

EUCLID'S ELEMENTS

BOOK V

Ratio and Proportion

18 Definitions and 25 Propositions

A classical illustrated edition of the Eudoxian theory of proportion

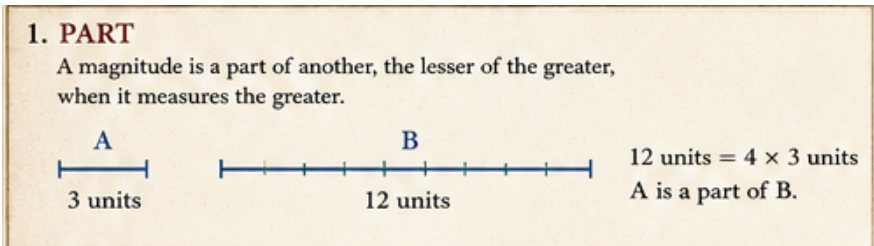
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DEFINITIONS

Definition 1 — Part

A magnitude is a part of another magnitude, the lesser of the greater, when it measures the greater. Thus, if A measures B , then for some positive integer n ,

$$B = nA.$$



Definition 2 — Multiple

The greater magnitude is a multiple of the lesser when it is measured by the lesser. Thus, if

$$B = nA,$$

then A is a part of B , while B is a multiple of A .

2. MULTIPLE

A magnitude is a multiple of another when it is measured by a part of it.

A
|-----|
3 units

B
|-----|
12 units

$B = 4 \times A$
B is a multiple of A.

Definition 3 — Ratio

A ratio is a certain relation with respect to size between two magnitudes of the same kind. The ratio $A : B$ expresses their comparative size without requiring the ratio itself to be treated as a rational number.

3. RATIO

A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.

For example, the ratio of A to B.
(A and B are both lengths.)

A **B**
|-----| |-----|
5 units 8 units

Definition 4 — Magnitudes Capable of Having a Ratio

Magnitudes have a ratio to one another when, being multiplied, each is capable of exceeding the other. Thus there exist positive integers m, n such that

$$mA > B, \quad nB > A.$$

4. WHEN TWO MAGNITUDES HAVE A RATIO

Two magnitudes have a ratio to one another when, if multiples of the one be taken, some will exceed the other, and if multiples of the other be taken, some will exceed the one.

A

 2 units

B

 5 units

$3 \times A = 6 > 5 = B$
 (so some multiple of A exceeds B)
 $3 \times B = 15 > 2 = A$
 (so some multiple of B exceeds A)
 Therefore, A and B have a ratio.

Definition 5 — Equality of Ratios

Magnitudes are in the same ratio, the first to the second and the third to the fourth, when arbitrary equal multiples of the first and third and arbitrary equal multiples of the second and fourth preserve the same comparison. Thus

$$A : B = C : D$$

means that for every positive m, n ,

$$mA > nB \iff mC > nD,$$

$$mA = nB \iff mC = nD,$$

and

$$mA < nB \iff mC < nD.$$

This is the central Eudoxian definition and applies even when the magnitudes are incommensurable.

5. EQUALITY OF RATIOS

The ratios of A to B and of C to D are equal when, for every pair of equal multiples, the multiples of the first are greater than, equal to, or less than one another, according as the multiples of the second are greater than, equal to, or less than one another.

Example: 

Take any m, n (multiples)	$m = 1, n = 1$	$m = 2, n = 1$	$m = 3, n = 2$
mA vs nB	$3 < 5$	$6 > 5$	$9 = 10$
mC vs nD	$6 < 10$	$12 > 10$	$18 = 20$

In every case the comparisons agree. Therefore $A : B = C : D$.

Definition 6 — Proportional Magnitudes

Magnitudes having the same ratio are called proportional. Thus if

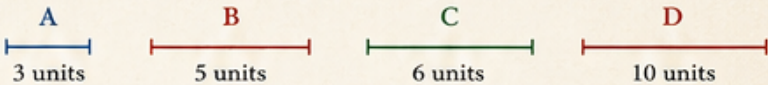
$$A : B = C : D,$$

then A, B, C, D are proportional.

6. PROPORTIONAL MAGNITUDES

When four magnitudes are such that the ratio of the first to the second is equal to the ratio of the third to the fourth, they are called proportional.

$$A : B :: C : D$$



Since $A : B = C : D$, the four magnitudes are proportional.

Definition 7 — Greater Ratio

The first magnitude has a greater ratio to the second than the third has to the fourth when suitable equal multiples can be found such that

$$mA > nB, \quad mC \leq nD.$$

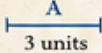
Then

$$A : B > C : D.$$

7. GREATER RATIO

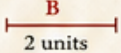
The ratio of A to B is greater than the ratio of C to D when equal multiples can be found such that the multiples of A exceed the multiples of B, while the multiples of C do not exceed the multiples of D.

A



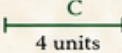
3 units

B



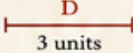
2 units

C



4 units

D



3 units

Take equal multiples $m = 2, n = 3$.

$mA = 2 \times 3 = 6$	$>$	$nB = 3 \times 2 = 6$
$mC = 2 \times 4 = 8$	$\leq$	$nD = 3 \times 3 = 9$

Therefore, $A : B > C : D$.

Definition 8 — Three-Term Proportion

A proportion in three terms is the smallest possible continued proportion:

$$A : B = B : C.$$

8. THE SMALLEST PROPORTION

A proportion in three terms is the smallest possible.

A



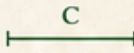
4 units

B



6 units

C



9 units

Here A, B, C are proportional, and the three-term proportion $A : B :: B : C$ is the smallest possible.

Definition 9 — Squared Ratio

When

$$A : B = B : C,$$

the ratio $A : C$ is called the squared ratio of $A : B$.

9. SQUARED RATIO

In a continued proportion ($A : B :: B : C$), the ratio of the first to the last is called the squared ratio.

A

2 units

B

3 units

C

4.5 units

$A : B = B : C = 2 : 3$
Then $A : C = 2 : 4.5 = 4 : 9$, which is the square of $2 : 3$.

Definition 10 — Cubed Ratio and Successive Powers

When

$$A : B = B : C = C : D,$$

the ratio $A : D$ is called the cubed ratio of $A : B$. The same manner of speech extends to longer continuous proportions.

10. CUBED RATIO AND HIGHER POWERS

In a continued proportion ($A : B :: B : C :: C : D$), the ratio of the first to the last is called the cubed ratio.

A

2 units

B

3 units

C

4.5 units

D

6.75 units

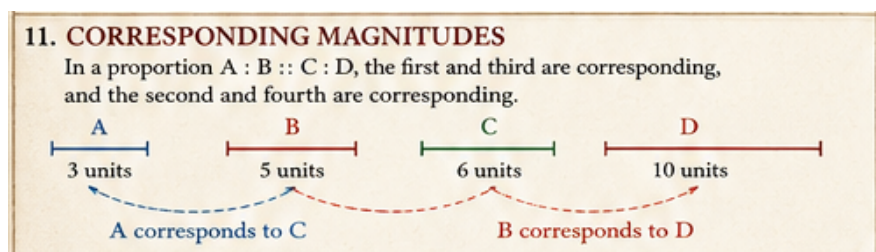
$A : B = B : C = C : D = 2 : 3$
Then $A : D = 2 : 6.75 = 8 : 27$, which is the cube of $2 : 3$.

Definition 11 — Corresponding Magnitudes

In

$$A : B = C : D,$$

the leading magnitudes A, C correspond, and the following magnitudes B, D correspond.



