles proportional.

DEFINITION 1 (Book VI). Similar rectilinear figures are those which have their angles separately equal and the corresponding sides about the equal angles proportional.

Figure 1: Triangle ABC



Angles:

$$\angle A = \angle A'$$

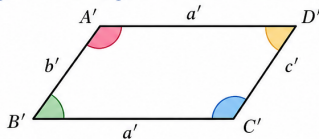
$$\angle B = \angle B'$$

$$\angle C = \angle C'$$

Corresponding sides are proportional:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$

Figure 2: Parallelogram A'B'C'D'



Angles:

$$\angle A' = \angle A$$

$$\angle B' = \angle B$$

$$\angle C' = \angle C$$

$$\angle D' = \angle D$$

Corresponding sides are proportional:

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CA} = \frac{D'A'}{A'B'}$$

Thus, the triangle ABC and the parallelogram A'B'C'D' are **similar**, because their **angles** are separately equal and their corresponding **sides** are proportional.

Definition 2

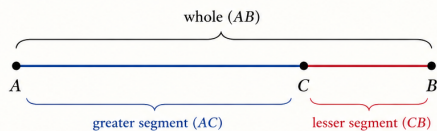
A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so the greater segment is to the lesser.

$$AB : AE = AE : EB$$

DEFINITION 2 (Book VI). A straight-line is said to have been cut in extreme and mean ratio when

$$\text{whole} : \text{greater segment} = \text{greater segment} : \text{lesser segment}.$$

Figure: Division in extreme and mean ratio



By definition,

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

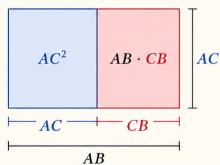
That is, the whole is to the greater segment as the greater segment is to the lesser segment.

UNDERSTANDING THE DEFINITION

Let AC be the greater segment and CB the lesser.

Since $\frac{AB}{AC} = \frac{AC}{CB}$, we have $AC^2 = AB \cdot CB$.

The square on the greater segment equals the rectangle contained by the whole and the lesser segment.



This division is also called the **GOLDEN SECTION**.

Numerically,

$$\frac{AB}{AC} = \frac{AC}{CB} = \varphi \approx 1.618033989\dots$$

where $\varphi = \frac{1 + \sqrt{5}}{2}$.



This ratio appears in many beautiful figures in geometry and in nature.



Definition 3

The height of any figure is the perpendicular drawn from its vertex to its base.

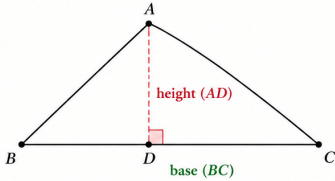
$$h \perp b$$

DEFINITION 3 (Book VI). The **height** of any figure is the perpendicular drawn from the **vertex** to the **base**.

WHAT IT MEANS

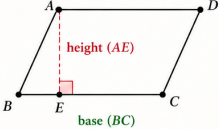
Choose a **vertex** (the top point) of the figure.
 From that **vertex**, draw a straight line **perpendicular** to the **base** (the supporting side or line).
 This perpendicular segment is called the **height**.

FIGURE: TRIANGLE ABC



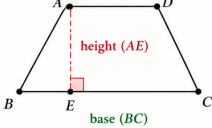
KEY FACTS
 $AD \perp BC$
 AD is the **height** of the triangle on base BC from vertex A.

PARALLELOGRAM ABCD

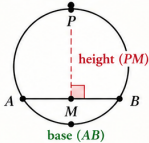


MORE EXAMPLES

TRAPEZOID ABCD



CIRCLE (with a chosen base)



In every case, the **height** is **perpendicular** to the **base**.



The **height** depends on the chosen **base** and **vertex**. A figure can have different heights with respect to different bases and vertices.



Proposition 1

Triangles and parallelograms of the same height are to one another as their bases.

Proof / Construction Actions

1. Set up triangles ABC and ACD with the same height h .
2. Produce the baseline BD in both directions to points H and L .
3. On the produced line mark $BG = GH = BC$.
4. Mark $DK = KL = CD$.
5. Draw AG, AH, AK, AL .
6. Since $CB = BG = GH$, triangles ABC, ABG , and AHG are equal (I.38).
7. However many times HC contains BC , triangle AHC contains triangle ABC the same number of times.
8. However many times LC contains CD , triangle ALC contains triangle ACD the same number of times.
9. If $HC = CL$, then $\triangle AHC = \triangle ACL$.
10. If $HC > CL$, then $\triangle AHC > \triangle ACL$.
11. If $HC < CL$, then $\triangle AHC < \triangle ACL$.
12. By Definition V.5, the corresponding equal multiples compare alike; therefore $BC : CD = \triangle ABC : \triangle ACD$.
13. Introduce parallelograms EC and FC : EC is twice $\triangle ABC$, and FC is twice $\triangle ACD$ (I.34).
14. By V.15, doubling preserves the ratio, so $\triangle ABC : \triangle ACD = EC : FC$.
15. Combine the ratios: $BC : CD = EC : FC$.

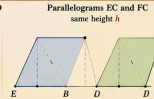
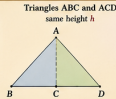
Final Conclusion

$$\text{Figure}_1 : \text{Figure}_2 = \text{base}_1 : \text{base}_2$$

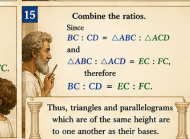
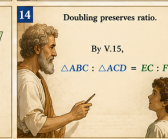
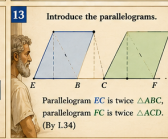
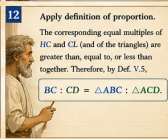
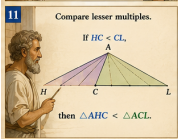
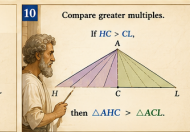
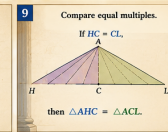
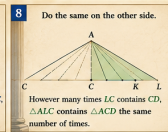
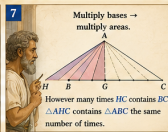
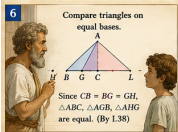
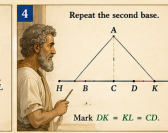
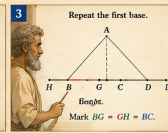
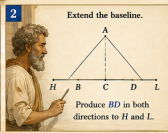
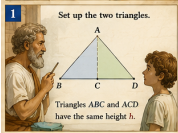


BOOK VI – PROPOSITION 1

Triangles and parallelograms which are of the same height are to one another as their bases.



To prove:
 $BC : CD = \triangle ABC : \triangle ACD$
and
 $BC : CD = EC : FC$

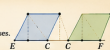


SAME HEIGHT:
Perpendicular from A to base BD (or its extension).

TRIANGLES:
Ratio of areas = ratio of bases.



PARALLELOGRAMS:
Ratio of areas = ratio of bases.



Remember: When figures stand on the same height, the area is determined only by the base.

Proposition 2

If a straight line is drawn parallel to one side of a triangle, it cuts the other two sides proportionally; and conversely.

Proof / Construction Actions

1. Take triangle ABC , with D on AB and E on AC .
2. Draw $DE \parallel BC$.
3. Join BE .
4. Join CD .
5. Triangles BDE and CDE stand on the same base DE and between the same parallels, so $\triangle BDE = \triangle CDE$ (I.38).
6. By V.7, equals have the same ratio to the same magnitude: $\triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE$.
7. By VI.1 on the left pair, $\triangle BDE : \triangle ADE = BD : DA$.
8. By VI.1 on the right pair, $\triangle CDE : \triangle ADE = CE : EA$; hence $BD : DA = CE : EA$.
9. For the converse, suppose $BD : DA = CE : EA$, and join DE .
10. By VI.1 on the left pair, $BD : DA = \triangle BDE : \triangle ADE$.
11. By VI.1 on the right pair, $CE : EA = \triangle CDE : \triangle ADE$.
12. Transfer the given proportion to obtain $\triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE$.
13. By V.9, $\triangle BDE = \triangle CDE$.
14. The equal triangles BDE and CDE stand on the same base DE .
15. By I.39 (converse), equal triangles on the same base lie between the same parallels; therefore $BC \parallel DE$.
16. Hence the converse is proved: proportional cuts on the two sides are joined by a line parallel to the remaining side.

Final Conclusion

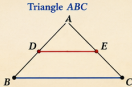
$$AD : DB = AE : EC$$



BOOK VI – PROPOSITION 2

If a straight-line is drawn parallel to one side of a triangle, it cuts the other two sides proportionally. And conversely, if two sides are cut proportionally, the straight-line joining the cutting points is parallel to the remaining side.

Triangle ABC



To prove:

(i) If $DE \parallel BC$,
then $BD:DA = CE:EA$.

(ii) If $BD:DA = CE:EA$,
then $DE \parallel BC$.

PART I – A PARALLEL PRODUCES PROPORTIONAL DIVISIONS ($DE \parallel BC \Rightarrow BD:DA = CE:EA$)

1 Draw the triangle.  Take triangle ABC. Let D be on AB and E be on AC.	2 Draw the parallel.  Draw DE parallel to BC.	3 Join BE.  Join the straight line BE.	4 Join CD.  Join the straight line CD.	5 Compare $\triangle BDE$ and $\triangle CDE$.  Triangles BDE and CDE stand on the same base DE and between the same parallels ($DE \parallel BC$). By L38, $\triangle BDE = \triangle CDE$.	6 Compare both with $\triangle ADE$.  By V7 equals have the same ratio to the same, $\triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE$.	7 Use VL1 on the left pair.  $\triangle BDE$ and $\triangle ADE$ have the same height from E to AB. By VL1, $\triangle BDE : \triangle ADE = BD:DA$.	8 Use VL1 on the right pair. Combine.  $\triangle CDE$ and $\triangle ADE$ have the same height from D to AC. By VL1, $\triangle CDE : \triangle ADE = CE:EA$. Therefore, $BD:DA = CE:EA$.
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PART II – PROPORTIONAL DIVISIONS PRODUCE A PARALLEL ($BD:DA = CE:EA \Rightarrow DE \parallel BC$)

9 Assume the proportion.  Suppose $BD:DA = CE:EA$. Join DE.	10 Use VL1 on the left pair.  $\triangle BDE$ and $\triangle ADE$ have the same height from E to AB. By VL1, $BD:DA = \triangle BDE : \triangle ADE$.	11 Use VL1 on the right pair.  $\triangle CDE$ and $\triangle ADE$ have the same height from D to AC. By VL1, $CE:EA = \triangle CDE : \triangle ADE$.	12 Transfer the given proportion.  Since $BD:DA = CE:EA$, therefore $\triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE$.	13 Conclude equality.  By V8, if two magnitudes have the same ratio to a third, they are equal. Thus, $\triangle BDE = \triangle CDE$.	14 They stand on the same base.  Triangles BDE and CDE stand on the same base DE.	15 Apply L39 (converse).  Equal triangles standing on the same base lie between the same parallels. By L39, B and C lie on a line parallel to DE.	16 Therefore, parallel.  Hence $DE \parallel BC$.
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Thus, a line drawn parallel to one side of a triangle cuts the other two sides proportionally, and conversely, proportional cuts on two sides are joined by a line parallel to the remaining side.




Proposition 3

An angle bisector of a triangle divides the opposite side in the ratio of the sides containing the angle; and conversely.

Proof / Construction Actions

1. Draw triangle ABC .
2. Bisect $\angle BAC$ with AD , so $\angle BAD = \angle DAC$.
3. Through C , draw $CE \parallel AD$.
4. Extend BA to meet CE at E .
5. Since $CE \parallel AD$ and AC is a transversal, $\angle ACE = \angle CAD$ (I.29).
6. Since AD bisects $\angle BAC$, $\angle CAD = \angle BAD$.
7. Since AE cuts the parallels AD and CE , $\angle AEC = \angle BAD$ (I.29).
8. Therefore $\angle ACE = \angle AEC$.
9. Hence $AE = AC$ (I.6).
10. In triangle BCE , since $AD \parallel CE$, VI.2 gives $BD : DC = BA : AE$.
11. Substituting $AE = AC$, obtain $BD : DC = BA : AC$.
12. Conversely, assume $BD : DC = BA : AC$, and join AD .
13. Through C , draw $CE \parallel AD$, extending BA to meet CE at E .
14. In triangle BCE , VI.2 gives $BD : DC = BA : AE$.
15. But $BD : DC = BA : AC$ is given; therefore $BA : AC = BA : AE$.
16. By V.9, $AC = AE$.
17. In triangle ACE , equal sides give equal opposite angles: $\angle AEC = \angle ACE$ (I.5).
18. Since $AD \parallel CE$, $\angle AEC = \angle BAD$ and $\angle ACE = \angle CAD$ (I.29).
19. Therefore $\angle BAD = \angle CAD$, so AD bisects $\angle BAC$.

Final Conclusion

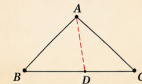
$$BD : DC = BA : AC$$



BOOK VI – PROPOSITION 3

If an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And conversely, if the segments of the base have the same ratio as the remaining sides, then the straight-line joining the vertex to the cutting point will bisect the angle.

THE TRIANGLE



TO PROVE

- (i) If AD bisects $\angle BAC$, then $BD:DC = BA:AC$.
- (ii) If $BD:DC = BA:AC$, then AD bisects $\angle BAC$.

PART I – THE ANGLE BISECTOR PRODUCES PROPORTIONAL BASE SEGMENTS ($\angle BAD = \angle DAC \Rightarrow BD:DC = BA:AC$)

1 Draw triangle ABC.	2 Bisect angle BAC with AD. $\angle BAD = \angle DAC$. To prove $BD:DC = BA:AC$.	3 Through C, draw $CE \parallel AD$.	4 Extend BA to meet CE at E.	5 Since $CE \parallel AD$ and AC cuts them, $\angle ACE = \angle CAD$ (By I.29)	6 But AD bisects $\angle BAC$. $\angle CAD = \angle BAD$. Therefore $\angle ACE = \angle BAD$.	7 Since AE cuts the parallels AD and CE, $\angle AEC = \angle BAD$ (By I.29)	8 Thus $\angle ACE = \angle AEC$.	9 Therefore $AE = AC$. (By I.6)	10 In $\triangle BCE$, since $AD \parallel CE$, $BD:DC = BA:AE$ (By VI.2)	11 Therefore $BD:DC = BA:AC$. (By I.6)
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PART II – PROPORTIONAL BASE SEGMENTS PRODUCE AN ANGLE BISECTOR ($BD:DC = BA:AC \Rightarrow \angle BAD = \angle DAC$)

12 Assume $BD:DC = BA:AC$. Join AD. To prove $\angle BAD = \angle DAC$.	13 Through C, draw $CE \parallel AD$, and extend BA to meet CE at E.	14 In $\triangle BCE$, since $AD \parallel CE$, $BD:DC = BA:AE$ (By VI.2)	15 But we are given $BD:DC = BA:AC$. Therefore $BA:AC = BA:AE$.	16 Hence $AC = AE$. (By V.9)	17 In $\triangle ACE$, $AC = AE$, so the opposite angles are equal. (By I.5) $\angle AEC = \angle ACE$.	18 Since $AD \parallel CE$, $\angle AEC = \angle BAD$ and $\angle ACE = \angle CAD$. (By I.29)	19 Therefore $\angle BAD = \angle CAD$. AD bisects $\angle BAC$.
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Thus, if an angle is bisected and the bisector meets the base,

$$BD:DC = BA:AC.$$

And conversely, if the base is cut so that $BD:DC = BA:AC$, then the line from the vertex to the cutting point bisects the angle.

Proposition 4

Equiangular triangles have their corresponding sides proportional.

Proof / Construction Actions

1. Place BC and CE on one straight line.
2. In $\triangle ABC$, $\angle ABC + \angle ACB < 2$ right angles (I.17).
3. Since $\angle ACB = \angle DEC$, we have $\angle ABC + \angle DEC < 2$ right angles.
4. Produce BA and ED ; they meet at F .
5. Since $\angle DCE = \angle ABC$, by I.28 we have $BF \parallel CD$.
6. Since $\angle ACB = \angle DEC$, by I.28 we have $AC \parallel FE$.
7. Thus $FACD$ is a parallelogram.
8. In parallelogram $FACD$, by I.34, $FA = DC$ and $AC = FD$.
9. Since $AC \parallel FE$, in triangle FBE , VI.2 gives $BA : AF = BC : CE$.
10. But $AF = DC$; therefore $BA : DC = BC : CE$.
11. By V.16 (alternation), $AB : BC = DC : CE$.
12. Since $CD \parallel BF$, in triangle FCE , VI.2 gives $BC : CE = FD : DE$.
13. But $FD = AC$; therefore $BC : CE = AC : DE$, and by V.16, $BC : CA = CE : ED$.
14. We now have $AB : BC = DC : CE$ and $BC : CA = CE : ED$.
15. By V.22, $BA : AC = CD : DE$. Thus all corresponding sides are proportional.