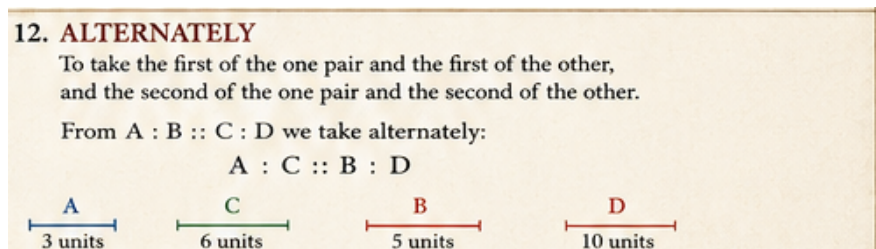
Definition 12 — Alternate Ratio

An alternate ratio takes leading to leading and following to following. Thus from

$$A : B = C : D$$

the alternate arrangement is

$$A : C = B : D.$$



Definition 13 — Inverse Ratio

The inverse of $A : B$ is

$$B : A.$$

13. INVERSELY

To invert a ratio is to turn it around.
From $A : B$ we take inversely $B : A$.

A

3 units

B

5 units

→

B

5 units

A

3 units

(If $A : B = C : D$, then inversely $B : A = D : C$.)

Definition 14 — Composition of a Ratio

Composition takes the leading and following magnitudes together, as one, to the following magnitude:

$$(A + B) : B.$$

14. COMPOSITION

To add the first and second magnitudes, and take the sum to the second.

From $A : B$ we take $(A + B) : B$.

A

3 units

B

5 units

→

A + B

8 units

B

5 units

Definition 15 — Separation of a Ratio

Separation takes the excess of the leading over the following to the following:

$$(A - B) : B.$$

15. SEPARATION

To subtract the second magnitude from the first, and take the remainder to the second.

From $A : B$ (where $A > B$) we take $(A - B) : B$.



Definition 16 — Conversion of a Ratio

Conversion takes the leading magnitude to the excess by which it exceeds the following:

$$A : (A - B).$$

16. CONVERSION

To take the first magnitude to the remainder when the second is subtracted from the first.

From $A : B$ (where $A > B$) we take $A : (A - B)$.



Definition 17 — Ratio via Equality — Ex Aequali

When several magnitudes and another equal number are in the same ratio taken two by two, ratio via equality compares the first with the last in each set. Thus

$$A : B = D : E, \quad B : C = E : F$$

leads via equality to

$$A : C = D : F.$$

17. RATIO VIA EQUALITY (EX AEQUALI)

If $A : B = D : E$ and $B : C = E : F$, then $A : C = D : F$.

A

2 units

B

3 units

C

4 units

D

6 units

E

9 units

F

12 units

$A : B = D : E$ ($2 : 3 = 6 : 9$)

$B : C = E : F$ ($3 : 4 = 9 : 12$)

Therefore, $A : C = D : F$
($2 : 4 = 6 : 12$).

Definition 18 — Perturbed Proportion

A perturbed proportion occurs when the proportional links of two triples are crossed:

$$A : B = E : F, \qquad B : C = D : E.$$

18. PERTURBED PROPORTION

If $A : B = E : F$ and $B : C = D : E$, then $A : C = D : F$.

A

2 units

B

3 units

C

4 units

E

6 units

F

9 units

D

12 units

$A : B = E : F$ ($2 : 3 = 6 : 9$)

$B : C = D : E$ ($3 : 4 = 12 : 6$)

Therefore, $A : C = D : F$
($2 : 4 = 12 : 9$).

PROPOSITION 1

If any number of magnitudes are respectively equal multiples of an equal number of other magnitudes, then as many times as one of the former contains one of the latter, so many times will all the former together contain all the latter together.

Proof.

1. Choose two smaller magnitudes E and F .
2. Construct larger magnitudes AB and CD so that

$$AB = mE, \quad CD = mF.$$

3. Divide AB into m equal pieces, each equal to E . For example, if $m = 3$,

$$AB = AG + GH + HB, \quad AG = GH = HB = E.$$

4. Divide CD into the same number m of equal pieces, each equal to F :

$$CD = CK + KL + LD, \quad CK = KL = LD = F.$$

5. Pair corresponding pieces:

$$AG \leftrightarrow CK, \quad GH \leftrightarrow KL, \quad HB \leftrightarrow LD.$$

Each pair contains one copy of E and one copy of F .

6. Each paired block equals $E + F$. Since there are m such blocks, $AB + CD$ contains m copies of $E + F$.
7. Hence

$$AB + CD = m(E + F).$$

Since $AB = mE$ and $CD = mF$,

$$mE + mF = m(E + F).$$

Therefore. If several magnitudes are corresponding equal multiples of several others, their totals are in that same multiple relation.

1 CHOOSE TWO SMALLER MAGNITUDES E AND F .

Let these be two magnitudes of the same kind.

Let the blue segment be E and the red segment be F .

2 CONSTRUCT LARGER MAGNITUDES AB AND CD AS THE SAME MULTIPLE.

Let both be three times as great.

Take $m = 3$.
So $AB = 3E$ and $CD = 3F$.

3 DIVIDE AB INTO m EQUAL PIECES, EACH EQUAL TO E .

Since $AB = 3E$, let us cut it into three equal parts, each equal to E .

$AG = GH = HB = E$

4 DIVIDE CD INTO m EQUAL PIECES, EACH EQUAL TO F .

Now divide CD also into three equal parts, each equal to F .

$CK = KL = LD = F$

5 PAIR THE PIECES.

Match each piece of AB with the piece of CD directly beneath it.

$AG \leftrightarrow CK, \quad GH \leftrightarrow KL, \quad HB \leftrightarrow LD$

6 OBSERVE ONE PAIRED BLOCK.

Each pair contains one E and one F . So each pair is $E + F$.

$E + F \quad E + F \quad E + F$

7 COUNT THE PAIRED BLOCKS AND CONCLUDE.

There are $m = 3$ such pairs.

The whole $AB + CD$ contains three copies of $E + F$.

3 copies

Therefore,

$AB + CD = 3(E + F)$

But $AB = 3E$ and $CD = 3F$.

$3E + 3F = 3(E + F)$

This is Book V, Proposition 1:
If several magnitudes are respectively the same multiple of several other magnitudes,
then their totals are also in that same multiple relation.

PROPOSITION 2

If a first and third magnitude are equal multiples of a second and fourth respectively, and a fifth and sixth are also equal multiples of the second and fourth respectively, then the sums of the first and fifth and of the third and sixth are equal multiples of the second and fourth respectively.

Proof.

1. Choose the two basic magnitudes C and F . For illustration let

$$C = 2, \quad F = 3.$$

2. Build the first pair with the same number of copies. Take $m = 2$:

$$AB = 2C, \quad DE = 2F.$$

3. Build a second pair with another matching number of copies. Take $n = 3$:

$$BG = 3C, \quad EH = 3F.$$

4. Join the first and second pieces:

$$AG = AB + BG, \quad DH = DE + EH.$$

5. Count the copies in each whole:

$$AG = 5C, \quad DH = 5F.$$

6. Thus

$$AG = C + C + C + C + C,$$

and

$$DH = F + F + F + F + F.$$

The number of copies is the same.

7. In general,

$$AB = mC, \quad DE = mF, \quad BG = nC, \quad EH = nF$$

gives

$$AG = (m + n)C, \quad DH = (m + n)F.$$

Therefore. The sums are equal multiples of the second and fourth magnitudes:

$$m\alpha + n\alpha = (m + n)\alpha.$$

BOOK V – PROPOSITION 2 EXPLAINED BY ARISTOTLE TO ALEXANDER (AGE 12)

If a first and third magnitude are equal multiples of a second and fourth,
and a fifth and sixth are also equal multiples of the second and fourth,
then the sums of the first and fifth and of the third and sixth are equal multiples of the second and fourth.

1 CHOOSE THE TWO BASIC MAGNITUDES.

Let these be our two basic magnitudes: C and F .

$C = 2$ units $F = 3$ units

Everything else we build by repeating them.

2 BUILD THE FIRST PAIR WITH THE SAME NUMBER OF COPIES.

Take $m = 2$. Construct $AB = 2C$ and $DE = 2F$.

$AB = C + C = 2C = 4$ units
 $DE = F + F = 2F = 6$ units

3 BUILD A SECOND PAIR WITH ANOTHER MATCHING NUMBER OF COPIES.

Now take $n = 3$. Construct $BG = 3C$ and $EH = 3F$.

$BG = C + C + C = 3C = 6$ units
 $EH = F + F + F = 3F = 9$ units

4 JOIN THE FIRST AND SECOND PIECES.

Place BG after AB . Place EH after DE .

$AB = 2C$ $BG = 3C$
 $DE = 2F$ $EH = 3F$

5 COUNT HOW MANY COPIES ARE NOW IN EACH WHOLE.

Add the copies.

$AG = AB + BG = 2C + 3C = 5C$
 $DH = DE + EH = 2F + 3F = 5F$

6 COMPARE THE MULTIPLICITIES.

How many copies of C in AG ? Five!
And how many copies of F in DH ? Five!
Both wholes contain their respective basic magnitudes the same number of times.

7 STATE THE GENERAL RULE.

Therefore, if $AB = mC$ and $BG = nC$, then $AG = (m + n)C$.
And if $DE = mF$ and $EH = nF$, then $DH = (m + n)F$.

In general:
 $AB = mC$ $BG = nC$ $DE = mF$ $EH = nF$
 $AG = AB + BG = (m + n)C$ $DH = DE + EH = (m + n)F$

PROPOSITION 2
The sums of the first and fifth and of the third and sixth are equal multiples of the second and fourth.

EXAMPLE USED: $m = 2$, $n = 3$, $C = 2$ units, $F = 3$ units | $AG = 5C = 5 \times 2 = 10$ units | $DH = 5F = 5 \times 3 = 15$ units

PROPOSITION 3

If a first and third magnitude are equal multiples of a second and fourth respectively, and equal multiples are taken of the first and third, then the magnitudes taken are equal multiples of the second and fourth respectively.

Proof.

1. Choose two basic magnitudes B and D .

2. Let

$$A = mB, \quad C = mD.$$

For a visible example take $m = 2$.

3. Take equal multiples:

$$EF = nA, \quad GH = nC.$$

For illustration take $n = 3$.

4. Open each large block into smaller pieces:

$$EF = (B + B) + (B + B) + (B + B),$$

$$GH = (D + D) + (D + D) + (D + D).$$

5. Count the original basic magnitudes:

$$EF = nmB, \quad GH = nmD.$$

6. The number of copies is a product because each of the n groups contains m basic magnitudes:

$$n(mB) = (nm)B.$$

7. Thus, in general,

$$A = mB, \quad C = mD, \quad EF = nA, \quad GH = nC$$

implies

$$EF = nmB, \quad GH = nmD.$$

Therefore. Equal multiples of equal multiples are equal multiples of the original magnitudes:

$$m(n\alpha) = (mn)\alpha.$$

1

CHOOSE THE TWO BASIC MAGNITUDES.

These will be our basic magnitudes, from which everything else will be built.

Let the basic magnitudes be B and D .

B 2 units D 3 units

2

BUILD THE FIRST PAIR AS EQUAL MULTIPLES.

Let $m = 2$. Construct $A = 2B$ and $C = 2D$.

$A = 2B = 4$ units
 $C = 2D = 6$ units

3

TAKE EQUAL MULTIPLES OF THESE NEW MAGNITUDES.

Now let $n = 3$. Construct $EF = 3A$ and $GH = 3C$.

$EF = 3 \times 4 = 12$ units $GH = 3 \times 6 = 18$ units

4

OPEN EACH LARGE BLOCK INTO ITS SMALLER PIECES.

Each A is two B 's.
Each C is two D 's.

$EF = 6B$
 $GH = 6D$

5

COUNT THE ORIGINAL BASIC MAGNITUDES.

In EF there are six B 's.
In GH there are six D 's.

$EF = 6B$
 $GH = 6D$

6

SEE WHY THE NUMBER OF COPIES IS A PRODUCT.

Three groups of two make six.

We took:

- $m = 2$ copies inside each block ($A = 2B$, $C = 2D$)
- $n = 3$ blocks

Total copies = $n \times m = 3 \times 2 = 6$

$EF = (3 \times 2) B = 6B$ $GH = (3 \times 2) D = 6D$

7

STATE THE GENERAL RULE.

Therefore, if $A = mB$ and $C = mD$, and $EF = nA$ and $GH = nC$, then $EF = nmB$ and $GH = nmD$.

In general:

$A = mB$ $C = mD$ $EF = nA$ $GH = nC$

m copies of B m copies of D n copies of A n copies of C

$EF = nA = n(mB) = (nm)B$ $GH = nC = n(mD) = (nm)D$

PROPOSITION 3

If the first and third are equal multiples of the second and fourth, then equal multiples of the first and third are equal multiples of the second and fourth.

Example used throughout: $m = 2$, $n = 3$, $B = 2$ units, $D = 3$ units So $nm = 6$. $3(2B) = 6B = 12$ units $3(2D) = 6D = 18$ units

PROPOSITION 4

If a first magnitude has the same ratio to a second that a third has to a fourth, then equal multiples of the first and third have the same ratio to equal multiples of the second and fourth, taken in corresponding order.

Proof.

1. Begin with

$$A : B = C : D.$$

2. Take equal multiples of the leading magnitudes:

$$E = mA, \quad F = mC.$$

3. Take equal multiples of the following magnitudes:

$$G = nB, \quad H = nD.$$

4. Compare the first new pair:

$$E \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} G.$$

5. Since $A : B = C : D$, Definition V.5 requires the corresponding comparison

$$F \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} H$$

to agree exactly.

6. This agreement holds for every positive m, n , so the new ratios satisfy Definition V.5.

7. Therefore

$$E : G = F : H,$$

that is,

$$mA : nB = mC : nD.$$

Therefore. Equal multiples of corresponding terms of equal ratios remain in equal ratios.

1

BEGIN WITH TWO EQUAL RATIOS.

Suppose the first is to the second as the third is to the fourth.

Let $A : B :: C : D$.

$A = 2$ units
 $B = 3$ units
 $C = 4$ units
 $D = 6$ units

So $A : B = 2 : 3$ and $C : D = 4 : 6 = 2 : 3$.
The two ratios are equal.

2

TAKE EQUAL MULTIPLES OF THE LEADING MAGNITUDES.

Let us take the same number of copies of the first and of the third. Let $m = 3$.

$E = 3A$
 $F = 3C$

$E = 6$ units
 $F = 12$ units

E is three copies of A . F is three copies of C .

3

TAKE ANOTHER EQUAL NUMBER OF COPIES OF THE FOLLOWING MAGNITUDES.

Now take the same number of copies of the second and of the fourth. Let $n = 2$.

$G = 2B$
 $H = 2D$

$G = 6$ units
 $H = 12$ units

G is two copies of B . H is two copies of D .

4

COMPARE THE FIRST NEW PAIR.

Compare E with G .

$E = 3A$
 $G = 2B$

$E = 6$ units
 $G = 6$ units

$E = G$ ($3A = 2B$).

5

COMPARE THE CORRESPONDING SECOND PAIR.

Compare F with H .

$F = 3C$
 $H = 2D$

$F = 12$ units
 $H = 12$ units

$F = H$ ($3C = 2D$).

The comparison agrees on both sides: $E = G$ and $F = H$.

6

REASON IN GENERAL.

Because the original ratios are equal, for any positive numbers m and n :

if $mA > nB$ then $mC > nD$
if $mA = nB$ then $mC = nD$
if $mA < nB$ then $mC < nD$

Thus the ratios $mA : nB$ and $mC : nD$ are equal (by Definition 5).

7

STATE THE CONCLUSION.

Therefore, if $A : B = C : D$, then for any positive m and n , we have $mA : nB = mC : nD$.