Final Conclusion

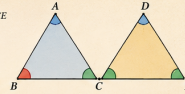
Corresponding sides are proportional



BOOK VI – PROPOSITION 4

In equiangular triangles, the sides about the equal angles are proportional, and the sides subtending equal angles correspond.

Given:
 $\triangle ABC$ and $\triangle DCE$
 are equiangular:
 $\angle ABC = \angle DCE$
 $\angle BAC = \angle CDE$
 $\angle ACB = \angle CED$



TO PROVE:
 $AB : BC = DC : CE$
 $BC : CA = CE : ED$
 $BA : AC = CD : DE$

PANEL 1 – CONSTRUCTION AND PARALLEL LINES

1 Place BC and CE in one straight line.

2 In $\triangle ABC$, $\angle ABC + \angle ACB < 2$ right angles. (By I.17)

3 Since $\angle ACB = \angle DEC$, $\angle ABC + \angle DEC < 2$ right angles.

4 Produce BA and ED . They meet at F .

5 Since $\angle DCE = \angle ABC$, by I.28, $BF \parallel CD$.

6 Since $\angle ACB = \angle DEC$, by I.28, $AC \parallel FE$.

7 $BF \parallel CD$ and $AC \parallel FE$. Therefore, $FACD$ is a parallelogram.

PANEL 2 – USING THE PARALLELOGRAM AND PROPOSITION VI.2

8 In parallelogram $FACD$, by I.34,
 $FA = DC$
 $AC = FD$

9 Since $AC \parallel FE$, in $\triangle FBE$, by VI.2,
 $BA : AF = BC : CE$.

10 But $AF = DC$. Therefore,
 $BA : DC = BC : CE$.

11 By V.16 (alternation),
 $AB : BC = DC : CE$.

12 Since $CD \parallel BF$, in $\triangle FCE$, by VI.2,
 $BC : CE = FD : DE$.

13 But $FD = AC$. Therefore,
 $BC : CE = AC : DE$.
 By V.16,
 $BC : CA = CE : ED$.

PANEL 3 – COMBINING THE PROPORTIONS

14 From panel 2 we have:
 $AB : BC = DC : CE$
 $BC : CA = CE : ED$

15 By V.22 (equality),
 $BA : AC = CD : DE$.

Hence, in equiangular triangles $\triangle ABC$ and $\triangle DCE$,
 $AB : BC = DC : CE$, $BC : CA = CE : ED$, $BA : AC = CD : DE$.

Thus the sides about equal angles are proportional, and the sides subtending equal angles correspond.

Proposition 5

Triangles whose corresponding sides are proportional are equiangular.

Proof / Construction Actions

1. Start with triangles ABC and DEF , with the three proportionalities given.
2. On EF , construct $\angle FEG = \angle ABC$ (I.23).
3. At F , construct $\angle EFG = \angle ACB$ (I.23).
4. Then in triangle EGF , $\angle EGF = \angle BAC$ (I.32).
5. Therefore triangles ABC and EGF are equiangular.
6. By VI.4, $AB : BC = GE : EF$.
7. But $AB : BC = DE : EF$ is given; therefore $DE : EF = GE : EF$.
8. By V.9, $DE = GE$.
9. Similarly, using the other proportionalities and VI.4, obtain $DF = GF$.
10. Now $DE = GE$, EF is common, and $DF = GF$; thus triangles DEF and GEF have their three sides respectively equal.
11. By I.8 (SSS), the triangles are equal; therefore $\angle DEF = \angle GEF$.
12. Also $\angle DFE = \angle GFE$ and $\angle EDF = \angle EGF$.
13. By construction, $\angle GEF = \angle ABC$.
14. By construction, $\angle GFE = \angle ACB$.
15. And $\angle EGF = \angle BAC$; therefore triangles ABC and DEF are equiangular.

Final Conclusion

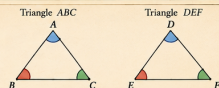
The triangles are equiangular



BOOK VI – PROPOSITION 5

If two triangles have proportional sides, then the triangles will be equiangular, and the angles which corresponding sides subtend will be equal.

GIVEN
Triangles ABC and DEF
 $AB : BC = DE : EF$
 $BC : CA = EF : FD$
 $BA : AC = ED : DF$



PANEL 1 – CONSTRUCT AUXILIARY TRIANGLE EGF

- Start with triangles ABC and DEF , with the three proportionalities given.
- On EF , construct $\angle EFG = \angle ABC$. (By I.23)
- At F , construct $\angle EFG = \angle ACB$. (By I.23)
- Then in triangle EGF , $\angle EGF = \angle BAC$. (By I.32)
- Therefore, triangles ABC and EGF are equiangular.

PANEL 2 – SHOW $DE = GE$ AND $DF = GF$

- Since ABC and EGF are equiangular, by Prop. VI.4, $AB : BC = GE : EF$.
- But given $AB : BC = DE : EF$. Therefore, $DE : EF = GE : EF$.
- Hence, by Def. V.9, $DE = GE$.
- Similarly, using the other proportionalities and VI.4, we obtain $DF = GF$.
- Now $DE = GE$, $EF = EF$ (common), and $DF = GF$. So triangles DEF and GEF have three sides equal respectively.

PANEL 3 – CONCLUDE EQUAL ANGLES

- By I.8 (SSS), the triangles DEF and GEF are equal. Therefore, $\angle DEF = \angle GEF$.
- Also, $\angle DFE = \angle GFE$ and $\angle EDF = \angle EGF$.
- But by construction, $\angle GEF = \angle ABC$.
- And $\angle GFE = \angle ACB$.
- And $\angle EGF = \angle BAC$.

Therefore, $\angle ABC = \angle DEF$ $\angle BCA = \angle EFD$ $\angle BAC = \angle EDF$

Triangles ABC and DEF are equiangular.

Proposition 6

If two triangles have one angle equal and the sides about those equal angles proportional, then the triangles are equiangular.

Proof / Construction Actions

1. Start with triangles ABC and DEF , with $\angle BAC = \angle EDF$ and $BA : AC = ED : DF$.
2. On the straight line DF , construct $\angle FDG = \angle BAC$ (which also equals $\angle EDF$) (I.23).
3. At F , construct $\angle DFG = \angle ACB$ (I.23).
4. Then triangles ABC and DGF have two angles equal; by I.32 the remaining angles are equal as well.
5. Thus triangles ABC and DGF are equiangular.
6. By VI.4, their corresponding sides are proportional; in particular $BA : AC = GD : DF$.
7. But the hypothesis gives $BA : AC = ED : DF$; hence $ED : DF = GD : DF$.
8. By V.9, $DE = DG$.
9. In triangles DEF and DGF , $DE = DG$, DF is common, and $\angle EDF = \angle GDF$; by I.4 (SAS), the triangles are equal.
10. Therefore $EF = GF$, and in particular $\angle DFE = \angle DFG$.
11. By construction $\angle DFG = \angle ACB$; hence $\angle DFE = \angle ACB$.
12. Also from the hypothesis, $\angle BAC = \angle EDF$.
13. Since triangles DEF and DGF are equal, their remaining angles are equal; and from step 4, $\angle DGF = \angle ABC$. Hence $\angle DEF = \angle ABC$.
14. Thus $\angle BAC = \angle EDF$, $\angle ACB = \angle DFE$, and $\angle ABC = \angle DEF$.
15. Therefore triangles ABC and DEF are equiangular, and corresponding angles are equal.

Final Conclusion

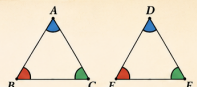
The triangles are equiangular



BOOK VI – PROPOSITION 6

If two triangles have one angle equal to one angle,
and the sides about the equal angles proportional,
then the triangles will be equiangular, and will have
the angles which corresponding sides subtend equal.

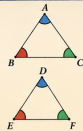
GIVEN
In triangles ABC and DEF ,
 $\angle BAC = \angle EDF$
and
 $BA : AC = ED : DF$



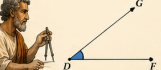
TO PROVE
 $\angle ABC = \angle DEF$
 $\angle ACB = \angle DFE$
Therefore,
 $\triangle ABC \sim \triangle DEF$

PANEL 1 – CONSTRUCT AUXILIARY TRIANGLE DGF

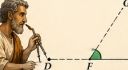
1 Start with triangles ABC and DEF , with $\angle BAC = \angle EDF$ and $BA : AC = ED : DF$.



2 On the straight line DF , construct $\angle FDG = \angle BAC$ (which also equals $\angle EDF$). (By I.23)



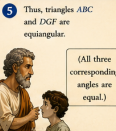
3 At F , construct $\angle DFG = \angle ACB$. (By I.23)



4 Then in triangles ABC and DGF , two angles are equal: $\angle BAC = \angle FDG$ and $\angle ACB = \angle DFG$. Therefore, by I.32, $\angle ABC = \angle DGF$.

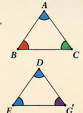


5 Thus, triangles ABC and DGF are equiangular. (All three corresponding angles are equal.)



PANEL 2 – SHOW $DE = DG$ AND $EF = GF$

6 Since the triangles are equiangular, by Prop. VI.4, their corresponding sides are proportional. In particular, $BA : AC = GD : DF$.



7 But the hypothesis gives $BA : AC = ED : DF$. Hence, $ED : DF = GD : DF$.



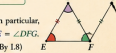
8 Since ED and GD have the same ratio to DF , by V.9, $DE = DG$.



9 In triangles DEF and DGF , we have $DE = DG$, $DF = DF$ (common), and $\angle EDF = \angle GDF$. (By construction) By I.4 (SAS), $\triangle DEF \cong \triangle DGF$.



10 Therefore, corresponding sides and angles are equal. Hence, $EF = GF$. And in particular, $\angle DEF = \angle DFG$. (By I.8)

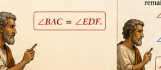


PANEL 3 – CONCLUDE ALL CORRESPONDING ANGLES ARE EQUAL

11 By construction, $\angle DFG = \angle ACB$. From step 10, $\angle DFE = \angle DFG$. Therefore, $\angle DFE = \angle ACB$.



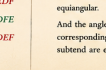
12 Also from the hypothesis, $\angle BAC = \angle EDF$.



13 Triangles DEF and DGF are equal, so the remaining angles are equal: $\angle DEF = \angle DGF$. But from step 4, $\angle DGF = \angle ABC$. Therefore, $\angle DEF = \angle ABC$.



14 Thus we have $\angle BAC = \angle EDF$, $\angle ACB = \angle DFE$, $\angle ABC = \angle DEF$. (By I.32)



15 Therefore, triangles ABC and DEF are equiangular. And the angles which corresponding sides subtend are equal.



Hence, if two triangles have one angle equal to one angle,
and the sides about the equal angles proportional,
then the triangles are equiangular,
and corresponding angles are equal.

Proposition 7

If two triangles have one angle equal, the sides about two other angles proportional, and the remaining angles both less than or both not less than right angles, then the triangles are equiangular.

Proof / Construction Actions

1. Start with triangles ABC and DEF , where $\angle BAC = \angle EDF$ and $AB : BC = DE : EF$. First assume the remaining angles C and F are less than a right angle.
2. Suppose $\angle ABC \neq \angle DEF$; let $\angle ABC > \angle DEF$.
3. At B , construct $\angle ABG = \angle DEF$ (I.23).
4. Since $\angle BAG = \angle EDF$ and $\angle ABG = \angle DEF$, by I.32 $\angle AGB = \angle DFE$.
5. Thus triangles ABG and DEF are equiangular.
6. By VI.4, $AB : BG = DE : EF$.
7. But $AB : BC = DE : EF$ is given; hence $AB : BG = AB : BC$.
8. By V.9, $BG = BC$.
9. Therefore triangle BGC is isosceles, so $\angle BGC = \angle BCG$ (I.5).
10. Since $\angle C$ is less than a right angle, $\angle BCG < 90^\circ$.
11. Hence $\angle BGC < 90^\circ$.
12. Angles BGC and AGB are adjacent on a straight line; therefore by I.13, $\angle AGB > 90^\circ$.
13. But $\angle AGB = \angle DFE$; hence $\angle DFE > 90^\circ$, contradicting the assumption that F is less than a right angle.
14. Therefore the supposition was wrong, and $\angle ABC = \angle DEF$.
15. Since $\angle BAC = \angle EDF$, the remaining angles are equal: $\angle ACB = \angle DFE$.

16. Now assume instead that both remaining angles are not less than a right angle; repeat the same construction.
17. As before, proportionality again gives $BG = BC$.
18. Therefore $\angle BGC = \angle BCG$; but now $\angle C \geq 90^\circ$, so $\angle BCG \geq 90^\circ$ and also $\angle BGC \geq 90^\circ$.
19. Then in triangle BGC , $\angle BGC + \angle BCG \geq 180^\circ$, impossible by I.17. Contradiction.
20. Therefore $\angle ABC = \angle DEF$.
21. Since $\angle BAC = \angle EDF$, the remaining angles are equal: $\angle ACB = \angle DFE$.
22. In either case all three corresponding angles are equal; therefore triangles ABC and DEF are equiangular.

Final Conclusion

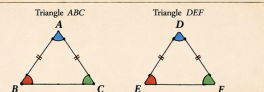
The triangles are equiangular



BOOK VI – PROPOSITION 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right angles, then the triangles will be equiangular, and will have the angles about which the sides are proportional equal.

GIVEN
In triangles ABC and DEF ,
 $\angle BAC = \angle EDF$
 $AB : BC = DE : EF$
The remaining angles at C and F are both acute, or both not acute.



PANEL 1 – CONSTRUCTION AND SETTING UP (ACTIONS 1–8)

- Start with triangles ABC and DEF , where $\angle BAC = \angle EDF$ and $AB : BC = DE : EF$. Assume both $\angle C$ and $\angle F$ are less than a right angle.
- Suppose $\angle ABC \neq \angle DEF$. Then one is greater. Let $\angle ABC > \angle DEF$.
- At B , construct $\angle ABG = \angle DEF$. (By I.23)
- At B , we have $\angle BAG = \angle EDF$, and $\angle ABC = \angle DEF$. Therefore, $\angle AGB = \angle DFE$. (By I.32)
- Thus triangles ABG and DEF are equiangular.
- By Prop. VI.4, for equiangular triangles, $AB : BG = DE : EF$.
- But given $AB : BC = DE : EF$. Hence $AB : BG = AB : BC$.
- By V.9 (equal ratios), $BG = BC$.

PANEL 2 – REACHING A CONTRADICTION (ACTIONS 9–15)

- Since $BG = BC$, triangle BGC is isosceles. Therefore, by I.5, $\angle BGC = \angle BCG$.
- But $\angle C$ is less than a right angle. Therefore $\angle BCG < 90^\circ$.
- Since $BG = BC$, $\angle BGC = \angle BCG$. Thus $\angle BGC < 90^\circ$.
- Angles BGC and AGB are adjacent on a straight line. Therefore, by I.13, $\angle AGB > 90^\circ$.
- But $\angle AGB = \angle DFE$. Hence $\angle DFE > 90^\circ$. This contradicts the assumption that $\angle F$ is less than a right angle.
- Therefore our assumption was wrong. Hence $\angle ABC = \angle DEF$.
- Since $\angle BAC = \angle EDF$, the remaining angles are equal. Therefore $\angle ACB = \angle DFE$. Thus triangles ABC and DEF are equiangular.

PANEL 3 – SECOND CASE: ANGLES NOT LESS THAN A RIGHT ANGLE (ACTIONS 16–22)

- Now assume both remaining angles are not less than a right angle. Repeat the same construction as before.
- As before, from the proportionality, we again obtain $BG = BC$.
- Therefore, $\angle BGC = \angle BCG$. But $\angle C$ is not less than a right angle. Hence $\angle BCG \geq 90^\circ$, and so $\angle BGC \geq 90^\circ$.
- Then in triangle BGC , $\angle BGC + \angle BCG \geq 180^\circ$, which is impossible. (By I.17). Contradiction.
- Therefore, $\angle ABC = \angle DEF$.
- And since $\angle BAC = \angle EDF$, the remaining angles are equal. Thus $\angle ACB = \angle DFE$.
- In other cases, we have $\angle BAC = \angle EDF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Therefore, triangles ABC and DEF are equiangular.

Therefore, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right angles, then the triangles are equiangular, and corresponding angles are equal.

Proposition 8

In a right triangle, the perpendicular drawn from the right angle to the hypotenuse creates two triangles similar to the whole and to one another.

Corollary. The perpendicular from the right angle to the hypotenuse is a mean proportional between the two segments of the hypotenuse.

Proof / Construction Actions

1. Draw right triangle ABC with $\angle BAC = 90^\circ$.
2. From A , draw $AD \perp BC$.
3. In $\triangle ABC$, $\angle BAC = 90^\circ$; in $\triangle ABD$, $\angle ADB = 90^\circ$. Therefore $\angle BAC = \angle ADB$.
4. Triangles ABC and ABD have angle B in common: $\angle ABC = \angle ABD$.
5. By I.32, $\angle ACB = \angle BAD$.
6. Thus $\triangle ABC$ and $\triangle ABD$ are equiangular.
7. By VI.4, corresponding sides are proportional: $BC : BA = AB : BD = AC : AD$.
8. By Definition VI.1, since the triangles are equiangular and corresponding sides proportional, $\triangle ABC \sim \triangle ABD$.
9. By the same reasoning for triangles ABC and ADC , $\angle BAC = \angle CDA = 90^\circ$, and they have angle C in common.
10. By I.32, $\angle ABC = \angle CAD$.
11. Thus $\triangle ABC$ and $\triangle ADC$ are equiangular.
12. By VI.4, $BC : AC = AC : DC = BA : AD$; hence $\triangle ABC \sim \triangle ADC$.
13. In triangles ABD and ADC , $\angle BDA = \angle ADC = 90^\circ$.
14. From panel 1, $\angle BAD = \angle ACB$.

15. By I.32, $\angle ABD = \angle DAC$.
16. Thus $\triangle ABD$ and $\triangle ADC$ are equiangular; by VI.4, $BD : AD = AD : DC = BA : AC$.
17. From the preceding steps, all three triangles ABC , ABD , and ADC are similar to one another.
18. Since $\triangle ABD \sim \triangle ADC$, $BD : AD = AD : DC$; therefore $AD^2 = BD \cdot DC$, so AD is the mean proportional between the two segments of the hypotenuse.

Final Conclusion

$\triangle ABD \sim \triangle ADC \sim \triangle ABC$



BOOK VI – PROPOSITION 8
If, in a right-angled triangle, a straight-line is drawn from the right angle perpendicular to the base, then the triangles around the perpendicular are similar to the whole triangle and to one another.

PANEL 1 – CONSTRUCT THE TRIANGLES AND SHOW $\triangle ABC \sim \triangle ABD$

1 Draw right triangle ABC with $\angle BAC = 90^\circ$.



2 From A draw $AD \perp BC$.



3 In $\triangle ABC$, $\angle BAC = 90^\circ$. In $\triangle ABD$, $\angle ADB = 90^\circ$. Therefore, $\angle BAC = \angle ADB$.



4 Triangles ABC and ABD have the angle at B in common. $\angle ABC = \angle ABD$.



5 Therefore, by I.32, $\angle ACB = \angle BAD$.



6 Thus $\triangle ABC$ and $\triangle ABD$ are equiangular.



7 By Prop. VI.4, corresponding sides are proportional:
 $BC : BA = AB : BD = AC : AD$.

Correspondence of angles
 $\angle BAC \leftrightarrow \angle ADB$
 $\angle ABC \leftrightarrow \angle ABD$
 $\angle ACB \leftrightarrow \angle BAD$



8 By Def. VI.1, since the triangles are equiangular and corresponding sides are proportional, $\triangle ABC \sim \triangle ABD$.



9 By the same reasoning for triangles ABC and ADC , $\angle BAC = \angle CDA = 90^\circ$. They have also the angle at C in common.



PANEL 2 – SHOW $\triangle ABC \sim \triangle ADC$ AND $\triangle ABD \sim \triangle ADC$

10 Therefore, by I.32, $\angle ABC = \angle CAD$.



11 Thus $\triangle ABC$ and $\triangle ADC$ are equiangular.



12 By Prop. VI.4, $BC : AC = AC : DC = BA : AD$.

Hence,
 $\triangle ABC \sim \triangle ADC$.

13 In triangles ABD and ADC , $\angle BDA = \angle ADC = 90^\circ$.



14 From panel 1, $\angle BAD = \angle ACD$.



15 Therefore, by I.32, $\angle ABD = \angle DAC$.



16 Thus $\triangle ABD$ and $\triangle ADC$ are equiangular. By Prop. VI.4, $BD : AD = AD : DC = BA : AC$. Hence, $\triangle ABD \sim \triangle ADC$.



17 From panels 1 and 2 we have:

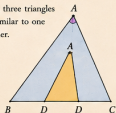
$\triangle ABC \sim \triangle ABD$

$\triangle ABC \sim \triangle ADC$

$\triangle ABD \sim \triangle ADC$

PANEL 3 – ALL THREE TRIANGLES ARE SIMILAR AND THE GEOMETRIC MEAN

So all three triangles are similar to one another.



18 Since $\triangle ABD \sim \triangle ADC$, $BD : AD = AD : DC$.

Therefore,
 $AD^2 = BD \cdot DC$.

Geometric Mean
 AD is the mean proportional (geometric mean) between the two segments BD and DC of the hypotenuse.

$BD : AD = AD : DC$

$AD^2 = BD \cdot DC$



29

Proposition 9

To cut off a prescribed part from a given straight line.

Proof / Construction Actions

1. Let AB be the given straight line.
2. We want to cut off one third of AB ; equivalently, construct F on AB such that $AF = \frac{1}{3}AB$.
3. From A , draw any straight line AC making an angle with AB .
4. Take any point D on AC .
5. On AC , construct $DE = AD$.
6. Construct $EC = AD$; hence $AD = DE = EC$.
7. Thus AC is divided into three equal parts, so $CD = DE + EC = 2AD$.
8. Join B to C .
9. Through D , draw $DF \parallel BC$, meeting AB at F (I.31).
10. In triangle ABC , since $DF \parallel BC$, VI.2 gives $CD : DA = BF : FA$.
11. But $CD = 2DA$; hence $CD : DA = 2 : 1$, and therefore $BF : FA = 2 : 1$.
12. Since $BF = 2FA$, $AB = AF + FB = AF + 2AF = 3AF$.
13. Hence $AF = \frac{1}{3}AB$; thus F cuts off one third of the given straight line.

Final Conclusion

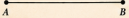
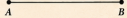
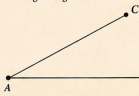
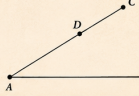
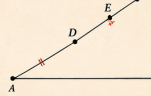
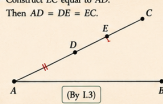
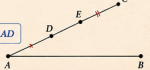
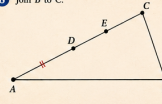
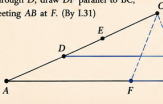
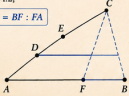
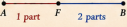
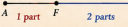
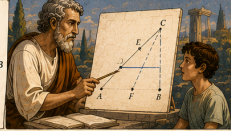
The prescribed part has been cut off



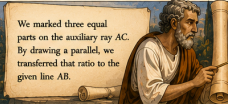
BOOK VI – PROPOSITION 9

To cut off a prescribed part from a given straight-line.

EXAMPLE
Cut off one third of the given straight line AB.
That is, construct F on AB such that $AF = \frac{1}{3} AB$.

1 Let AB be the given straight line. 	2 We want to cut off one third of AB. Goal: $AF = \frac{1}{3} AB$ 	3 From A, draw any straight line AC making an angle with AB. 	4 Take any point D on AC. 	5 On AC, construct DE equal to AD. 
6 Construct EC equal to AD. Then $AD = DE = EC$. 	7 Since $AD = DE = EC$, the whole AC is divided into three equal parts. Therefore $CD = DE + EC = 2AD$. $CD = 2AD$ 	8 Join B to C. 	9 Through D, draw DF parallel to BC, meeting AB at F. (By I.31) 	
10 In triangle ABC, since $DF \parallel BC$, by Prop. VI.2, $CD : DA = BF : FA$ 	11 But $CD = 2DA$. Therefore, $CD : DA = 2 : 1$. Hence, $BF : FA = 2 : 1$ 	12 Since $BF = 2FA$, $AB = AF + FB = AF + 2AF = 3AF$. $AB = 3AF$ 	13 Hence, $AF = \frac{1}{3} AB$ Thus F is the point on AB which cuts off one third of the given straight line. 	

We marked three equal parts on the auxiliary ray AC. By drawing a parallel, we transferred that ratio to the given line AB.



General Method

To cut off a fraction $\frac{1}{n}$ of a line AB:

1. On a ray from A, mark n equal segments.
2. Join the last point to B.
3. Through the (n-1)-th point, draw a parallel to meet AB. The intersection divides AB in the ratio 1 : (n-1).

Geometric Mean (Corollary)

In Prop. VI.8 we proved that $BD : AD = AD : DC$. Therefore, $AD^2 = BD \cdot DC$.



Proposition 10

To divide an uncut straight line similarly to a given divided straight line.

Proof / Construction Actions

1. Let AB be the given uncut straight line.
2. Let AC be another straight line cut at D and E .
3. Place the two lines so that they make any convenient angle at A .
4. Join C to B .
5. Through D , draw $DF \parallel CB$, meeting AB at F .
6. Through E , draw $EG \parallel CB$, meeting AB at G .
7. Through D , draw $DHK \parallel AB$; it meets EG at H and CB at K .
8. Quadrilaterals $DFGH$ and $HGBK$ are parallelograms; therefore by I.34, $DH = FG$ and $HK = GB$.
9. In triangle DKC , since $EH \parallel KC$, VI.2 gives $CE : ED = KH : HD$. Using $HK = GB$ and $HD = GF$, obtain $CE : ED = BG : GF$.
10. In triangle AGE , since $DF \parallel GE$, VI.2 gives $ED : DA = GF : FA$.
11. Thus both ratios from AC have been transferred to AB : $CE : ED = BG : GF$ and $ED : DA = GF : FA$.
12. Hence the successive parts of AB are in the same ratios as the successive parts of AC .
13. Therefore the uncut straight line AB has been cut similarly to the given cut straight line AC .

Final Conclusion

The target line is divided similarly



BOOK VI – PROPOSITION 10

To cut a given uncut straight-line similarly to a given cut straight-line.

Let AB be the uncut straight-line.
Let AC be another straight-line cut at D, E.
We will cut AB at F, G so that

$CE : ED = BG : GF$
 $ED : DA = GF : FA$

Given uncut line



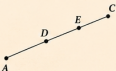
Given cut line



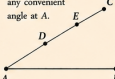
1 Let AB be the uncut straight-line.



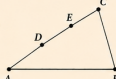
2 Let AC be another straight-line cut at D and E.



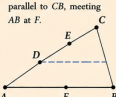
3 Place the two lines so that they make any convenient angle at A.



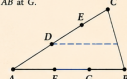
4 Join C to B.



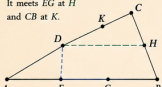
5 Through D, draw DF parallel to CB, meeting AB at F.



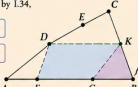
6 Through E, draw EG parallel to CB, meeting AB at G.



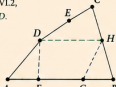
7 Through D, draw DHK parallel to AB. It meets EG at H and CB at K.



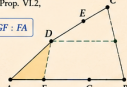
8 Quadrilaterals DFGH and HGBK are parallelograms. Therefore, by I.34,
 $DH = FG$
 $HK = GB$



9 In triangle DKC, EH is parallel to KC. Therefore, by Prop. VI.2, $CE : ED = KH : HD$. But $HK = GB$ and $HD = GF$. Hence,
 $CE : ED = BG : GF$



10 In triangle AGE, DF is parallel to GE. Therefore, by Prop. VI.2, $ED : DA = GF : FA$

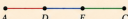


11 Thus we have transferred both ratios from AC to AB.
 $CE : ED = BG : GF$
 $ED : DA = GF : FA$



12 Hence the successive parts of AB are in the same ratio as the successive parts of AC.

AC :



AB :



13 Therefore the uncut straight-line AB has been cut similarly to the given cut straight-line AC.

AB is cut similarly to AC






General Idea

Divide an auxiliary line (AC) in the required pattern.

Join last point to the end of AB.



Draw parallels through the division points.



The same ratios are transferred to AB.

Thus AB is cut similarly to AC.

Proposition 11

To find a third proportional to two given straight lines.

Proof / Construction Actions

1. Place the two given straight lines BA and AC so that they form any convenient angle at A .
2. Produce BA beyond B to a point D .
3. Produce AC beyond C to a point E .
4. On the extension of BA , construct $BD = AC$ (I.3).
5. Join B to C .
6. Through D , draw $DE \parallel BC$, meeting the produced line AC at E (I.31).
7. In triangle ADE , since $BC \parallel DE$, VI.2 gives $AB : BD = AC : CE$.
8. Thus $AB : BD = AC : CE$.
9. But $BD = AC$; substituting gives $AB : AC = AC : CE$.
10. Hence CE is the third proportional to the two given straight lines AB and AC .

Final Conclusion

$$\boxed{A : B = B : C}$$



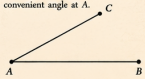
BOOK VI – PROPOSITION 11

To find a third straight-line proportional to two given straight-lines.

GOAL
Given straight-lines AB and AC ,
construct CE such that

$AB : AC = AC : CE$

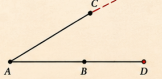
1 Place the two given straight-lines BA and AC so that they form any convenient angle at A .



2 Produce BA beyond B to a point D .

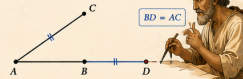


3 Produce AC beyond C to a point E .

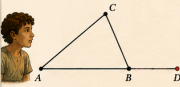


4 On the extension of BA , construct BD equal to AC . (By I.3)

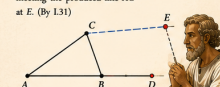
$BD = AC$



5 Join B to C .



6 Through D , draw DE parallel to BC , meeting the produced line AC at E . (By I.31)



7 In triangle ADE , since $BC \parallel DE$, by Prop. VI.2,



8 $AB : BD = AC : CE$.

$AB : BD = AC : CE$



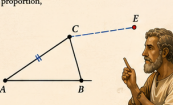
9 But $BD = AC$ (from step 4). Therefore, substituting in the proportion,
 $AB : AC = AC : CE$.

SUBSTITUTION

$$AB : BD = AC : CE$$

↓

$$AB : AC = AC : CE$$



10 Hence CE is the third straight-line proportional to the two given straight-lines AB and AC . That is, we have constructed CE such that

$AB : AC = AC : CE$

THIRD PROPORTIONAL
 CE is the third proportional to AB and AC .



WHAT WE DID

We copied AC onto BA to get $BD = AC$, then used a parallel to transfer the ratio to the other line.

GENERAL IDEA



RESULT

$AB : AC = AC : CE$
 CE is the third proportional.

Proposition 12

To find a fourth proportional to three given straight lines.

Proof / Construction Actions

1. Take the three given straight lines A, B, C ; we must construct a fourth proportional.
2. Draw two straight lines DE and DF from D , making any convenient angle EDF .
3. On DE , mark G so that $DG = A$.
4. On DE , mark E so that $GE = B$.
5. On DF , mark H so that $DH = C$.
6. Join G to H .
7. Through E , draw $EF \parallel GH$, meeting DF at F (I.31).
8. In triangle DEF , since $GH \parallel EF$, VI.2 gives $DG : GE = DH : HF$.
9. But $DG = A$, $GE = B$, and $DH = C$; therefore $A : B = C : HF$.
10. Hence HF is the fourth proportional to the three given straight lines A, B, C .

Final Conclusion

$$A : B = C : D$$



BOOK VI – PROPOSITION 12

To find a fourth straight-line proportional to three given straight-lines.

GIVEN THREE STRAIGHT-LINES



CONSTRUCT A FOURTH STRAIGHT-LINE HF SUCH THAT

$$A : B = C : HF$$

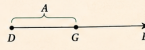
- 1 Take the three given straight-lines A, B, C . We must construct a fourth proportional.



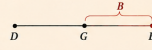
- 2 Draw two straight-lines DE and DF from D , making any convenient angle EDF .