PROPOSITION 4 (Book V)

$A : B :: C : D \rightarrow mA : nB :: mC : nD$

Equal ratios remain equal when the corresponding terms are taken in equal multiples.

OUR EXAMPLE:

$m = 3, n = 2$
 $3A : 2B :: 3C : 2D$
 $6 : 6 :: 12 : 12$
Equal ratios.

This is Book V, Proposition 4: From equal ratios, equal multiples of corresponding terms give equal ratios again.

PROPOSITION 5

If a magnitude is the same multiple of another magnitude as a part taken away is of a corresponding part taken away, then the remainder is the same multiple of the remainder as the whole is of the whole.

Proof.

1. Choose whole magnitudes with

$$AB = mCD.$$

For illustration take $m = 3$.

2. Mark parts to be removed so that

$$AE = mCF.$$

3. Remove the corresponding parts:

$$EB = AB - AE, \quad FD = CD - CF.$$

4. Compare what remains. Since both the wholes and removed parts use the same multiplier, the remainders must use that multiplier too.
5. Structurally,

$$EB = AB - AE = mCD - mCF = m(CD - CF) = mFD.$$

6. As repeated groups,

$$AB = \underbrace{CD + \cdots + CD}_{m \text{ copies}},$$

and

$$AE = \underbrace{CF + \cdots + CF}_{m \text{ copies}}.$$

Removing one CF -part from each CD -copy leaves m copies of FD .

7. Hence

$$AB = mCD, \quad AE = mCF$$

implies

$$EB = mFD.$$

Therefore.

$$m\alpha - m\beta = m(\alpha - \beta).$$

1

CHOOSE TWO WHOLE MAGNITUDES WITH THE SAME MULTIPLIER.

Let AB be three times as great as CD . We take $m = 3$.

$AB = 12$ units
 $CD = 4$ units
 $m = 3$
 $AB = 3CD$
 $CD = 4$
 $AB = 12$

AB contains CD exactly three times.

2

MARK PARTS TO BE REMOVED THAT ALSO HAVE THE SAME MULTIPLIER.

Mark part AE in AB and corresponding part CF in CD so that $AE = 3CF$.

$AE = 3$ units
 $CF = 1$ unit
 $AE = 3CF$

3

REMOVE THOSE CORRESPONDING PARTS.

Subtract AE from AB to get EB . Subtract CF from CD to get FD .

$EB = 12 - 3 = 9$
 $FD = 4 - 1 = 3$

4

COMPARE WHAT REMAINS.

How many times does FD go into EB ?

$EB = 9$ units
 $FD = 3$ units
 $EB = 9 = 3 \times 3$
 $FD = 3$
Therefore, $EB = 3FD$.

5

EXPLAIN WHY THE SAME MULTIPLIER SURVIVES.

Since $AB = 3CD$ and $AE = 3CF$, then
 $EB = AB - AE = 3CD - 3CF = 3(CD - CF) = 3FD$.

6

SEE IT AS GROUPS.

AB is three copies of CD . AE is three copies of CF . Remove one CF from each CD , and three FD 's remain.

Three copies minus three copies leaves three copies. So $EB = 3FD$.

7

STATE THE GENERAL RULE.

Therefore, if $AB = mCD$ and $AE = mCF$, then $EB = mFD$.

$AB = mCD$
 $AE = mCF$ } $\Rightarrow EB = mFD$

In general:
 $m\alpha - m\beta = m(\alpha - \beta)$

If the wholes are built with the same number of copies, and the parts removed are built with that same number of copies, then what remains must also contain that same number of copies.

BOOK V – PROPOSITION 5: If one whole magnitude is the same multiple of another whole magnitude as a part taken away is of a corresponding part taken away, then the remainders are also in that same multiple relationship.

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PROPOSITION 6

If two magnitudes are equal multiples of two other magnitudes, and parts taken away are equal multiples of those same magnitudes, then the remainders are either equal to those magnitudes or equal multiples of them.

Proof.

1. Choose the basic magnitudes E and F .
2. Build equal multiples:

$$AB = mE, \quad CD = mF.$$

For illustration take $m = 4$.

3. Mark parts to remove:

$$AG = nE, \quad CH = nF.$$

For illustration take $n = 2$.

4. Remove them:

$$GB = AB - AG, \quad HD = CD - CH.$$

5. Count the remaining copies:

$$GB = (m - n)E, \quad HD = (m - n)F.$$

6. If $m - n = 1$, then

$$GB = E, \quad HD = F.$$

Thus Euclid's special case, where the remainders are equal to the basic magnitudes themselves, is included.

7. Therefore

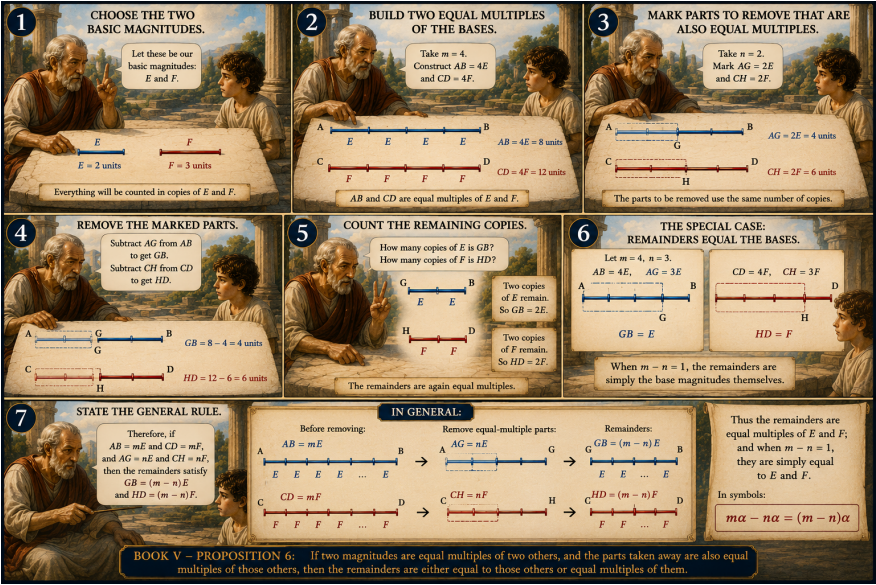
$$AB = mE, \quad CD = mF, \quad AG = nE, \quad CH = nF$$

gives

$$GB = (m - n)E, \quad HD = (m - n)F.$$

Therefore. The remainders are either equal to the bases or equal multiples of them:

$$m\alpha - n\alpha = (m - n)\alpha.$$



PROPOSITION 7

Equal magnitudes have the same ratio to the same magnitude, and the same magnitude has the same ratio to equal magnitudes.

Proof.

1. Choose equal magnitudes

$$A = B$$

and another magnitude C .

2. Take equal multiples

$$D = mA, \quad E = mB.$$

Because $A = B$, we have $D = E$.

3. Take any multiple

$$F = nC.$$

4. Since $D = E$, whatever comparison D has with F , E has the same:

$$D \gtrless F \iff E \gtrless F.$$

By Definition V.5,

$$A : C = B : C.$$

5. Reverse the viewpoint while retaining

$$D = mA, \quad E = mB, \quad F = nC.$$

6. Again $D = E$, so

$$F \gtrless D \iff F \gtrless E.$$

Hence

$$C : A = C : B.$$

7. Therefore

$$A = B \implies A : C = B : C$$

and

$$C : A = C : B.$$

As a corollary, proportional magnitudes are also proportional inversely:

$$A : B = C : D \implies B : A = D : C.$$

Therefore. Equal magnitudes are indistinguishable in ratio when compared with the same magnitude, in either position.

1

CHOOSE TWO EQUAL MAGNITUDES AND ANOTHER MAGNITUDE.

Let A and B be equal magnitudes.
Let C be any magnitude.

$A = 4$ units $B = 4$ units
 $C = 6$ units

$A = B = 4$ units, $C = 6$ units.