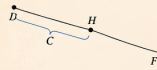
- 3 On DE , mark G so that $DG = A$.



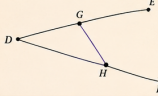
- 4 On DE , mark E so that $GE = B$. Thus $GE = B$.



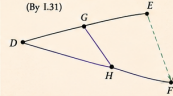
- 5 On DF , mark H so that $DH = C$.



- 6 Join G to H . Draw the straight line GH .



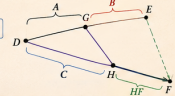
- 7 Through E , draw EF parallel to GH , meeting DF at F . (By I.31)



- 8 In triangle DEF , since $GH \parallel EF$, by Prop. VI.2, $DG : GE = DH : HF$.

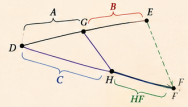
- 9 But $DG = A$, $GE = B$, and $DH = C$. Therefore,

$$A : B = C : HF$$



- 10 Hence HF is the fourth straight-line proportional to the three given straight-lines A, B, C .

$$A : B = C : HF$$



CONCLUSION

The straight-line HF satisfies

$$A : B = C : HF$$

Therefore HF is the fourth proportional to A, B , and C .



THE GENERAL METHOD (IDEA)

1. Copy the first given line A .



2. Copy the second given line B .



3. Copy the third given line C .



4. Join G to H .



5. Draw through E a parallel to GH .



6. By VI.2, the ratio is transferred.



- Thus HF is the fourth proportional:

$$A : B = C : HF$$

Proposition 13

To find a mean proportional between two given straight lines.

Proof / Construction Actions

1. Take the two given straight lines AB and BC ; we must find the mean proportional between them.
2. Lay AB and BC straight-on with respect to one another, so that A, B, C are collinear.
3. With AC as diameter, draw semicircle ADC .
4. From B , draw $BD \perp AC$, meeting the semicircle at D .
5. Join A to D .
6. Join D to C .
7. Angle ADC stands in a semicircle; therefore $\angle ADC = 90^\circ$ (III.31).
8. Triangle ADC is therefore right-angled at D , and BD is drawn from the right angle perpendicular to the hypotenuse AC .
9. By the corollary to VI.8, the perpendicular is in mean proportion to the two segments of the hypotenuse: $AB : BD = BD : BC$.
10. Therefore BD is the required mean proportional between AB and BC .

Final Conclusion

$$AB : BD = BD : BC$$



BOOK VI – PROPOSITION 13

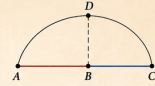
To find the straight-line in mean proportion to two given straight-lines.

GOAL

Given straight-lines AB and BC ,

construct BD such that

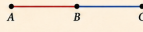
$$AB : BD = BD : BC.$$



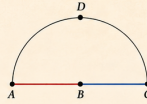
- 1 Take the two given straight-lines AB and BC . We must find the mean proportional between them.



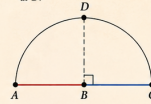
- 2 Lay AB and BC straight-on with respect to one another, so that A, B, C are collinear.



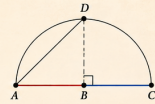
- 3 With AC as diameter, draw the semicircle ADC .



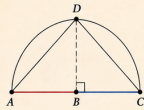
- 4 From B , draw BD perpendicular to AC , meeting the semicircle at D .



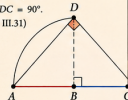
- 5 Join A to D .



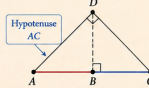
- 6 Join D to C .



- 7 Angle ADC stands in a semicircle. Therefore, $\angle ADC = 90^\circ$. (By III.31)



- 8 Triangle ADC is therefore a right-angled triangle, right-angled at D . And BD is drawn from the right angle perpendicular to the hypotenuse AC .



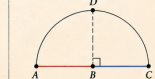
- 9 By the corollary to Proposition VI.8, the perpendicular from the right angle to the hypotenuse is in mean proportion to the segments of the hypotenuse.

$$AB : BD = BD : BC.$$

Thus BD is the mean proportional between AB and BC .

- 10 Therefore BD is the required straight-line in mean proportion to AB and BC . That is,

$$AB : BD = BD : BC.$$



WHAT WE HAVE DONE

We placed AB and BC on one straight line. With AC as diameter, we drew a semicircle. From B we erected a perpendicular to AC to meet the semicircle at D . This perpendicular BD is the mean proportional.

GEOMETRIC MEAN

From $AB : BD = BD : BC$ we get $BD^2 = AB \cdot BC$. Hence $BD = \sqrt{AB \cdot BC}$.

IN WORDS

BD is that straight-line which, being placed between AB and BC makes the ratio of the first to it equal to the ratio of it to the last.



Proposition 14

In equal and equiangular parallelograms, the sides about equal angles are reciprocally proportional; and conversely.

Proof / Construction Actions

1. Take equal and equiangular parallelograms AB and BC , with their equal angles at B .
2. Lay DB and BE in one straight line.
3. Since the angles at B are equal, FB and BG are also straight-on (I.14).
4. Complete the surrounding parallelogram FE .
5. Since $AB = BC$, by V.7, $AB : FE = BC : FE$.
6. By VI.1, $AB : FE = DB : BE$.
7. By VI.1, $BC : FE = GB : BF$.
8. Therefore $DB : BE = GB : BF$: the sides about the equal angles are reciprocally proportional.
9. Conversely, suppose $DB : BE = GB : BF$.
10. By VI.1, $DB : BE = AB : FE$.
11. By VI.1, $GB : BF = BC : FE$.
12. Therefore, by V.11, $AB : FE = BC : FE$.
13. By V.9, $AB = BC$; hence the equiangular parallelograms are equal.

Final Conclusion

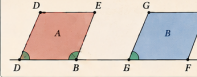
Equality $\iff$ reciprocal side proportionality



BOOK VI – PROPOSITION 14

In equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional.
And conversely, equiangular parallelograms whose sides about the equal angles are reciprocally proportional are equal.

DIAGRAM OF THE FIGURES

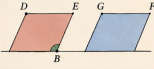


Parallelogram $AB =$
parallelogram BC .
Equal angles at B .

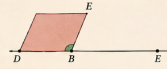
To prove: $DB : BE = GB : BF$

PART I – FROM EQUAL PARALLELOGRAMS TO RECIPROCAL PROPORTIONALITY

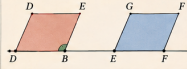
- 1 Take equal and equiangular parallelograms AB and BC , with their equal angles at B .



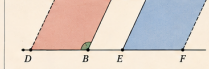
- 2 Lay DB and BE in one straight line.



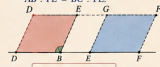
- 3 Since the angles at B are equal, FB and BC are also straight-on. (By I.14)



- 4 Complete the surrounding parallelogram FE .

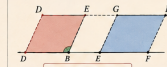


- 5 Since AB and BC are equal, by Prop. V.7,



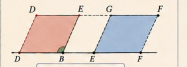
$$AB : FE = BC : FE$$

- 6 By Prop. VI.1,
 $AB : FE = DB : BE$.



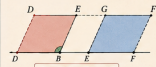
$$AB : FE = DB : BE$$

- 7 By Prop. VI.1,
 $BC : FE = GB : BF$.



$$BC : FE = GB : BF$$

- 8 Therefore,
 $DB : BE = GB : BF$.



$$DB : BE = GB : BF$$

RESULT OF PART I
In equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional.

$$DB : BE = GB : BF$$

PART II – FROM RECIPROCAL PROPORTIONALITY TO EQUALITY

- 9 Conversely, suppose that
 $DB : BE = GB : BF$.



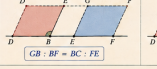
$$DB : BE = GB : BF$$

- 10 By Prop. VI.1,
 $DB : BE = AB : FE$.



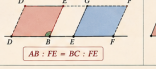
$$DB : BE = AB : FE$$

- 11 By Prop. VI.1,
 $GB : BF = BC : FE$.



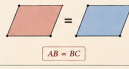
$$GB : BF = BC : FE$$

- 12 Therefore, by Prop. V.11,
 $AB : FE = BC : FE$.



$$AB : FE = BC : FE$$

- 13 Therefore, by Prop. V.9,
 $AB = BC$.



$$AB = BC$$

RESULT OF PART II
If equiangular parallelograms have their sides about the equal angles reciprocally proportional, then they are equal.

$$AB = BC$$

FINAL CONCLUSION

If equal and equiangular parallelograms have the equal angles at B , then
 $DB : BE = GB : BF$.



If equiangular parallelograms have $DB : BE = GB : BF$, then $AB = BC$.

GEOMETRIC MEANING

If we increase DB , the adjacent side BE must decrease in the same ratio, so that the area of the parallelogram remains equal to the other one. This is the principle of reciprocal proportionality.



Proposition 15

In equal triangles having one angle equal, the sides about the equal angles are reciprocally proportional; and conversely.

Proof / Construction Actions

1. Take equal triangles ABC and ADE , with $\angle BAC = \angle DAE$.
2. Place CA and AD in one straight line.
3. Since $\angle BAC = \angle DAE$ and CA is straight-on with AD , by I.14 EA and AB are also straight-on.
4. Join B to D .
5. Since $\triangle ABC = \triangle ADE$, by V.7, $\triangle CAB : \triangle BAD = \triangle EAD : \triangle BAD$.
6. By VI.1, $CA : AD = \triangle ABC : \triangle BAD$.
7. By VI.1, $EA : AB = \triangle EAD : \triangle BAD$.
8. Therefore $CA : AD = EA : AB$; the sides about the equal angles are reciprocally proportional.
9. Conversely, suppose $CA : AD = EA : AB$.
10. Again place CA and AD in one straight line, and EA and AB in one straight line; join B to D .
11. By VI.1, $CA : AD = \triangle ABC : \triangle BAD$.
12. By VI.1, $EA : AB = \triangle EAD : \triangle BAD$.
13. Since $CA : AD = EA : AB$, we have $\triangle ABC : \triangle BAD = \triangle EAD : \triangle BAD$.
14. The two triangles have the same ratio to triangle BAD ; therefore by V.9, $\triangle ABC = \triangle EAD$.
15. Hence the converse is proved: triangles having one equal angle whose sides about that angle are reciprocally proportional are equal.

Final Conclusion

Equality $\iff$ reciprocal side proportionality



BOOK VI – PROPOSITION 15

In equal triangles having one angle equal to one angle, the sides about the equal angles are reciprocally proportional. And conversely, triangles having one angle equal whose sides about those equal angles are reciprocally equal.

DIAGRAM OF THE FIGURES

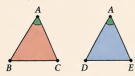


Equal triangles ABC and ADE with equal angle at A.

To prove: $CA : AD = EA : AB$

PART I – FROM EQUAL TRIANGLES TO RECIPROCAL PROPORTIONALITY

- 1 Take equal triangles ABC and ADE with angle BAC equal to angle DAE.



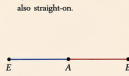
$$\triangle ABC = \triangle ADE$$

$$\angle BAC = \angle DAE$$

- 2 Place CA and AD in one straight line.



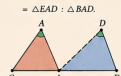
- 3 Since $\angle BAC = \angle DAE$, and CA is straight-on with AD, by I.14, EB and AB are also straight-on.



- 4 Join B to D.

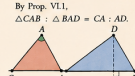


- 5 Since $\triangle ABC = \triangle ADE$, by Prop. V.7, $\triangle CAB : \triangle BAD = \triangle EAD : \triangle BAD$.



$$\triangle CAB : \triangle BAD = \triangle EAD : \triangle BAD$$

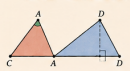
- 7 Triangle CAB and BAD stand on the same altitude from B to the straight line CAD. By Prop. VI.1, $\triangle CAB : \triangle BAD = CA : AD$.



$$\triangle CAD : \triangle BAD = EA : AB$$

- 6 Therefore, $CA : AD = EA : AB$. The sides about the equal angles are reciprocally proportional.

- 8 By Prop. VI.1, $CA : AD = \triangle ABC : \triangle BAD$.



- 7 Therefore, since $CA : AD = EA : AB$, we have $\triangle ABC : \triangle BAD = \triangle EAD : \triangle BAD$.



RESULT OF PART I
In equal triangles having one angle equal to one angle, the sides about the equal angles are reciprocally proportional.

$$CA : AD = EA : AB$$

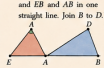
PART II – FROM RECIPROCAL PROPORTIONALITY TO EQUALITY

- 9 Conversely suppose that $CA : AD = EA : AB$.

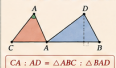


$$CA : AD = EA : AB$$

- 10 Again place CA and AD in one straight line, and EB and AB in one straight line. Join B to D.



- 11 By Prop. VI.1, $CA : AD = \triangle ABC : \triangle BAD$.



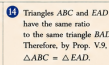
$$CA : AD = \triangle ABC : \triangle BAD$$

- 12 By Prop. VI.1, $EA : AB = \triangle EAD : \triangle BAD$.



$$EA : AB = \triangle EAD : \triangle BAD$$

- 13 Therefore, since $CA : AD = EA : AB$, we have $\triangle ABC : \triangle BAD = \triangle EAD : \triangle BAD$.



$$\triangle ABC = \triangle EAD$$

- 14 Triangles ABC and EAD have the same ratio to the same triangle BAD. Therefore, by Prop. V.9, $\triangle ABC = \triangle EAD$.

RESULT OF PART II
 $CA : AD = EA : AB$
 $\Rightarrow \triangle ABC = \triangle EAD$



FINAL CONCLUSION

If equal triangles have one angle equal, then the sides about those equal angles are reciprocally proportional.

$$\triangle ABC = \triangle ADE \Rightarrow CA : AD = EA : AB$$

If triangles have one angle equal whose sides about those equal angles are reciprocally proportional, then they are equal.

$$CA : AD = EA : AB \Rightarrow \triangle ABC = \triangle ADE$$

GEOMETRIC MEANING

For triangles with the included angle equal, increasing one side about that angle must be compensated by decreasing the other side in inverse ratio, if the triangles are to remain equal.



Proposition 16

If four straight lines are proportional, the rectangle contained by the outer terms equals the rectangle contained by the middle terms; and conversely.

Proof / Construction Actions

1. Assume $AB : CD = E : F$.
2. Erect $AG \perp AB$ and $CH \perp CD$.
3. Take $AG = F$ and $CH = E$.
4. Complete rectangles BG and DH .
5. Rewrite the proportion as $AB : CD = CH : AG$.
6. Recognize reciprocal proportionality about equal right angles.
7. By VI.14, $BG = DH$.
8. Interpret $BG = AB \cdot F$ and $DH = CD \cdot E$.
9. Hence $AB \cdot F = CD \cdot E$.
10. For the converse, assume $AB \cdot F = CD \cdot E$.
11. Repeat the construction.
12. Then $BG = DH$.
13. The equal rectangles are equiangular.
14. By the converse of VI.14, $AB : CD = CH : AG$.
15. Substitute $CH = E$ and $AG = F$.
16. Therefore $AB : CD = E : F$.

Final Conclusion

$AB : CD = E : F \iff AB \cdot F = CD \cdot E$
--



BOOK VI – PROPOSITION 16

If four straight-lines are proportional, then the rectangle contained by the two outermost is equal to the rectangle contained by the middle two. And conversely, if those rectangles by, equal, then the four straight-lines are proportional.

GIVEN FOUR STRAIGHT-LINES



TO PROVE

$$AB : CD = E : F \\ \Leftrightarrow \\ AB \cdot F = CD \cdot E$$

PART I – FROM PROPORTION TO EQUALITY OF RECTANGLES

Assume $AB : CD = E : F$

- 1 Begin with
 $AB : CD = E : F$
- 2 From A, erect AG perpendicular to AB.
- 3 From C, erect CH perpendicular to CD.
- 4 On AG, construct AG = F.
- 5 Complete rectangle BG on AB, AG and rectangle DH on CD, CH.

By 1.3 we copied F and E on AG and CH.

PART I (continued)

- 6 Since $AG = F$ and $CH = E$, so $AB : CD = CH : AG$.
- 7 Therefore the sides about the equal right angles are reciprocally proportional: $AB : CD = CH : AG$.
- 8 By Proposition VI.14, equal and equiangular parallelograms with reciprocally proportional sides are equal. Hence $BG = DH$.
- 9 But BG is the rectangle contained by AB and AG, and $AG = F$. So $BG = AB \cdot F$.
- 10 Similarly DH is the rectangle contained by CD and CH, and $CH = E$. So $DH = CD \cdot E$.
- 11 Since $BG = DH$, we have $AB \cdot F = CD \cdot E$.

Thus if $AB : CD = E : F$, then $AB \cdot F = CD \cdot E$.

PART II – FROM EQUALITY OF RECTANGLES TO PROPORTION

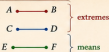
Assume $AB \cdot F = CD \cdot E$

- 12 Assume $AB \cdot F = CD \cdot E$. We must prove $AB : CD = E : F$.
- 13 As before, construct $AG = F$ perpendicular to AB and $CH = E$ perpendicular to CD, and complete rectangles BG and DH.
- 14 Then $BG = AB \cdot F$ and $DH = CD \cdot E$. But $AB \cdot F = CD \cdot E$ by hypothesis. Therefore $BG = DH$.
- 15 Rectangles BG and DH are equiangular.
- 16 By the other direction of Proposition VI.14, equal and equiangular parallelograms have reciprocally proportional sides. Hence $AB : CD = CH : AG$.
- 17 Since $CH = E$ and $AG = F$, therefore $AB : CD = E : F$.

Thus if $AB \cdot F = CD \cdot E$, then $AB : CD = E : F$.

GEOMETRIC MEANING

For four magnitudes in proportion, multiplying the outer ones equals multiplying the inner ones.



FINAL CONCLUSION

$$AB : CD = E : F \Leftrightarrow AB \cdot F = CD \cdot E$$

(The product of the extremes equals the product of the means.)

ALGEBRAIC FORM

If $a : b = c : d$, then $ad = bc$. Conversely, if $ad = bc$ and all are positive, then $a : b = c : d$.

Proposition 17

If three straight lines are proportional, the rectangle contained by the outer terms equals the square on the middle term; and conversely.

Proof / Construction Actions

1. Assume $A : B = B : C$.
2. Construct $D = B$.
3. Rewrite the proportion as $A : B = D : C$.
4. By VI.16, $A \cdot C = B \cdot D$.
5. Since $D = B$, $B \cdot D = B^2$.
6. Thus $A \cdot C = B^2$.
7. For the converse, assume $A \cdot C = B^2$.
8. Again construct $D = B$.
9. Then $B^2 = B \cdot D$.
10. So $A \cdot C = B \cdot D$.
11. By the converse of VI.16, $A : B = D : C$.
12. Substitute $D = B$.
13. Therefore $A : B = B : C$.

Final Conclusion

$A : B = B : C \iff A \cdot C = B^2$



BOOK VI – PROPOSITION 17

If three straight-lines are proportional, then the rectangle contained by the two outermost is equal to the square on the middle one. And conversely, if the rectangle contained by the two outermost is equal to the square on the middle one, then the three straight-lines are proportional.

THE GIVEN THREE STRAIGHT-LINES



TO PROVE

$$A : B = B : C$$



$$A \cdot C = B^2$$

PART I – FROM PROPORTION TO EQUALITY OF FIGURES

- Begin with three proportional straight-lines:

$$A : B = B : C$$



- Construct a fourth straight-line D equal to B . (By I.3)

$$D = B$$

- Replace the second B in the proportion by D . Since $D = B$,

$$A : B = D : C$$



- Thus we have four proportional straight-lines:

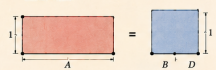


GEOMETRIC MEAN

If B is in mean proportion between A and C , then B is called the geometric mean of A and C . Then $B^2 = A \cdot C$.

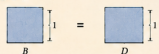
PART I (continued)

- By Proposition VI.16, the rectangle on the two outer terms equals the rectangle on the two middle terms.

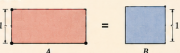


$$A \cdot C = B \cdot D$$

- But $D = B$.

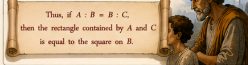


- Therefore the rectangle on B and D is the square on B . Hence



$$A \cdot C = B^2$$

Thus, if $A : B = B : C$, then the rectangle contained by A and C is equal to the square on B .



FINAL CONCLUSION

$$A : B = B : C \iff A \cdot C = B^2$$

(The product of the extremes equals the square on the mean proportion.)

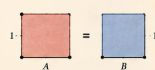
PART II – FROM EQUALITY OF FIGURES TO PROPORTION

- Conversely, suppose that

$$A \cdot C = B^2$$

We must prove

$$A : B = B : C$$



- Construct D equal to B .

$$D = B$$

- Since $D = B$, the square on B is the rectangle contained by B and D .



- Therefore $A \cdot C = B \cdot D$.

- By the converse of Proposition VI.16, the four straight-lines are proportional.



- But $D = B$. Therefore,

$$A : B = B : C$$

Thus, if the rectangle on A and C is equal to the square on B , then $A : B = B : C$.



ALGEBRAIC FORM

If $a : b = b : c$, then $b^2 = a \cdot c$.

Conversely, if $b^2 = a \cdot c$ (all positive), then $a : b = b : c$.



Proposition 18

To construct on a given straight line a rectilinear figure similar and similarly laid out to a given rectilinear figure.

Proof / Construction Actions

1. Take the given line AB and the given rectilinear figure.
2. Divide the given figure into triangles by a diagonal.
3. Copy the first corresponding angle at A .
4. Copy the second corresponding angle at B .
5. By I.32, obtain equality of the third angles.
6. The first pair of triangles is therefore equiangular.
7. By VI.4, their corresponding sides are proportional.
8. Copy the next angle of the original figure.
9. Copy the adjacent angle.
10. By I.32, infer the remaining angle.
11. By VI.4, obtain the second proportional side system.
12. Unite the two proportional systems.
13. Combine adjacent equal angles to obtain equality of whole polygonal angles.
14. Repeat at the remaining composite vertices.
15. Recall the copied endpoint angles.
16. Conclude the complete figures are equiangular.
17. Together with proportional corresponding sides, apply Definition VI.1.
18. The constructed figure is similar and similarly laid out to the given figure.

Final Conclusion

$$AH \sim CE$$



BOOK VI – PROPOSITION 18

To describe on a given straight-line a rectilinear figure similar, and similarly laid down, to a given rectilinear figure.

Given:

Straight-line AB

Given rectilinear figure CE



To construct on AB
a rectilinear figure AH
similar to CE .
($AH \sim CE$)

KEY EQUAL ANGLES COPIED

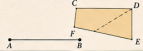
$\angle C = \angle A$
 $\angle CDE = \angle ABG$
 $\angle DFE = \angle BGH$
 $\angle FDE = \angle GBH$

$\angle CFD = \angle AGB$
 $\angle FED = \angle GHB$
 $\angle CFE = \angle AGH$
 $\angle CDE = \angle ABH$
 $\angle E = \angle H$

Therefore
 $AH \sim CE$

PANEL 1 – COPYING THE FIRST TRIANGLE

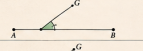
- 1 Begin with the given straight-line AB and the given rectilinear figure CE .



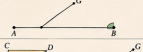
- 2 Join DF . This divides CE into two triangles: $\triangle FCD$ and $\triangle FDE$.



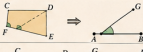
- 3 At A , construct an angle GAB equal to angle C . (By I.23)



- 4 At B , construct an angle ABH equal to angle CDE . (By I.23)



- 5 Therefore, by I.32, $\angle CFD = \angle AGB$.



- 6 Thus triangles FCD and GAB are equiangular.



- 7 By Proposition VI.4,

$$FD : GB = FC : GA = CD : AB$$

FIRST RESULT

$$FD : GB = FC : GA = CD : AB$$

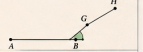


PANEL 2 – COPYING THE SECOND TRIANGLE

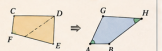
- 8 At G , construct an angle BGH equal to angle DFE . (By I.23)



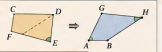
- 9 At B , construct an angle GBH equal to angle FDE . (By I.23)



- 10 Therefore, by I.32, $\angle FED = \angle GHB$.



- 11 Thus triangles FDE and GHB are equiangular.



By Proposition VI.4,

$$FD : GB = FE : ED = HB$$

- 12 From (7) and (11), $FD : GB = FC : GA = CD : AB = FE : ED = HB$.

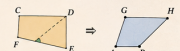
ALL CORRESPONDING SIDES ARE PROPORTIONAL

IMPORTANT: The triangles are constructed so that their corresponding sides are proportional and their corresponding angles are equal.

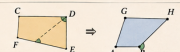


PANEL 3 – REASSEMBLING THE WHOLE FIGURES

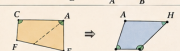
- 13 Since $\angle CFD = \angle AGB$ and $\angle DFE = \angle BGH$, therefore $\angle CFE = \angle AGH$.



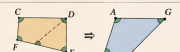
- 14 For the same reason, $\angle CDE = \angle ABH$.



- 15 By construction, $\angle C = \angle A$ and $\angle E = \angle H$.



- 16 Thus every corresponding angle of figures CE and AH is equal. Therefore the two figures are equiangular.



- 17 Their corresponding angles are equal and their corresponding sides are proportional. Therefore, by Definition VI.1,

$AH \sim CE$.

- 18 Hence the rectilinear figure AH , constructed on the given straight-line AB , is similar and similarly laid down to the given rectilinear figure CE .



GEOMETRIC MEANING

This proposition shows how any rectilinear figure can be reproduced on a new base line so that the new figure has the same shape (up to size) and the same orientation.

CE
(given figure)

FINAL CONCLUSION

$\longleftrightarrow$ AH

(constructed on AB)

$AH \sim CE$

Similar and similarly laid down.

PRACTICAL USE

It allows the scaling of any polygonal shape from one base line to another while preserving its form and proportions.



Proposition 19

Similar triangles are to one another in the squared ratio of their corresponding sides.

Proof / Construction Actions

1. Take similar triangles ABC and DEF .
2. Write a corresponding side proportion.
3. Construct a third proportional BG with $BC : EF = EF : BG$.
4. Join AG .
5. Alternate the original proportion.
6. Insert the third proportional to obtain $AB : DE = EF : BG$.
7. Recognize reciprocal proportionality about equal angles.
8. By VI.15, $\triangle ABG = \triangle DEF$.
9. Return to $BC : EF = EF : BG$.
10. By Definition V.9, $BC : BG$ is the squared ratio of $BC : EF$.
11. By VI.1, $\triangle ABC : \triangle ABG = BC : BG$.
12. Substitute the squared ratio.
13. Replace $\triangle ABG$ by the equal triangle DEF .

Final Conclusion

$$\triangle ABC : \triangle DEF = (BC : EF)^2$$

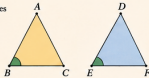


BOOK VI – PROPOSITION 19

Similar triangles are to one another
in the squared ratio of their corresponding sides.

PANEL 1 – SETTING UP THE CONSTRUCTION

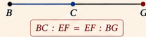
- 1 Take similar triangles $\triangle ABC \sim \triangle DEF$, with $\angle B = \angle E$. BC corresponds to EF .



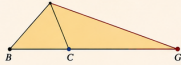
- 2 Since the triangles are similar, $AB : BC = DE : EF$.

$$AB : BC = DE : EF$$

- 3 By Prop. VI.11, construct BG so that $BC : EF = EF : BG$.



- 4 Join A to G.



Goal: Show that $\triangle ABC \sim \triangle DEF = (BC : EF)^2$

GEOMETRIC MEANING

If corresponding sides of similar triangles are in the ratio k , then their areas are in the ratio k^2 .



Sides scale by k
Areas scale by k^2

FINAL RESULT

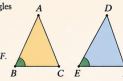
$$\triangle ABC \sim \triangle DEF \implies \triangle ABC : \triangle DEF = (BC : EF)^2$$

Given: Similar triangles

$\triangle ABC \sim \triangle DEF$

with $\angle B = \angle E$.

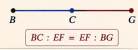
BC corresponds to EF .



To Prove:
 $\triangle ABC : \triangle DEF = (BC : EF)^2$

PANEL 3 – FINISHING THE PROOF

- 5 From (3), $BC : EF = EF : BG$.
Thus BC, EF, BG are three proportional.

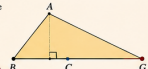


- 6 By Def. V.9, $BC : BG$ is the squared ratio of $BC : EF$.

$$BC : BG = (BC : EF)^2$$

- 7 Triangles ABC and ABG have the same altitude from A to line BG .
By Prop. VI.1,

$$\triangle ABC : \triangle ABG = BC : BG$$



- 8 From (6) and (7),

$$\triangle ABC : \triangle ABG = (BC : EF)^2$$

- 9 But from (8), $\triangle ABG = \triangle DEF$.

Therefore,

$$\triangle ABC : \triangle DEF = (BC : EF)^2$$



CONCLUSION

Similar triangles are to one another in the squared ratio of their corresponding sides.

ALGEBRAIC VIEW

If corresponding sides are in the ratio $k = BC : EF$, then $\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle DEF)} = k^2$.

Area = k^2
units²

k

:

Area = 1
unit²

Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding proportionally to the wholes, and the polygons are in the squared ratio of corresponding sides.

Corollary. Similar rectilinear figures are to one another in the squared ratio of corresponding sides.

Proof / Construction Actions

1. Take similar polygons $ABCDE$ and $FGHKL$.
2. Triangulate the first polygon.
3. Triangulate the second polygon correspondingly.
4. Use polygon similarity to obtain equal corner angles and proportional adjacent sides.
5. By VI.6, prove the first triangle pair similar.
6. Record the corresponding angle equality.
7. Subtract equal angle parts from equal whole polygonal angles.
8. Obtain the next equal included angle.
9. Use the first triangle similarity for a side proportion.
10. Use polygon similarity for the adjacent side proportion.
11. Combine the proportions by V.22.
12. By VI.6, prove the second triangle pair similar.
13. Repeat for the third pair.
14. Draw additional diagonals for proportional comparison.
15. Prove the resulting larger triangle pair similar.
16. Introduce intersections of the diagonals.
17. Establish smaller auxiliary triangles as similar.

18. Convert similarity into segment proportions.
19. Use VI.1 to convert segment ratios into area ratios.
20. By V.12 and alternation, obtain equal ratios of corresponding component triangles.
21. Include the third pair.
22. Sum corresponding triangles to pass to the whole polygons.
23. Use VI.19 for the squared ratio of one corresponding triangle pair.
24. Transfer that squared ratio to the whole polygons.

Final Conclusion

$$ABCDE : FGHLK = (AB : FG)^2$$



BOOK VI – PROPOSITION 20

Similar polygons can be divided into equal numbers of similar triangles corresponding proportionally to the wholes; and the polygons are to one another in the squared ratio of their corresponding sides.

PANEL 1 – DIVIDING INTO CORRESPONDING SIMILAR TRIANGLES (I)

- Take similar polygons $ABCDE$ and $FGHKL$, with AB corresponding to FG .
- Join BE and EC .
- Join GL and LH .
- Since polygons are similar, $\angle BAE = \angle FGL$ and $AB : AE = FG : FL$.
- By Prop. VI.6, $\triangle ABE \sim \triangle FGL$.
- Therefore $\angle ABE = \angle FGL$.
- Since $\angle ABC = \angle FGH$, therefore $\angle EBC = \angle LGH$.
- Then $EB : BA = LG : GF$.

$$EB : BA = LG : GF$$

$$\text{So } \triangle ABE \sim \triangle FGL$$

- WHAT THIS PROPOSITION GIVES
- Any similar polygons can be cut into the same number of similar triangles.
 - Corresponding triangles have the same ratio.
 - The whole polygons have that same ratio.



PANEL 2 – DIVIDING INTO CORRESPONDING SIMILAR TRIANGLES (II)

- Also $AB : BC = FG : GH$.
- Hence, by Prop. V.22, $EB : BC = LG : GH$.
- Since $\angle EBC = \angle LGH$ and $EB : BC = LG : GH$, by Prop. VI.6, $\triangle EBC \sim \triangle LGH$.
- Again, $\angle ECD = \angle LHK$, and $EC : CD = LH : HK$. Therefore, by Prop. VI.6, $\triangle ECD \sim \triangle LHK$.
- Thus the polygons are divided into three pairs of similar triangles.
- Join AC in the first polygon and FN in the second.
- Let the diagonals meet BE , EC at M , and GL , LH at N .
- By similar reasoning, $\triangle ABM \sim \triangle FGN$ and $\triangle BMC \sim \triangle GNL$.

$$\begin{aligned} \triangle ABE &\sim \triangle FGL \\ \triangle EBC &\sim \triangle LGH \\ \triangle ECD &\sim \triangle LHK \end{aligned}$$

The three pairs of triangles are similar.