2

TAKE EQUAL MULTIPLES OF A AND B .

Take the same whole number of copies of A and of B .
Let $m = 3$.

$D = 3A = 12$ units
 $E = 3B = 12$ units

Since $A = B$, we have $D = E$.

3

TAKE ANY MULTIPLE OF C AND COMPARE.

Take any multiple of C .
Let $F = 2C$.

$F = 2C = 12$ units

Compare D with F , and E with F .
Here $D = F$, and therefore $E = F$.

4

WHATEVER RELATION D HAS TO F , E HAS THE SAME RELATION.

If $D > F$, $\Rightarrow E > F$.

If $D = F$, $\Rightarrow E = F$.

If $D < F$, $\Rightarrow E < F$.

By Definition 5, $A : C = B : C$.

5

NOW REVERSE THE VIEWPOINT.

Now compare a multiple of C with the multiples of A and B .

$D = 3A = 12$ units $E = 3B = 12$ units $F = 2C = 12$ units

Compare F with D , and F with E .

6

THE REVERSED COMPARISONS ALSO AGREE.

If $F > D$, $\Rightarrow F > E$.

If $F = D$, $\Rightarrow F = E$.

If $F < D$, $\Rightarrow F < E$.

By Definition 5 again,
 $C : A = C : B$.

7

STATE THE RULE.

Therefore, if $A = B$, then for any magnitude C :

(1) $A : C = B : C$

(2) $C : A = C : B$

Equal magnitudes have the same ratio to the same magnitude, and conversely.

COROLLARY (Immediately from the above)

If $A : B = C : D$, then the proportion may be inverted.

$A : B = C : D \Rightarrow B : A = D : C$

BOOK V – PROPOSITION 7: Equal magnitudes keep the same ratio when compared with the same magnitude, and the same magnitude keeps the same ratio when compared with equal magnitudes.

ἴσα τὰ μέρη, ἴση καὶ ἡ πρὸς τὰ ἴσα σχέσις.

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PROPOSITION 8

For unequal magnitudes, the greater has a greater ratio than the lesser to the same magnitude; and the same magnitude has a greater ratio to the lesser than to the greater.

Proof.

1. Choose unequal magnitudes

$$A > B$$

and another magnitude C .

2. Separate the excess:

$$A = B + E.$$

3. By Definition V.4, choose m so that

$$mE > C.$$

4. Take the same multiple of A and B :

$$mA = mB + mE.$$

5. Choose a multiple nC such that

$$mA > nC, \quad mB \leq nC.$$

By Definition V.7,

$$A : C > B : C.$$

6. Reverse the comparison. The same separator gives

$$nC > mB, \quad nC \not> mA.$$

Hence

$$C : B > C : A.$$

7. Therefore

$$A > B \implies A : C > B : C$$

and

$$C : B > C : A.$$

Therefore. With the comparison magnitude fixed, the greater magnitude gives the greater ratio; with the first magnitude fixed, the lesser following magnitude gives the greater ratio.

1

CHOOSE TWO UNEQUAL MAGNITUDES AND ANOTHER.

Let A be greater than B .
Let C be any magnitude.

$A = 5$ units
 $B = 3$ units
 $C = 4$ units

Clearly, $A > B$.

2

SEPARATE THE EXCESS.

Since $A > B$, write $A = B + E$, where E is the excess.

$A = 5$ units = $B = 3$ + $E = 2$

E is the part by which A exceeds B .

3

MULTIPLY THE EXCESS UNTIL IT EXCEEDS C .

By Definition 4, some multiple of E is greater than C . Take $m = 3$.

$E = 2$ units
 $C = 4$ units

$3E = 6$ units > 4 units = C

4

TAKE THE SAME MULTIPLE OF A AND B .

Take the same multiple $n = 3$ of A and B .

$3A = 3 \times 5 = 15$ units
 $3B = 3 \times 3 = 9$ units
 $3E = 3 \times 2 = 6$ units

Since $A = B + E$, $3A = 3B + 3E = 9 + 6 = 15$.

5

CHOOSE A MULTIPLE OF C THAT SEPARATES THEM.

Choose n so that $nA > nC$, but $nB \leq nC$.

Take $n = 3$.
 $nC = 3C = 12$ units

$3B = 9$
 $3A = 15$

$9 < 12$
 $15 > 12$

Thus $nA > nC$ while nB not $> nC$. Therefore, by Definition 7, $A : C > B : C$.

6

REVERSE THE COMPARISON.

Now compare $C : B$ with $C : A$ using the same multiples.

$nC = 12$
 $3B = 9$
 $3A = 15$

$12 > 9$
 12 not > 15

Thus $nC > nB$ while nC not $> nA$. Therefore, by Definition 7, $C : B > C : A$.

7

STATE THE GENERAL RULE.

Whenever $A > B$, then for any magnitude C ,
 $A : C > B : C$
and
 $C : B > C : A$

$A > B \Rightarrow A : C > B : C$
 $A > B \Rightarrow C : B > C : A$

THE IDEA:
A larger magnitude gives a larger ratio when the comparison magnitude is fixed.
But if the first magnitude is fixed, dividing it by a smaller magnitude gives the larger ratio.

BOOK V – PROPOSITION 8: For unequal magnitudes, the greater magnitude has a greater ratio than the lesser to the same magnitude; and the same magnitude has a greater ratio to the lesser than to the greater.

1

CHOOSE TWO UNEQUAL MAGNITUDES AND A THIRD MAGNITUDE.

Let A be greater than B . Take any magnitude C .

$A = 5$ units
 $B = 3$ units
 $C = 4$ units

$A > B$
 C is any magnitude.

2

SEPARATE THE EXCESS.

Since $A > B$, write $A = B + E$, where E is the excess.

$A = 5$
 $B = 3$
 $E = 2$

$A = B + E$
 $E = 2$

3

MULTIPLY THE EXCESS UNTIL IT BEATS C .

By Definition 4, there is a multiple of E that exceeds C . Let $m = 3$.

$E = 2$
 $3E = 6$
 $C = 4$

$6 > 4 = C$

4

TAKE THE SAME MULTIPLE OF A AND B .

Now take the same multiple $m = 3$ of A and B . Since $A = B + E$, we have $mA = mB + mE$.

$3A = 15$
 $3B = 9$
 $3E = 6$

$3A = 3B + 6$

5

CHOOSE A MULTIPLE OF C THAT SEPARATES THEM.

Pick n so that $mA > nC$ but $mB \leq nC$. Take $n = 3$.

$3A = 15$
 $3C = 12$
 $3B = 9$

$15 > 12$ but $9 < 12$

Therefore, $A : C > B : C$.

6

REVERSE THE COMPARISON.

Using the same multiples, we get $nC > mB$, but nC is not greater than mA . So $C : B > C : A$.

$3C = 12$
 $3B = 9$
 $3A = 15$

$12 > 9$ but $12 < 15$

7

STATE THE GENERAL RULE.

Therefore, if $A > B$, then $A : C > B : C$ and $C : B > C : A$.

IN GENERAL

If $A > B$, then
 $A : C > B : C$
and
 $C : B > C : A$.

The idea:

- A larger magnitude gives a larger ratio when the comparison magnitude is fixed.
- With the same first magnitude, dividing by the smaller magnitude gives the larger ratio.

BOOK V – PROPOSITION 8

Inequality of magnitudes produces inequality of ratios.

PROPOSITION 9

Magnitudes having the same ratio to the same magnitude are equal to one another; and magnitudes to which the same magnitude has the same ratio are equal.

Proof.

1. Assume

$$A : C = B : C.$$

2. Suppose, for contradiction, that

$$A \neq B.$$

If necessary interchange them, and suppose $A > B$.