FINAL RESULT
If polygons are similar and $AB \propto FG$,

$$ABCDE : FGHKL = (AB : FG)^2$$

Sides in ratio k
 $\Rightarrow$
Areas in ratio k^2

Given: Similar polygons

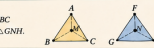


To Prove:
 $ABCDE : FGHKL = (AB : FG)^2$

AB corresponds to FG .

PANEL 3 – FROM TRIANGLES TO POLYGONS

- From $\triangle ABM \sim \triangle FGN$ and $\triangle BMC \sim \triangle GNL$, $AM : MB = FN : NG$ and $BM : MC = GN : NH$.
- Hence by Prop. V.22, $AM : MC = FN : NH$.
- By Prop. VI.1, $AM : MC = \triangle ABM : \triangle BMC$ and $FN : NH = \triangle FGN : \triangle GNL$.
- Therefore, by Prop. V.12, $\triangle ABE : \triangle EBC = \triangle FGL : \triangle LGH$.
- Again, by similar reasoning, $\triangle EBC : \triangle ECD = \triangle LGH : \triangle LHK$. Hence all three ratios are equal:
- Therefore, by Prop. V.12, $\triangle ABE : \triangle EBC = \triangle FGL : \triangle LGH$.
- But $\triangle ABE \sim \triangle FGL$, so by Prop. 6.14 (VI.19), $\triangle ABE : \triangle FGL = (AB : FG)^2$.
- Therefore, $ABCDE : FGHKL = (AB : FG)^2$.



Thus similar polygons are to one another in the squared ratio of their corresponding sides.

GENERAL COROLLARY (by Euclid)

The same holds for similar quadrilaterals, and therefore for all similar rectilinear figures.



Proposition 21

Rectilinear figures similar to the same rectilinear figure are similar to one another.

Proof / Construction Actions

1. Take figures A, B, C with $A \sim C$ and $B \sim C$.
2. Unpack $A \sim C$: equal corresponding angles and proportional sides.
3. Record the corresponding angle equalities.
4. Record the corresponding side proportions.
5. Unpack $B \sim C$ in the same way.
6. Compare corresponding angles of A and B through C .
7. Compare corresponding side ratios of A and B through C .
8. Thus A and B satisfy both conditions of similarity.
9. Apply Definition VI.1.

Final Conclusion

$$A \sim C, B \sim C \implies A \sim B$$



BOOK VI – PROPOSITION 21

Rectilinear figures similar to the same rectilinear figure are also similar to one another.

GIVEN:

Three rectilinear figures



Assume

$A \sim C$

$B \sim C$

TO PROVE:

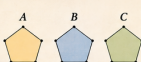
$A \sim B$

1 BEGIN WITH THREE FIGURES.

Let rectilinear figures A, B, C be such that

$A \sim C$ and

$B \sim C$.



2 UNPACK THE SIMILARITY $A \sim C$.

By Definition VI.1,

- Figures A and C are equiangular.
- Their corresponding sides are proportional.



$$A_1A_2 : C_1C_2 = A_2A_3 : C_2C_3 = \dots$$

3 VISUALIZE THE EQUAL CORRESPONDING ANGLES.

Each angle of A equals the corresponding angle of C .

$$\angle A_1 = \angle C_1,$$

$$\angle A_2 = \angle C_2, \dots \text{etc.}$$



Equal angles by similarity (Definition VI.1)

4 VISUALIZE THE PROPORTIONAL CORRESPONDING SIDES.

The corresponding sides of A and C are proportional.

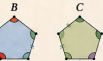


$$A_1A_2 : C_1C_2 = A_2A_3 : C_2C_3 = \dots = A_5A_1 : C_5C_1$$

5 UNPACK THE SIMILARITY $B \sim C$.

By Definition VI.1,

- Figures B and C are equiangular.
- Their corresponding sides are proportional.



$$B_1B_2 : C_1C_2 = B_2B_3 : C_2C_3 = \dots = B_5B_1 : C_5C_1$$

6 COMPARE THE ANGLES OF A AND B THROUGH C .

An angle of A equals the corresponding angle in C , and the corresponding angle of B also equals that angle in C .

Therefore corresponding angles of A and B are equal.



Hence A is equiangular to B .

7 COMPARE THE SIDE RATIOS THROUGH C .

The sides of A are proportional to the corresponding sides of C , and the sides of B are also proportional to those same corresponding sides of C . Therefore the corresponding sides of A and B are proportional to one another.



$$A_1A_2 : B_1B_2 = A_2A_3 : B_2B_3 = \dots = A_5A_1 : B_5B_1$$

8 ASSEMBLE THE TWO REQUIREMENTS.

We have:

- Corresponding angles of A and B are equal.
- Corresponding sides of A and B are proportional.



9 APPLY DEFINITION VI.1.

Since corresponding angles are equal and corresponding sides are proportional, by Definition VI.1,

$$A \sim B.$$



Thus,
 $A \sim C$,
 $B \sim C$
 $\Rightarrow$
 $A \sim B$.

FINAL RESULT

$$A \sim C \text{ and } B \sim C \Rightarrow A \sim B$$

If two figures each have the same shape as a third figure, then they have the same shape as each other.



Proposition 22

If four straight lines are proportional, similar and similarly described rectilinear figures constructed on them are proportional; and conversely.

Proof / Construction Actions

1. Assume $AB : CD = EF : GH$.
2. Construct similar figures on AB and CD .
3. Construct corresponding similar figures on EF and GH .
4. Construct a third proportional to AB, CD .
5. Construct a third proportional to EF, GH .
6. Combine the two chains of proportions.
7. Convert the first line ratio into a figure ratio.
8. Convert the second line ratio into a figure ratio.
9. Conclude the forward proportionality of the figures.
10. For the converse, assume the figures proportional.
11. Assume the desired line proportion does not hold and construct a fourth proportional.
12. Construct on it a similar figure.
13. Apply the forward direction.
14. Compare with the original hypothesis.
15. Use V.9 to infer equality of the auxiliary and original figures.
16. Use their similarity to infer equality of corresponding bases.
17. Substitute that equality.
18. Conclude the desired proportion of the four straight lines.

Final Conclusion

$$AB : CD = EF : GH \iff KAB : LCD = MF : NH$$



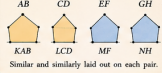
BOOK VI – PROPOSITION 22

If four straight-lines are proportional, then similar and similarly described rectilinear figures drawn on them will also be proportional. And conversely, if the similar and similarly described figures are proportional, then the straight-lines themselves will also be proportional.

THE GIVEN FOUR STRAIGHT LINES



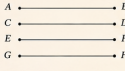
FIGURES DRAWN ON THEM



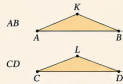
PANEL 1 – PROPORTIONAL LINES ⇔ PROPORTIONAL SIMILAR FIGURES

1 Given four proportional straight lines:

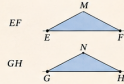
$$AB : CD = EF : GH$$



2 On AB and CD, construct similar and similarly laid out figures KAB and LCD.

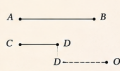


3 On EF and GH, construct similar and similarly laid out figures MF and NH.



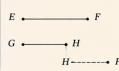
4 By Prop. VI.11, on CD take O so that

$$AB : CD = CD : O$$



5 By Prop. VI.11, on GH take P so that

$$EF : GH = GH : P$$



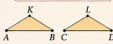
6 Since $AB : CD = EF : GH$ and $CD : O = GH : P$, by Prop. V.22,

$$AB : O = EF : P$$

PANEL 2 – PROPORTIONAL LINES ⇔ PROPORTIONAL FIGURES (CONTINUED) AND THE CONVERSE – SETUP

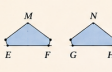
7 By the corollary of Prop. VI.19 (the square ratio of similar figures on proportional lines),

$$AB : O = KAB : LCD$$



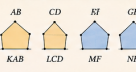
8 Similarly,

$$EF : P = MF : NH$$



9 Therefore,

$$KAB : LCD = MF : NH$$



10 Conversely, assume

$$KAB : LCD = MF : NH$$

We must prove

$$AB : CD = EF : GH$$



11 Suppose not. Then construct QR so that by Prop. VI.12,

$$AB : CD = EF : QR$$



12 On QR, describe SR similar and similarly laid out to MF and NH.



(By Prop. VI.18 and VI.21)

PANEL 3 – THE CONVERSE (CONTINUED) AND CONCLUSION

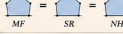
13 Since $AB : CD = EF : QR$, by the first part proved,

$$KAB : LCD = MF : SR$$



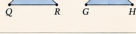
14 But $KAB : LCD = MF : NH$ (assumption). Therefore,

$$MF : SR = MF : NH$$



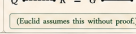
15 By Prop. V.9 (equal ratios),

$$SR = NH$$



16 Since SR and NH are similar and similarly laid out, equal figures have equal bases.

$$QR = GH$$



(Euclid assumes this without proof.)

17 From $AB : CD = EF : QR$ and $QR = GH$, by substitution,

$$AB : CD = EF : GH$$



18 Thus we have proved both ways:

$$AB : CD = EF : GH$$

$$KAB : LCD = MF : NH$$

$$KAB : LCD = MF : NH$$

$$AB : CD = EF : GH$$

WHAT THIS PROPOSITION GIVES

- Proportional lengths produce proportional similar figures.
- Proportional similar figures imply proportional lengths.
- The square of the scale factor governs the area ratio.

FINAL RESULT

For four straight lines AB, CD, EF, GH and similar, similarly laid out figures on them (KAB, LCD, MF, NH), the following are equivalent:

$$AB : CD = EF : GH$$



$$KAB : LCD = MF : NH$$



Proposition 23

Equiangular parallelograms have to one another the ratio compounded from the ratios of their sides.

Proof / Construction Actions

1. Take equiangular parallelograms AC and CF .
2. Place BC and CG on one straight line.
3. Deduce that DC and CE are straight-on.
4. Complete the intermediate parallelogram.
5. Introduce a magnitude K .
6. Construct L so that $BC : CG = K : L$.
7. Construct M so that $DC : CE = L : M$.
8. Recognize $K : M$ as the compounded ratio.
9. By VI.1, compare AC with the intermediate parallelogram.
10. Replace the first side ratio by $K : L$.
11. By VI.1, compare the intermediate parallelogram with CF .
12. Replace the second side ratio by $L : M$.
13. Form the linked chains.
14. By V.22, obtain $K : M = AC : CF$.
15. Recall what $K : M$ represents.
16. Transfer the compounded ratio to the two parallelograms.

Final Conclusion

$$AC : CF = (BC : CG)(DC : CE)$$

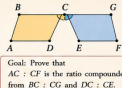


BOOK VI – PROPOSITION 23

Equiangular parallelograms have to one another the ratio compounded out of the ratios of their sides.

Given: Parallelograms AC and CF are equiangular, with $\angle BCD = \angle ECG$.

THE GIVEN FIGURE



KEY TO COLORS



Goal: Prove that $AC : CF$ is the ratio compounded from $BC : CG$ and $DC : CE$.

PANEL 1 – SETTING UP THE FIGURE

- Take equiangular parallelograms AC and CF such that $\angle BCD = \angle ECG$.
- Place BC and CG on one straight line.
- Since the angles at C are equal, by I.14, D, C, E are on one straight line.
- Complete the parallelogram DG .
- Take an arbitrary straight line K .

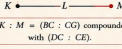
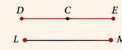
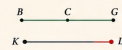
$K \leftarrow \rightarrow$



We will use K, L, M to represent the compounded ratio of the two pairs of sides.

PANEL 2 – REPRESENTING THE SIDE RATIOS

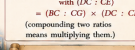
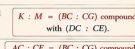
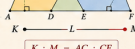
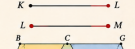
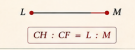
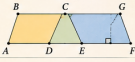
- By Prop. VI.12, construct L so that $BC : CG = K : L$.
- By Prop. VI.12, construct M so that $DC : CE = L : M$.
- The ratio $K : M$ is compounded from $K : L$ and $L : M$. But $K : L = BC : CG$ and $L : M = DC : CE$. Therefore, $K : M$ is the ratio compounded from $BC : CG$ and $DC : CE$.
- By Prop. VI.1, AC and CH have the same altitude.
- But $BC : CG = K : L$. Therefore, $AC : CH = BC : CG$.
- But $BC : CG = K : L$. Therefore, $AC : CH = K : L$.



We have related AC to CH using the first pair of sides.

PANEL 3 – FINISHING THE PROOF

- By Prop. VI.1, CH and CF have the same altitude.
- But $DC : CE = L : M$. Therefore, $CH : CF = L : M$.
- Thus we have $AC : CH = K : L$ and $CH : CF = L : M$.
- By Prop. V.22 (through equality), $K : M = AC : CF$.
- But $K : M$ is the ratio compounded from $BC : CG$ and $DC : CE$.
- Therefore, $AC : CF$ is the ratio compounded from $BC : CG$ and $DC : CE$.



Hence equiangular parallelograms have to one another the ratio compounded out of the ratios of their sides.

FINAL RESULT

For equiangular parallelograms AC and CF ,

$$AC : CF = (BC : CG) \text{ compounded with } (DC : CE)$$

$$\Rightarrow AC : CF = (BC : CG) \times (DC : CE)$$

Proposition 24

In any parallelogram, the parallelograms about the diagonal are similar to the whole parallelogram and to one another.

Proof / Construction Actions

1. Take parallelogram $ABCD$ with diagonal AC .
2. Use a parallel in triangle ABC and VI.2 for the first segment proportion.
3. Use a parallel in triangle ACD and VI.2 for the second segment proportion.
4. Link the two proportions.
5. Compound the parts.
6. Alternate the resulting proportion.
7. Use the first pair of parallels for an angle equality.
8. With the common angle, show the first triangle pair equiangular.
9. Repeat on the other side.
10. Conclude the whole and first smaller parallelogram are equiangular.
11. Use VI.4 on the first similar triangle pair.
12. Continue the side-ratio chain.
13. Use the second similar triangle pair.
14. Continue once more.
15. Combine the middle ratios.
16. Recognize full side proportionality.
17. Apply Definition VI.1 to prove the first similarity.
18. Repeat for the second smaller parallelogram.
19. Use VI.21.

20. Conclude both smaller parallelograms are similar to the whole and to one another.

Final Conclusion

$$EG \sim ABCD, \quad HK \sim ABCD, \quad EG \sim HK$$

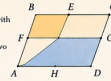


BOOK VI – PROPOSITION 24

In any parallelogram, the parallelograms about the diagonal are similar to the whole parallelogram and to one another.

Given:

Parallelogram $ABCD$ with diagonal AC .
 EG and HK are the two parallelograms about the diagonal.



To Prove:

- $EG \sim ABCD$
- $HK \sim ABCD$
- $EG \sim HK$

PANEL 1 – PREPARATION AND KEY PROPORTIONS

1 Take parallelogram $ABCD$ with diagonal AC. Inside it are the parallelograms about the diagonal: EG and HK.	2 In $\triangle ABC$, since $EF \parallel BC$, by Prop. VI.2, $BE : EA = CF : FA$	3 In $\triangle ACD$, since $FG \parallel CD$, by Prop. VI.2, $CF : FA = DG : GA$	4 From (2) and (3), $BE : EA = DG : GA$	5 By Prop. V.18, $BA : AE = DA : AG$	6 By Prop. V.16 (alternando), $BA : AD = EA : AG$	7 Set out an arbitrary line K. We will use K, L, M to represent the compounded ratio.
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PANEL 2 – EQUAL ANGLES

8 Since $GF \parallel DC$, by Prop. I.29, $\angle AFG = \angle DCA$	9 Also $\angle GAF = \angle DAC$ (common angle at A). Therefore, by Prop. I.32, $\triangle ADC \sim \triangle AGF$.	10 Likewise, $\triangle ACB \sim \triangle AFE$	11 Thus the whole parallelogram $ABCD$ and parallelogram EG are equiangular.	Every angle of EG equals the corresponding angle of $ABCD$.	12 Therefore EG is equiangular to the whole parallelogram.	13 Similarly, HK is equiangular to the whole parallelogram.
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PANEL 3 – PROPORTIONAL CORRESPONDING SIDES AND CONCLUSION

14 From $\triangle ADC \sim \triangle AGF$ [Prop. VI.4], $AD : DC = AG : GF$ $DC : CA = GF : FA$	15 From $\triangle ACB \sim \triangle AFE$: $AC : CB = AF : FE$ $CB : BA = FE : EA$	16 From (15) second ratio and (16) first ratio, by Prop. V.22, $DC : CB = GF : FE$	17 Thus corresponding sides are proportional: • $BA : AD = EA : AG$ • $DC : CB = GF : FE$ (and hence the whole sets of adjacent sides are proportional.)	18 Since EG and $ABCD$ are equiangular and their corresponding sides are proportional, by Def. VI.1, $EG \sim ABCD$.	19 By the same reasoning, $HK \sim ABCD$. And by Prop. VI.21, figures similar to the same figure are similar to one another. $EG \sim HK$	FINAL RESULT The parallelograms about the diagonal are each similar to the whole parallelogram and to one another: • $EG \sim ABCD$ • $HK \sim ABCD$ • $EG \sim HK$
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WHAT THIS PROPOSITION GIVES
A powerful relation between areas: the parts about the diagonal are similar to the whole and to each other.