3. By Prop. V.8,

$$A : C > B : C.$$

4. This contradicts

$$A : C = B : C.$$

Thus $A > B$ is impossible; similarly $B > A$ is impossible, so

$$A = B.$$

5. Now assume

$$C : A = C : B.$$

6. If $A > B$, then Prop. V.8 gives

$$C : B > C : A,$$

contradicting the assumed equality. Likewise $B > A$ is impossible.

7. Hence

$$A = B.$$

Thus

$$A : C = B : C \implies A = B,$$

and

$$C : A = C : B \implies A = B.$$

Therefore. Equal ratios to or from the same magnitude force equality of the compared magnitudes.

1

ASSUME TWO MAGNITUDES HAVE THE SAME RATIO TO THE SAME MAGNITUDE.

Suppose A and B have the same ratio to the same magnitude C . Can they be different?

Let $C = 4$ units.
Assume $A : C = B : C$.

We do not yet know whether A and B are equal.

2

SUPPOSE, FOR CONTRADICTION, THAT A AND B ARE UNEQUAL.

Assume the opposite of what we want to prove: A and B are not equal.
Suppose $A > B$.

Since A and B are unequal, one must be greater.

3

APPLY PROPOSITION 8.

From Proposition 8, if $A > B$, then with the same C we must have $A : C > B : C$.

Take any multiples.

mA mB nC

$>$

Therefore $A : C > B : C$.

4

REACH THE CONTRADICTION.

But we assumed $A : C = B : C$.
Now we have both $A : C = B : C$ and $A : C > B : C$.

Assumption: $A : C = B : C$
From Proposition 8: $A : C > B : C$

IMPOSSIBLE!

Thus our assumption that $A > B$ is false.
Similarly, $B > A$ is also false. Therefore, $A = B$.

5

REVERSE THE SETUP.

Now suppose $C : A = C : B$.
Can A and B still be different?

Assume $C : A = C : B$.

We again ask whether A and B could be unequal.

6

APPLY PROPOSITION 8 IN THE REVERSED DIRECTION.

From Proposition 8, if $A > B$, then the same C has a greater ratio to B than to A . So we get $C : B > C : A$.

Take any multiples.

nC mA mB

$>$

Therefore $C : B > C : A$.

7

CONCLUDE IN BOTH CASES.

In the first case, assuming A and B unequal led to a contradiction. In the second case, the same contradiction arises.

Therefore:

(1) If $A : C = B : C$, then $A = B$.
(2) If $C : A = C : B$, then $A = B$.

THE GENERAL RULE

Magnitudes having the same ratio to the same magnitude are equal to one another; and magnitudes to which the same magnitude has the same ratio are equal.

BOOK V – PROPOSITION 9:

Magnitudes having the same ratio to the same magnitude are equal.
Magnitudes to which the same magnitude has the same ratio are equal.

PROPOSITION 10

Of magnitudes having a ratio to the same magnitude, the one having the greater ratio is greater; and the magnitude to which the same magnitude has the greater ratio is the lesser.

Proof.

1. Assume

$$A : C > B : C.$$

2. If A were not greater than B , then either

$$A = B$$

or

$$A < B.$$

3. Equality is impossible, because Prop. V.7 would give

$$A : C = B : C.$$

4. Inferiority is impossible, because Prop. V.8 would give

$$A : C < B : C.$$

5. Therefore

$$A > B.$$

6. For the second half assume

$$C : B > C : A.$$

If $B = A$, Prop. V.7 gives equality of ratios. If $B > A$, Prop. V.8 gives

$$C : B < C : A.$$

Both contradict the assumption. Hence

$$B < A.$$

7. Thus

$$A : C > B : C \implies A > B,$$

and

$$C : B > C : A \implies B < A.$$

Therefore. A greater ratio to the same magnitude reveals the greater leading magnitude; with the leading magnitude fixed, a greater ratio reveals the smaller following magnitude.

1

START WITH A GREATER RATIO.

Suppose $A : C > B : C$.
What does that tell us about A and B ?

Let $C = 4$ units.

A has a greater ratio to C than B has to C .

$A : C > B : C$

2

COULD A FAIL TO BE GREATER THAN B?

If A is not greater than B , then either $A = B$ or $A < B$.

The only possibilities:

$A > B$ $A = B$ $A < B$

We will show the last two are impossible.

3

RULE OUT EQUALITY.

If $A = B$, then by Proposition 7, equal magnitudes have the same ratio to the same magnitude.

If $A = B \Rightarrow A : C = B : C$

But we assumed $A : C > B : C$.
Contradiction! Therefore, $A \neq B$

4

RULE OUT $A < B$.

If $A < B$, then by Proposition 8, the lesser has the lesser ratio to the same magnitude.

If $A < B \Rightarrow A : C < B : C$

But we assumed $A : C > B : C$.
Contradiction! Therefore, A is not less than B .

5

CONCLUDE THE FIRST HALF.

A is neither equal to B nor less than B . Therefore it must be greater.

Therefore,
 $A > B$

So, if $A : C > B : C$, then $A > B$.

6

REVERSE THE POSITION.

Now suppose $C : B > C : A$. What does that tell us about B and A ?

Let $C = 4$ units.

$C : B > C : A$

We will show that this means $B < A$.

7

CONCLUDE THE SECOND HALF.

First, if $B \equiv A$, then by Prop. 7,
 $C : B = C : A$

$\times$ Contradicts $C : B > C : A$

Second, if $B > A$, then by Prop. 8 (reversed form),
 $C : B < C : A$

$\times$ Contradicts $C : B > C : A$

Therefore B is neither equal to nor greater than A .

Thus, $B < A$.

THE GENERAL RULE (PROPOSITION 10)

If two magnitudes have a ratio to the same magnitude, the greater ratio belongs to the greater magnitude.

$A : C > B : C \Rightarrow A > B$

◆

If the same magnitude has greater ratio to one magnitude than to another, that other magnitude is the lesser.

$C : B > C : A \Rightarrow B < A$

BOOK V – PROPOSITION 10:

Of magnitudes having a ratio to the same magnitude, the one having the greater ratio is greater; and the magnitude to which the same magnitude has the greater ratio is the lesser.

PROPOSITION 11

Ratios which are the same as the same ratio are also the same as one another.

Proof.

1. Assume

$$A : B = C : D$$

and

$$C : D = E : F.$$

2. Take equal multiples of the leading magnitudes:

$$G = mA, \quad H = mC, \quad K = mE.$$

3. Take equal multiples of the following magnitudes:

$$L = nB, \quad M = nD, \quad N = nF.$$

4. Since

$$A : B = C : D,$$

Definition V.5 gives

$$G \gtrless L \iff H \gtrless M.$$

5. Since

$$C : D = E : F,$$

Definition V.5 gives

$$H \gtrless M \iff K \gtrless N.$$

6. Therefore

$$G \gtrless L \iff K \gtrless N.$$

7. Since m, n were arbitrary, Definition V.5 gives

$$A : B = E : F.$$

Therefore. Ratios equal to the same ratio are equal to one another.

1

START WITH TWO EQUALITIES OF RATIO.

Suppose these two ratios are equal.

$$A : B = C : D$$

$$C : D = E : F$$

Do they also give us $A : B = E : F$?

EXAMPLE. (All three ratios are 2 : 3)

 $A = 2 \quad B = 3 \quad C = 4 \quad D = 6 \quad E = 6 \quad F = 9$

2

TAKE EQUAL MULTIPLES OF THE LEADING MAGNITUDES.

Take any whole number m .
Take equal multiples of A , C , and E .

Let $m = 3$.
 $G = 3A$
 $H = 3C$
 $K = 3E$

G, H, K are equal multiples of A, C, E .

3

TAKE EQUAL MULTIPLES OF THE FOLLOWING MAGNITUDES.

Take any whole number n .
Take equal multiples of B , D , and F .

Let $n = 2$.
 $L = 2B$
 $M = 2D$
 $N = 2F$

We will compare: $G ? L \quad H ? M \quad K ? N$

4

USE THE FIRST EQUALITY OF RATIOS.

Since $A : B = C : D$,
by Definition 5:

 $G > L \Rightarrow H > M$
 $G = L \Rightarrow H = M$
 $G < L \Rightarrow H < M$

Whenever relation holds between G and L , the same holds between H and M .

Bridge 1: the sign passes from the first pair (G, L) to the middle pair (H, M).

5

USE THE SECOND EQUALITY OF RATIOS.

Since $C : D = E : F$,
by Definition 5:

 $H > M \Rightarrow K > N$
 $H = M \Rightarrow K = N$
 $H < M \Rightarrow K < N$

Whenever relation holds between H and M , the same holds between K and N .

Bridge 2: the sign passes from the middle pair (H, M) to the last pair (K, N).

6

PASS STRAIGHT FROM THE FIRST PAIR TO THE LAST.

Combine the two bridges.

 $G > L \Rightarrow H > M \Rightarrow K > N$
 $G = L \Rightarrow H = M \Rightarrow K = N$
 $G < L \Rightarrow H < M \Rightarrow K < N$

The sign survives the whole journey!

So the relation between G and L is the same as between K and N .

7

CONCLUDE BY DEFINITION 5.

Since for arbitrary m and n , the comparisons between mA and nB always agree with those between mE and nF ...

By Definition 5,
 $A : B = E : F$.

Therefore, from $A : B = C : D$ and $C : D = E : F$, we get $A : B = E : F$.

$$\frac{A}{B} = \frac{C}{D} = \frac{C}{D} = \frac{E}{F}$$

$$\frac{A}{B} = \frac{E}{F}$$

BOOK V – PROPOSITION 11: Ratios which are the same as the same ratio are also the same as one another.

PROPOSITION 12

If any number of magnitudes are proportional, then as one of the leading magnitudes is to one of the following magnitudes, so all the leading magnitudes together are to all the following magnitudes together.

Proof.

1. Begin with

$$A : B = C : D = E : F.$$

2. Form

$$G = A + C + E, \quad H = B + D + F.$$

3. Take an arbitrary positive integer m . By Prop. V.1,

$$mG = mA + mC + mE.$$

4. Take another arbitrary positive integer n . Again by Prop. V.1,

$$nH = nB + nD + nF.$$

5. Since all original ratios are equal, Definition V.5 gives the same comparison in every pair:

$$mA \gtrless nB, \quad mC \gtrless nD, \quad mE \gtrless nF.$$

6. Adding the corresponding comparisons yields

$$mG \gtrless nH.$$

Thus the comparison of mG, nH agrees with that of mA, nB .

7. By Definition V.5,

$$A : B = G : H = (A + C + E) : (B + D + F).$$

Therefore. The sum of all leading magnitudes has to the sum of all following magnitudes the same ratio as each corresponding pair.

1 BEGIN WITH SEVERAL EQUAL RATIOS.

Suppose these three pairs of magnitudes are proportional.

A	B
$\frac{2}{3}$	$\frac{4}{6}$
C	D
$\frac{4}{6}$	$\frac{6}{9}$
E	F
$\frac{6}{9}$	$\frac{8}{12}$

Thus $A : B = C : D = E : F = 2 : 3$

What happens if we join all the first magnitudes together and all the second magnitudes together?

2 FORM THE TWO GREAT COMBINED MAGNITUDES.

Join the leading magnitudes: $G = A + C + E$

Join the following magnitudes: $H = B + D + F$

We want to prove $A : B = G : H$, that is, $2 : 3 = 12 : 18$.

3 TAKE AN ARBITRARY MULTIPLE m OF EVERY LEADING MAGNITUDE.

Let m be any whole number. Take the same multiple m of A , C , and E .

By Proposition 1, their sum is m times their sum.

$mA + mC + mE = m(A + C + E) = mG$

4 TAKE AN ARBITRARY MULTIPLE n OF EVERY FOLLOWING MAGNITUDE.

Let n be any whole number. Take the same multiple n of B , D , and F .

By Proposition 1, their sum is n times their sum.

$nB + nD + nF = n(B + D + F) = nH$

So, $nB + nD + nF = n(B + D + F) = nH$.

5 COMPARE EVERY CORRESPONDING PAIR.

Since $A : B = C : D = E : F$, by Definition 5 the comparisons agree.

$mA > nB \Rightarrow mC > nD \Rightarrow mE > nF$
$mA = nB \Rightarrow mC = nD \Rightarrow mE = nF$
$mA < nB \Rightarrow mC < nD \Rightarrow mE < nF$

All three comparisons always point in the same direction.

6 ADD THE CORRESPONDING COMPARISONS.

Add the three comparisons.

$mA > nB, mC > nD, mE > nF \Rightarrow mG > nH$
$mA = nB, mC = nD, mE = nF \Rightarrow mG = nH$
$mA < nB, mC < nD, mE < nF \Rightarrow mG < nH$

Because $mG = mA + mC + mE$ and $nH = nB + nD + nF$, the comparison between mG and nH agrees with that between mA and nB .

7 CONCLUDE BY DEFINITION 5.

For arbitrary m and n , $mA \gtrless nB$ always gives the same comparison as $mG \gtrless nH$. By Definition 5,

$A : B = G : H$

THE GENERAL RULE (Proposition 12)

If any number of magnitudes are proportional, then as one of the leading magnitudes is to one of the following magnitudes, so all the leading magnitudes together are to all the following magnitudes together.

$A_1 : B_1 = A_2 : B_2 = \dots = A_k : B_k$

$\downarrow$

$(A_1 + A_2 + \dots + A_k) : (B_1 + B_2 + \dots + B_k)$

BOOK V – PROPOSITION 12: If any number of magnitudes are proportional, then as one of the leading magnitudes is to one of the following magnitudes, so all the leading magnitudes together are to all the following magnitudes together.

PROPOSITION 13

If a first magnitude has the same ratio to a second that a third has to a fourth, and the third has a greater ratio to the fourth than a fifth has to a sixth, then the first has a greater ratio to the second than the fifth has to the sixth.

Proof.

1. Assume

$$A : B = C : D$$

and

$$C : D > E : F.$$

2. By Definition V.7 choose positive integers m, n such that

$$mC > nD, \quad mE \leq nF.$$

3. Take the corresponding multiples

$$mA, \quad nB.$$

4. Since

$$A : B = C : D,$$

Definition V.5 transfers the strict comparison:

$$mC > nD \implies mA > nB.$$

5. We now have

$$mA > nB$$

while

$$mE \leq nF.$$

6. By Definition V.7,

$$A : B > E : F.$$

7. Hence

$$A : B = C : D, \quad C : D > E : F \implies A : B > E : F.$$

Therefore. Equality of ratio transfers a strict ratio inequality.

1

START WITH AN EQUALITY AND AN INEQUALITY.

Suppose
 $A : B = C : D$
and
 $C : D > E : F$.

We must prove: $A : B > E : F$.

2

USE DEFINITION 7 ON THE GREATER RATIO.

Since $C : D > E : F$, by Definition 7 there exist equal multiples m, n such that
 $mC > nD$ + $mE \leq nF$

Choose m and n so that this happens.

3

TAKE THE CORRESPONDING MULTIPLES OF A AND B.

Because $A : B = C : D$, Definition 5 says corresponding multiples compare the same way.

Take the same m and n .

Whatever holds between mC and nD holds between mA and nB .

4

TRANSFER THE STRICT COMPARISON.

We have $mC > nD$.

Therefore, since $A : B = C : D$,
 $mA > nB$.

Also from step 2, $mE \leq nF$.

5

COMPARE THE FIRST AND LAST RATIOS.

We have found equal multiples m, n for which

So the multiples of the first ratio exceed, while the multiples of the last ratio do not.

6

APPLY DEFINITION 7.

Definition 7: If for some equal multiples m, n , we have $mX > nY$ but $mU \leq nV$, then $X : Y > U : V$.