In modern terms:
The area of each small parallelogram is a fixed fraction of the whole.

So the two small parallelograms have the same shape and the same ratio to the whole.



Proposition 25

To construct a single rectilinear figure similar to one given rectilinear figure and equal to another given rectilinear figure.

Proof / Construction Actions

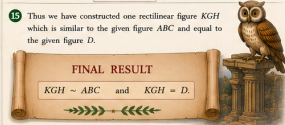
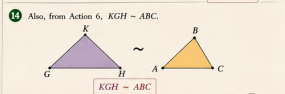
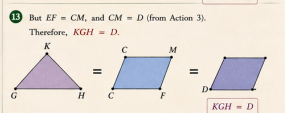
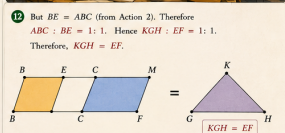
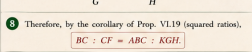
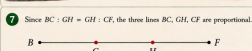
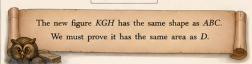
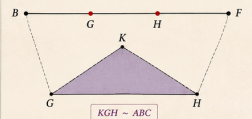
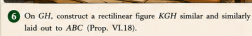
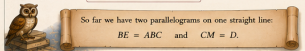
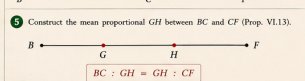
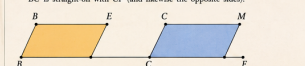
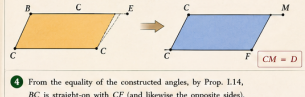
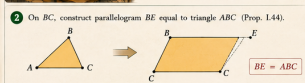
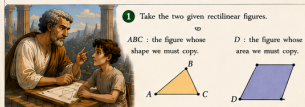
1. Take ABC for the required shape and D for the required area.
2. Construct a parallelogram equal to ABC .
3. Construct a second parallelogram equal to D .
4. Place their bases on one straight line.
5. Find the mean proportional between the relevant bases.
6. On it construct $KGH \sim ABC$.
7. Use the mean-proportional relation.
8. Convert the line proportion into a figure proportion.
9. Compare the two parallelograms by their bases using VI.1.
10. Equate the two ratios.
11. Alternate the proportion.
12. Use the first area equality to obtain $KGH = EF$.
13. Use the second area equality to obtain $KGH = D$.
14. Recall $KGH \sim ABC$.
15. Conclude the new figure has the shape of ABC and area of D .

Final Conclusion

$KGH \sim ABC, \quad KGH = D$

BOOK VI – PROPOSITION 25

To construct a single rectilinear figure similar to a given rectilinear figure, and equal to another given rectilinear figure.



Proposition 26

If from a parallelogram another parallelogram is subtracted which is similar and similarly laid out to the whole and has a common angle with it, then the smaller parallelogram lies about the same diagonal as the whole.

Proof / Construction Actions

1. Take whole parallelogram $ABCD$.
2. Take smaller parallelogram AF similar and similarly laid out to it.
3. Identify their common angle.
4. State that AF must lie about the same diagonal.
5. Assume the contrary.
6. Produce a side of the smaller parallelogram to meet the diagonal.
7. Through the meeting point, draw a parallel to a side of the whole parallelogram.
8. Form an auxiliary parallelogram about the true diagonal.
9. By VI.24, obtain the first side proportion.
10. By the assumed similarity, obtain the second side proportion.
11. Compare the proportions.
12. By V.9, infer equality of the two compared segments.
13. This makes the lesser equal to the greater, impossible.
14. Reject the contrary assumption.
15. Conclude the smaller parallelogram lies about the same diagonal as the whole.

Final Conclusion

AF is about the same diagonal as $ABCD$



BOOK VI – PROPOSITION 26

If from a parallelogram another parallelogram is subtracted which is similar and similarly laid out to the whole, and has a common angle with it, then the subtracted parallelogram is about the same diagonal as the whole.

GIVEN

$ABCD$ is a parallelogram.
 AF is a parallelogram
 similar and similarly laid
 out to $ABCD$, and has the
 common angle $\angle DAB$.

TO PROVE
 AF is about the
 same diagonal
 as $ABCD$.

PANEL 1 – SETTING UP THE PROBLEM

1. Take parallelogram $ABCD$.
2. Inside it is parallelogram AF .
3. They have the common angle $\angle DAB$.
4. We must prove that AF is about the same diagonal as $ABCD$.
5. Suppose the contrary: AE is NOT about the same diagonal. Take AHC as the diagonal of $ABCD$.

$AF \sim ABCD$
 (similar and
 similarly laid out)

$\angle DAB$ is
 common to
 both.



We will reach a contradiction.

PANEL 2 – CONSTRUCTING AND COMPARING

6. Produce GF to meet the diagonal AHC at H .
7. Through H draw HK parallel to AD (Prop. 1.31).
8. Then HG forms a parallelogram about the same diagonal as $ABCD$.
9. By Prop. VI.24 (parallelograms about the same diagonal),
 $DA : AB = GA : AK$.
10. But $AF \sim ABCD$ (given similarity).
 Therefore corresponding sides are proportional:
 $DA : AB = GA : AE$.

$$DA : AB = GA : AK$$

$$DA : AB = GA : AE$$



We have two proportions with the
 same first ratio $DA : AB$.

PANEL 3 – DERIVING THE CONTRADICTION

11. From (9) and (10),
 $GA : AK = GA : AE$.
 $DA : AB = GA : AK$
 $DA : AB = GA : AE$
 $\Downarrow$
 $GA : AK = GA : AE$
12. By Prop. V.9, since GA is the same magnitude, $AK = AE$.
13. But AE is the lesser segment and AK is the greater segment.
 $AE < AK$
14. This contradicts $AE = AK$.
 Therefore our supposition is false.
15. Hence AF is about the same diagonal as $ABCD$.

Contradiction!

Proved:
 AF is about the
 same diagonal
 as $ABCD$.



Proposition 27

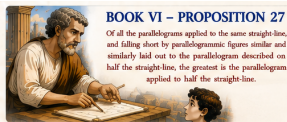
Of all parallelograms applied to the same straight line and falling short by similar and similarly situated parallelogrammic figures, the greatest is the parallelogram applied to half the straight line.

Proof / Construction Actions

1. Take the straight line AB .
2. Bisect it at C .
3. Construct the special applied parallelogram AD , falling short by the parallelogram on half CB .
4. State the maximum claim.
5. Construct an arbitrary competing parallelogram AF .
6. Assume its deficit is similar and similarly laid out to the midpoint deficit.
7. By VI.26, the deficits lie about the same diagonal.
8. Draw the common diagonal and complete the figure.
9. By I.43, the complements are equal.
10. Add the common parallelogram to obtain equality of larger regions.
11. Use $AC = CB$ and VI.1 for the next equality.
12. Transfer the equality.
13. Add the common part and identify AF with the gnomon.
14. Compare the gnomon with the larger midpoint parallelogram.
15. Since AF was arbitrary, every competitor is smaller.
16. Conclude the maximum occurs at the midpoint.

Final Conclusion

$$AD > AF$$



BOOK VI – PROPOSITION 27

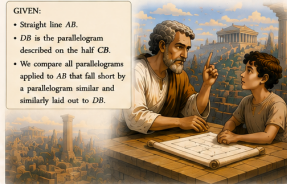
Of all the parallelograms applied to the same straight line, and falling short by parallelogrammic figures similar and similarly laid out to the parallelogram described on half the straight-line, the greatest is the parallelogram applied to half the straight-line.

PANEL 1 – SETTING UP THE FIGURE

- Take the straight line AB .
- Cut it in half at C (Prop. I.10).
 $AC = CB$.
- Apply parallelogram AD along AB . It falls short by parallelogram DB on the half CB .
- Claim: Among all such parallelograms, AD is the greatest.
 AD is applied to AB and falls short by DB , constructed on CB .

GIVEN:

- Straight line AB .
- DB is the parallelogram described on the half CB .
- We compare all parallelograms applied to AB that fall short by a parallelogram similar and similarly laid out to DB .



PANEL 2 – COMPARING WITH AN ARBITRARY PARALLELOGRAM

- Let another parallelogram AF be applied to AB , falling short by parallelogram FB .
- Assume $FB \sim DB$ and similarly laid out. We must prove $AD > AF$.
 $FB \sim DB$.
- By Prop. VI.26, since $FB \sim DB$ and similarly laid out, they are about the same diagonal.
- Draw their common diagonal and complete the figure.
- By Prop. I.43, the complements about the diagonal are equal.
 $CH = FE$.
- Parallelogram FB is common. Therefore, adding it to the equals gives:
 $CH = KE$.
- Since $AC = CB$, by Prop. VI.1,
 $CH = CG$.

From (10) and (11), we have:

$$CH = KE \quad \text{and} \quad CH = CG \quad \Rightarrow \quad CG = KE$$

PANEL 3 – FINISHING THE ARGUMENT

- Add parallelogram CG to both equal figures CH and KE . Therefore,
 $AF = \text{gnomon } LMN$.
- The gnomon LMN is part of the parallelogram DB . Therefore,
 $DB > LMN$.
- But $AF = LMN$ (from 12). Hence,
 $DB > AF$.
- Since AF was arbitrary (subject to the conditions), every such parallelogram is less than AD .
 All are less than AD .
- Therefore the greatest is precisely AD , the parallelogram applied to AB that falls short by DB on the half CB .

FINAL RESULT

AD is the greatest parallelogram.



Proposition 28

To apply to a given straight line a parallelogram equal to a given rectilinear figure, falling short by a parallelogrammic figure similar to a given parallelogram.

Condition. The given rectilinear figure must not be greater than the parallelogram described on half the given straight line and similar to the required deficit.

Proof / Construction Actions

1. Take the given line AB , area C , and parallelogram D defining the deficit shape.
2. Bisect AB at E .
3. On EB , construct a parallelogram similar and similarly laid out to D .
4. Complete the surrounding parallelogram.
5. If it already equals C , the construction is complete.
6. Otherwise use the stipulated size condition so that the midpoint configuration exceeds C .
7. Isolate the excess.
8. By VI.25, construct a parallelogram similar to D and equal to that excess.
9. By VI.21, relate it by similarity to the midpoint deficit.
10. Match corresponding sides.
11. Copy the smaller corresponding sides inside the midpoint deficit.
12. Complete the corresponding inner parallelogram.
13. By VI.26, place the similar parallelograms about one diagonal.
14. Subtract the constructed excess to obtain a gnomon equal to C .
15. Use equality of complements.

16. Add the common parallelogram.
17. Use $AE = EB$.
18. Add the remaining common region.
19. Substitute the area of the gnomon.
20. Identify the remaining deficit.
21. Conclude the applied parallelogram equals C and falls short by a parallelogram similar to D .

Final Conclusion

$ST = C, \quad QB \sim D$



BOOK VI – PROPOSITION 28

To apply to a given straight-line a parallelogram equal to a given rectilinear figure, falling short by a parallelogrammic figure similar to a given parallelogram.

GIVEN:

- AB = given straight line
- C = given rectilinear figure (area to be obtained)
- D = given parallelogram (shape of the deficit)

Condition: C must not be greater than the parallelogram constructed on half of AB and similar to D .

PANEL 1 – SETTING UP THE FIGURE

- Take the given straight line AB , the rectilinear figure C , and the parallelogram D .
- Cut AB in half at E (Prop. I.10).
 $AE = EB$
- On the half EB , construct parallelogram $EBFG$ similar and similarly laid out to D (Prop. VI.18).
- Complete the surrounding parallelogram AG .
- If $AG = C$, we are done.
- Otherwise, take $AH > C$. Since $AHE = GB$ (Prop. VI.1), we have $GB > C$.
- Let $(GB - C)$ be the excess.



PANEL 2 – BUILDING THE EXCESS AND EQUALITIES

- By Prop. VI.25, construct parallelogram $KLMN$ equal to $(GB - C)$ and similar to D .
- Since $GB = D$ and $KLMN = D$, by Prop. VI.21, $KLMN \sim GB$.
- Let KL correspond to GE and LM to GF . Since $GB > KLMN$, we have $GE > KL$ and $GF > LM$.
- On GB , construct $GO = KL$ and $GP = LM$. Complete parallelogram $OGPQ$.
- Then $GQ = KLMN$ and $GQ = GB$.
- By Prop. VI.26, since $GQ = GB$ and similarly laid out, they are about the same diagonal. Draw the common diagonal GQB and complete the figure.
- Since $GB = C + KLMN$ and $GQ = KLMN$, the remainder (gnomon GWV) equals C .

PANEL 3 – FINISHING THE ARGUMENT

- By Prop. I.43, the complements about the diagonal are equal: $PR = OS$.
- Add parallelogram QR to both equal figures PR and OS . Therefore, $PB = OR$.
- Since $AE = EB$, by Prop. VI.1, $OB = TE$. Hence, $PB = TE$.
- Add parallelogram OS to both equal figures PB and TE . Therefore, $TS = VWU$.
- But gnomon $VWU = C$ (from Action 14). Therefore, $TS = C$.
- Thus parallelogram ST (is applied to AB) and falls short by QB .
- And QB is similar to D . Construction is complete.

FINAL RESULT

Parallelogram ST is equal to the given rectilinear figure C and falls short by parallelogram QB similar to D .



Proposition 29

To apply to a given straight line a parallelogram equal to a given rectilinear figure, overshooting by a parallelogrammic figure similar to a given parallelogram.

Proof / Construction Actions

1. Take the given line AB , area C , and parallelogram D .
2. Bisect AB at E .
3. On EB , construct $BF \sim D$.
4. Form the combined magnitude $BF + C$.
5. By VI.25, construct $GH \sim D$ and equal to $BF + C$.
6. Match corresponding sides and note that those of GH are greater.
7. Produce the smaller sides until equal to the corresponding sides of GH .
8. Complete the enlarged parallelogram MN .
9. Establish $MN \sim BF$.
10. By VI.26, place the similar parallelograms about the same diagonal.
11. Recall $MN = BF + C$.
12. Subtract BF to obtain a gnomon equal to C .
13. Use $AE = EB$ for the equal-area relation.
14. Identify the corresponding complement.
15. Add the common parallelogram.
16. Substitute the gnomon's area to obtain $AO = C$.
17. Identify the overshoot QP .
18. By VI.24, prove $QP \sim D$.
19. Conclude the required construction.

Final Conclusion

$$AO = C, \quad QP \sim D$$

BOOK VI – PROPOSITION 29

To apply a parallelogram, equal to a given rectilinear figure, to a given straight line, the applied parallelogram overshooting by a parallelogram; a figure similar to a given parallelogram.

AB = given straight line

C = given rectilinear figure

D = given parallelogram

Required: A parallelogram **AO** equal to **C**, overshooting **AB** by a parallelogram **QP** similar to **D**.

PANEL 1 – SETTING UP THE CONSTRUCTION (ACTIONS 1-6)

- Take the given straight line **AB**, the rectilinear figure **C**, and the parallelogram **D**.
- Cut **AB** in half at **E** (Prop. I.10).
 $AE = EB$
- On **EB**, construct parallelogram **BF** similar and similarly laid out to **D** (Prop. VI.18).
- Consider the sum of the magnitudes **BF** + **C**.
- By Prop. VI.13, construct parallelogram **GH** equal to **(BF + C)** and similar to **D**.
- Let **KH** correspond to **FL** and **KG** correspond to **FE**. Since **GH** > **BF**, **KH** > **FL** and **KG** > **FE**.

PANEL 2 – BUILDING THE EXCESS AND USING EQUALITIES

- Produce **FL** and **FE**. On them, mark points **M** and **N** so that **FM** = **KH** and **FN** = **KG** (Prop. I.3).
- Complete parallelogram **MN**. Then **MN** = **GH** and **AN** = **GH**.
- Since **GH** = **D** and **BF** = **D**, by Prop. VI.11, **GH** ~ **BF**. Hence **AN** ~ **BF**.
- By Prop. VI.16, since **MN** ~ **BF** and similarly laid out, they are about the same diagonal. Draw the common diagonal **MB** and complete the figure.
- By construction, **GH** = **BF** + **C**, and **GH** = **AN**. Subtract **BF** from both. The remainder (gnomon **XWV**) equals **C**.
- Therefore, gnomon **XWV** = **C**.

PANEL 3 – TRANSFERRING THE AREA AND FINISHING (ACTIONS 13-19)

- Since **AE** = **EB**, by Prop. VI.1, **AN** = **NB**.
- By equality of complements about the diagonal, **NB** = **LP**. Hence **AN** = **LP**.
- Add the common parallelogram **EO** to both equal figures **AN** and **LP**. Then **AO** = **YOEK**.
- But gnomon **WEX** = **C** (from Action 12). Therefore, **AO** = **C**.
- Parallelogram **AO** is applied to **AB** and overshoots it by parallelogram **QP**.
- By Prop. VI.18, **QP** is similar to the given parallelogram **D**.
 $QP \sim D$
- Therefore we have constructed parallelogram **AO** equal to **C**, overshooting **AB** by a parallelogram **QP** similar to **D**.
 $AO = C$
 $QP \sim D$

CONSTRUCTION COMPLETE

Proposition 30

To cut a given finite straight line in extreme and mean ratio.

Proof / Construction Actions

1. Take the given line AB .
2. Construct a square on AB .
3. By VI.29, apply along the adjacent side a parallelogram equal to the square and overshooting by a similar figure.
4. The overshooting similar figure is also a square.
5. Subtract the common rectangle from the equal figures.
6. The remaining rectangle and square are equal and equiangular.
7. By VI.14, their sides about equal angles are reciprocally proportional.
8. Identify one side of the rectangle with the whole AB .
9. Identify the corresponding side of the square with AE .
10. Substitute these equalities into the proportion.
11. By V.14, determine $AE > EB$.
12. Conclude $AB : AE = AE : EB$.

Final Conclusion

$AB : AE = AE : EB, \quad AE > EB$

Modern note. This construction is commonly called the Golden Section.

BOOK VI – PROPOSITION 30

To cut a given finite straight-line in extreme and mean ratio.

Given AB , to find point E on AB such that $AB : AE = AE : EB$, with AE the greater.

PANEL 1 – CONSTRUCTION (ACTIONS 1–4)

- Take the given finite straight line AB .
- Describe square BC on AB (Prop. I.46).
- On AC , apply parallelogram CD equal to square BC , overshooting AC by a figure AD similar to BC (Prop. VI.29).
- Since BC is a square and $AD \sim BC$, the figure AD is also a square.

We will prove that E divides AB in extreme and mean ratio.

PANEL 2 – THE PROPORTION (ACTIONS 5–8)

- Since $BC = CD$, subtract the common rectangle CE from both. The remainders are equal: $BF = AD$.
- Rectangles BF and square AD are equal and equiangular. By Prop. VI.14, the sides about their equal angles are reciprocally proportional.
- Therefore, $FE : ED = AE : EB$.
- But from the construction we have: $FE = AB$ (whole line) and $ED = AE$ (since AD is a square).

PANEL 3 – CONCLUSION (ACTIONS 9–12)

- Substituting (8) into (7): $AB : AE = AE : EB$.
- Since $AB > AE$, from the proportion $AB : AE = AE : EB$, by Prop. V.14, $AE > EB$.
- Thus E divides AB in extreme and mean ratio.
- This construction is called the Golden Section. (See also Prop. II.11.)

Proved:
 $AB : AE = AE : EB$
with AE the greater.

Golden Ratio:
 $AB : AE = AE : EB = \phi$
($\phi = (1 + \sqrt{5}) / 2$)

CONSTRUCTION COMPLETE

Proposition 31

In a right-angled triangle, the figure described on the hypotenuse equals the sum of the similar and similarly described figures on the two sides containing the right angle.

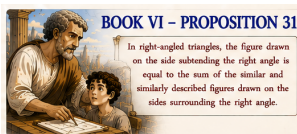
Interpretive note. This is Euclid's generalized form of the Pythagorean theorem.

Proof / Construction Actions

1. Take right triangle ABC with $\angle BAC = 90^\circ$.
2. Describe similar and similarly situated figures on BC, BA, AC .
3. Draw $AD \perp BC$.
4. By VI.8, the two smaller triangles are similar to the whole and to one another.
5. From $\triangle ABC \sim \triangle ABD$, obtain $CB : BA = AB : BD$.
6. Thus CB, BA, BD are three proportional lines.
7. By the corollary to VI.19, relate $CB : BD$ to the figures on CB and BA .
8. Similarly relate $BC : CD$ to the figures on BC and CA .
9. Combine the ratios by V.24.
10. Since $BD + DC = BC$, the line ratio is $1 : 1$.
11. By V.9, the corresponding figure ratio is equality.
12. Conclude the figure on the hypotenuse equals the sum of the two similar figures on the legs.

Final Conclusion

$$\text{Fig}(BC) = \text{Fig}(BA) + \text{Fig}(AC)$$



BOOK VI – PROPOSITION 31

In right-angled triangles, the figure drawn on the side subtending the right angle is equal to the sum of the similar and similarly described figures drawn on the sides surrounding the right angle.

PANEL 1 – SETTING UP THE FIGURE (ACTIONS 1–4)

- 1 Take right-angled triangle ABC with $\angle BAC = 90^\circ$.
- 2 On side BC take any rectilinear figure (yellow). On BA take a similar and similarly described figure (purple). On AC take another similar figure (blue). To prove: Figure on BC = Figure on BA + Figure on AC .
- 3 From the right angle vertex A , draw AD perpendicular to BC (Prop. I.12).
- 4 By Prop. VI.8, $\triangle ABD \sim \triangle ABC$, $\triangle ADC \sim \triangle ABC$, so $\triangle ABD \sim \triangle ADC$.



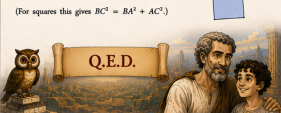
PANEL 2 – USING SIMILARITY AND PROPORTIONS (ACTIONS 5–8)

- 5 Since $\triangle ABC \sim \triangle ABD$, the corresponding sides are proportional:
 $\frac{CB}{BA} = \frac{AB}{BD}$
- 6 Thus CB , BA , BD are three proportional straight lines:
 $CB : BA = BA : BD$
- 7 By the corollary to Prop. VI.19, for three proportional lines,
 $\frac{CB}{BD} = \frac{\text{Figure on } BC}{\text{Figure on } BA}$
- 8 Similarly, using $\triangle ABC \sim \triangle ADC$, we obtain
 $\frac{BC}{CD} = \frac{\text{Figure on } BC}{\text{Figure on } AC}$



PANEL 3 – FINISHING THE ARGUMENT (ACTIONS 9–12)

- 9 By Prop. V.24 (composition of ratios),
 $\frac{BC}{BD+DC} = \frac{\text{Figure on } BC}{\text{Figure on } BA + \text{Figure on } AC}$
- 10 But $BD + DC = BC$.
Hence
 $\frac{BC}{BD+DC} = \frac{BC}{BC} = 1 = 1$.
- 11 Therefore the ratio on the right is also equality (Prop. V.9):
Figure on BC
Figure on $BA + \text{Figure on } AC = 1$.
Thus,
Figure on BC = Figure on $BA + \text{Figure on } AC$.
- 12 Hence, in right-angled triangle ABC , for any similar and similarly described figures on the three sides,
Figure on BC = Figure on $BA + \text{Figure on } AC$.
(For squares this gives $BC^2 = BA^2 + AC^2$.)



Q.E.D.

Proposition 32

If two triangles having two sides proportional are placed together at one angle so that corresponding sides are parallel, then their remaining sides are straight-on with respect to one another.

Proof / Construction Actions

1. Take triangles ABC and DCE .
2. Assume $AB : AC = DC : DE$.
3. Assume $AB \parallel DC$.
4. Assume $AC \parallel DE$.
5. From the first parallelism, obtain $\angle BAC = \angle ACD$.
6. From the second, obtain $\angle ACD = \angle CDE$.
7. Therefore $\angle BAC = \angle CDE$.
8. Combine that equal angle with the given proportional sides.
9. By VI.6, the triangles are equiangular.
10. Thus $\angle ABC = \angle DCE$.
11. Recall $\angle ACD = \angle BAC$.
12. Form the whole angle $\angle ACE$.
13. Add $\angle ACB$ to both sides.
14. By I.32, the three angles of triangle ABC equal two right angles.
15. Hence $\angle ACE + \angle ACB = 2$ right angles.
16. By I.14, BC and CE lie on one straight line.

Final Conclusion

$$\boxed{B - C - E}$$

BOOK VI – PROPOSITION 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on with respect to one another.

GIVEN

- Triangles ABC and DCE with common point C .
- $AB : AC = DC : DE$
- $AB \parallel DC$
- $AC \parallel DE$

To prove: $B - C - E$

PANEL 1 – SETTING UP AND EQUAL ANGLES (ACTIONS 1–5)

- Take the two triangles ABC and DCE with common point C .
- Assume the proportional sides:
 $AB : AC = DC : DE$
- Assume $AB \parallel DC$.
- Assume $AC \parallel DE$.
- Since $AB \parallel DC$, with AC as transversal, alternate angles are equal (Prop. 1.29):
 $\angle BAC = \angle ACD$

We will prove that BC and CE are in a straight line.

PANEL 2 – PROPORTIONS AND SIMILARITY (ACTIONS 6–11)

- Since $AC \parallel DE$, with CD as transversal, alternate angles are equal:
 $\angle ACD = \angle CDE$
- Therefore, from (5) and (6):
 $\angle BAC = \angle CDE$
- We have $AB : AC = DC : DE$, and the included angles $\angle BAC = \angle CDE$ are equal. By Prop. VI.6 (SAS similarity),
 $\triangle ABC \sim \triangle DCE$
- Therefore corresponding angles are equal. In particular,
 $\angle ABC = \angle DCE$
- From (5) we already have
 $\angle ACD = \angle BAC$
- So we have two equal angles:
 $\angle ABC = \angle DCE$ and $\angle ACD = \angle BAC$

From proportional sides and parallel lines we have shown the triangles are similar.

PANEL 3 – FINISHING THE ARGUMENT (ACTIONS 12–16)

- The whole angle $\angle ACE$ is made of the adjacent angles $\angle ACD$ and $\angle DCE$:
 $\angle ACE = \angle ACD + \angle DCE$
- Using (10), (11) in (12):
 $\angle ACE = \angle BAC + \angle ABC$
- Add $\angle ACB$ to both sides:
 $\angle ACE + \angle ACB = \angle BAC + \angle ABC + \angle ACB = \angle BAC + \angle ABC + \angle ACB$
- But in $\triangle ABC$, by Prop. 1.32,
 $\angle BAC + \angle ABC + \angle ACB = 2$ right angles.
Therefore,
 $\angle ACE + \angle ACB = 2$ right angles.
- By Prop. 1.14, if two adjacent angles are equal to two right angles, the straight lines forming them are in one straight line.
Hence,
 $B - C - E$

Q.E.D.
 BC and CE are straight-on with respect to one another.

Proposition 33

In equal circles, angles have the same ratio as the arcs or circumferences on which they stand, whether the angles are at the centers or at the circumferences.

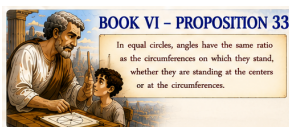
Proof / Construction Actions

1. Take equal circles ABC and DEF , with centers G, H , central angles $\angle BGC, \angle EHF$, and circumference angles $\angle BAC, \angle EDF$.
2. On the first circle, lay off any number of consecutive arcs equal to BC .
3. On the second circle, lay off any number of consecutive arcs equal to EF .
4. Join the new division points to the centers.
5. By III.27, equal arcs subtend equal central angles.
6. Therefore a multiple arc BL is accompanied by the same multiple of $\angle BGC$.
7. Likewise, a multiple arc EN is accompanied by the same multiple of $\angle EHF$.
8. If $BL = EN$, the corresponding multiple central angles are equal.
9. If $BL > EN$, the first multiple central angle is greater.
10. If $BL < EN$, the first multiple central angle is less.
11. Thus equal multiples of the arcs compare exactly as equal multiples of the central angles.
12. By Definition V.5, $BC : EF = \angle BGC : \angle EHF$.
13. By III.20, each central angle is double the corresponding circumference angle.
14. By V.15, doubling preserves the ratio.
15. Combine the two established proportions.
16. Hence $BC : EF = \angle BGC : \angle EHF = \angle BAC : \angle EDF$.

17. Therefore central angles have the same ratio as the arcs on which they stand.
18. And circumference angles have that same ratio; Q.E.D.

Final Conclusion

$$\text{arc } BC : \text{arc } EF = \angle BGC : \angle EHF = \angle BAC : \angle EDF$$

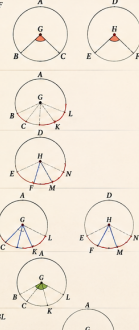


BOOK VI - PROPOSITION 33

In equal circles, angles have the same ratio as the circumferences on which they stand, whether they are standing at the centres or at the circumferences.

PANEL 1 - SETUP AND EQUAL ARCS (ACTIONS 1-6)

- Take two equal circles ABC and DEF with centres G and H . At the centres take angles $\angle BGC$ and $\angle EHF$. At the circumferences take angles $\angle BAC$ and $\angle EDF$.
- On the first circle, lay off any number of arcs $CK, KL, \dots$ each equal to arc BC .
- On the second circle, lay off any number of arcs $FM, MN, \dots$ each equal to arc EF .
- Join the new division points to the centres: draw GK, GL, HM, HN .
- Since arcs BC, CK, KL are equal, the corresponding central angles are equal (Prop. III.27). $\angle BGC = \angle CGK = \angle KGL$.
- Therefore, whatever multiple arc BL is of arc BC , angle $\angle BGL$ is the same multiple of $\angle BGC$.



We compare equal multiple arcs with equal multiple angles.

PANEL 2 - COMPARING MULTIPLES (ACTIONS 7-12)

- By the same reasoning in the second circle, whatever multiple arc EN is of arc EF , angle $\angle EHN$ is the same multiple of $\angle EHF$.
- If the multiple arc BL equals the multiple arc EN , then $\angle BGL = \angle EHN$. (Prop. III.27).
- If $BL > EN$, then $\angle BGL > \angle EHN$.
- If $BL < EN$, then $\angle BGL < \angle EHN$.
- Then equal multiples of arcs BC and EF compare exactly as the corresponding equal multiples of angles $\angle BGC$ and $\angle EHF$: greater with greater, equal with equal, lesser with lesser.
- Therefore, by Definition V.5, $BC : EF = \angle BGC : \angle EHF$.

Summary of Comparisons
 $BL = EN \Rightarrow \angle BGL = \angle EHN$
 $BL > EN \Rightarrow \angle BGL > \angle EHN$
 $BL < EN \Rightarrow \angle BGL < \angle EHN$



PANEL 3 - FROM CENTRAL TO CIRCUMFERENCE ANGLES (ACTIONS 13-16)

- By Prop. III.20, the central angle is double the angle at the circumference standing on the same arc: $\angle BGC = 2\angle BAC$, $\angle EHF = 2\angle EDF$.
- Equal multiples preserve ratio (Prop. V.15), hence $\angle BGC : \angle EHF = \angle BAC : \angle EDF$.
- We have already proved $BC : EF = \angle BGC : \angle EHF$.
- And we have $\angle BGC : \angle EHF = \angle BAC : \angle EDF$.
- Therefore, by transitivity of equality of ratios, $BC : EF = \angle BGC : \angle EHF = \angle BAC : \angle EDF$.
- Thus, in equal circles, angles have the same ratio as the circumferences on which they stand, whether at the centres or at the circumferences. Q.E.D.

PROVED
 $BC : EF = \angle BGC : \angle EHF = \angle BAC : \angle EDF$

In equal circles, arcs and angles stand in the same ratio!

Conclusion

Book VI concludes with Proposition 33:

$$\text{arc } BC : \text{arc } EF = \angle BGC : \angle EHF = \angle BAC : \angle EDF.$$