Here, $mA > nB$ but $mE \leq nF$.

Therefore,
 $A : B > E : F$.

The first ratio is greater than the last.

7

STATE THE GENERAL RULE.

If
 $A : B = C : D$
and
 $C : D > E : F$,
then
 $A : B > E : F$.

BOOK V – PROPOSITION 13

If the first magnitude has the same ratio to the second that the third has to the fourth, and the third has a greater ratio to the fourth than a fifth has to a sixth, then the first has a greater ratio to the second than the fifth has to the sixth.

Equality of ratio can carry inequality through to another ratio.

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PROPOSITION 14

If a first magnitude has the same ratio to a second that a third has to a fourth, then the first and third have the same ordering as the second and fourth.

Proof.

1. Assume

$$A : B = C : D.$$

2. Take the first case:

$$A > C.$$

3. By Prop. V.8,

$$A : B > C : B.$$

4. Since

$$A : B = C : D,$$

replace $A : B$ by $C : D$:

$$C : D > C : B.$$

5. By Prop. V.10,

$$D < B,$$

so

$$B > D.$$

6. Similarly,

$$A = C \implies B = D,$$

and

$$A < C \implies B < D.$$

7. Therefore

$$A \gtrless C \iff B \gtrless D.$$

Therefore. Equal ratios preserve the ordering of corresponding magnitudes.

1

START WITH EQUAL RATIOS.

Suppose $A : B = C : D$.

These two pairs of magnitudes have the same ratio.

Example: $A = 6, B = 4, C = 3, D = 2$
Then $6 : 4 = 3 : 2$

2

SUPPOSE THE FIRST MAGNITUDE IS GREATER.

Suppose $A > C$.
We will prove $B > D$.

In our example, $6 > 3$.

3

COMPARE A AND C TO THE SAME MAGNITUDE B.

Since $A > C$, by Proposition 8, the greater magnitude has the greater ratio to the same magnitude.

Therefore, $A : B > C : B$.

Both ratios have the same second magnitude B.

4

REPLACE A : B BY THE EQUAL RATIO C : D.

But $A : B = C : D$.
So from $A : B > C : B$ we get $C : D > C : B$.

The same magnitude C stands first in both ratios.

5

USE PROPOSITION 10.

By Proposition 10, if the same magnitude has a greater ratio to one magnitude than to another, then the magnitude in the greater ratio's second position is the smaller.

Therefore, $B > D$.

6

THE OTHER TWO CASES ARE SIMILAR.

If $A = C$,

From $A : B = C : D$, we get $B = D$.

If $A < C$,

From $A : B = C : D$, we get $B < D$.

(From step 5)
If $A > C$,

Then $B > D$.

The three possibilities correspond in the same way.

7

THE GENERAL RULE.

If $A : B = C : D$, then

$A > C \Rightarrow B > D$
 $A = C \Rightarrow B = D$
 $A < C \Rightarrow B < D$

Equal ratios preserve order: what is greater, equal, or less among the first magnitudes is the same among the corresponding second magnitudes.

BOOK V – PROPOSITION 14

If the first magnitude has the same ratio to the second that the third has to the fourth, and the first is greater (or equal to, or less than) the third, then the second is greater (or equal to, or less than) the fourth.

$$A : B = C : D \Rightarrow A \leq C \Leftrightarrow B \geq D$$

50

PROPOSITION 15

Parts have the same ratio as similar multiples, taken in corresponding order.

Proof.

1. Choose two basic magnitudes C and F .
2. Build equal multiples:

$$AB = mC, \quad DE = mF.$$

3. Divide the first into m equal parts each equal to C , and the second into m equal parts each equal to F . For example,

$$AG = GH = HB = C,$$

$$DK = KL = LE = F.$$

4. By Prop. V.7,

$$AG : DK = GH : KL = HB : LE.$$

5. By Prop. V.12,

$$AG : DK = (AG + GH + HB) : (DK + KL + LE).$$

6. Since

$$AG = C, \quad DK = F, \quad AG + GH + HB = AB, \quad DK + KL + LE = DE,$$

we obtain

$$C : F = AB : DE.$$

7. Hence

$$C : F = mC : mF.$$

Therefore. Similar multiples preserve ratio:

$$\alpha : \beta = m\alpha : m\beta.$$

1

CHOOSE TWO BASIC MAGNITUDES.

Let the magnitudes be C and F .

C F

Example: $C = 2$ units $F = 3$ units

We will see that their ratio is unchanged if we take equal multiples of both.

2

BUILD EQUAL MULTIPLES OF BOTH.

Take the same multiplier, say $m = 3$.

$AB = 3C$ $DE = 3F$

6 units 9 units

$AB = C + C + C$ and $DE = F + F + F$

3

DIVIDE THE LARGER MAGNITUDES INTO EQUAL PARTS.

Divide AB into three equal parts.

Each upper part equals C .

Divide DE into three equal parts.

Each lower part equals F .

4

COMPARE EACH CORRESPONDING PAIR.

All the upper parts are equal to one another.

All the lower parts are equal to one another.

A G H B

D K L E

$C : F$ $C : F$ $C : F$

By Proposition 7, each pair has the same ratio $C : F$.

5

COMBINE ALL LEADING PARTS AND ALL FOLLOWING PARTS.

By Proposition 12,

$AG : DK = (AG + GH + HB) : (DK + KL + LE)$

$\downarrow$

$AG + GH + HB = AB$ $DK + KL + LE = DE$

Therefore, $AG : DK = AB : DE$.

6

REPLACE THE SMALL PARTS BY THE ORIGINAL MAGNITUDES.

$AG = C$ $DK = F$

So,

$C : F = AB : DE$

Example: $2 : 3 = 6 : 9$

Tripling both magnitudes changes their size, not their ratio.

7

STATE THE GENERAL RULE.

If $AB = mC$ and $DE = nF$ that is, if AB and DE are equal multiples of C and F , then

$C : F = AB : DE$.

In general,

$\alpha : \beta :: m\alpha : m\beta$

Parts have the same ratio as similar multiples, taken in corresponding order.



BOOK V – PROPOSITION 15: Parts have the same ratio as similar multiples, taken in corresponding order.

PROPOSITION 16

If four magnitudes are proportional, then they will also be proportional alternately.

Proof.

1. Assume

$$A : B = C : D.$$

We seek to prove

$$A : C = B : D.$$

2. Take equal multiples of A and B :

$$E = mA, \quad F = mB.$$

By Prop. V.15,

$$A : B = E : F.$$

3. Since

$$A : B = C : D,$$

Prop. V.11 gives

$$C : D = E : F.$$

4. Take equal multiples of C and D :

$$G = nC, \quad H = nD.$$

By Prop. V.15,

$$C : D = G : H.$$

Hence, by Prop. V.11,

$$E : F = G : H.$$

5. By Prop. V.14,

$$E \gtrless G \iff F \gtrless H.$$

6. Since

$$E = mA, \quad G = nC, \quad F = mB, \quad H = nD,$$

for arbitrary m, n ,

$$mA \gtrless nC \iff mB \gtrless nD.$$

7. By Definition V.5,

$$A : C = B : D.$$

Therefore. If four magnitudes are proportional, they are proportional alternately:

$$A : B = C : D \implies A : C = B : D.$$

1 BEGIN WITH FOUR PROPORTIONAL MAGNITUDES.

Suppose A is to B as C is to D .

$A : B = C : D$

Can we cross them and say A is to C as B is to D ?

2 TAKE EQUAL MULTIPLES OF A AND B.

Take the same multiplier m .

$E = mA$ $F = mB$

By Proposition 15, because E and F are equal multiples of A and B , we have $A : B = E : F$.

3 TRANSFER THAT RATIO TO C AND D.

Since $A : B = C : D$, and $A : B = E : F$, by Proposition 11, $C : D = E : F$.

So the ratio of C to D is also the ratio of E to F .

4 TAKE EQUAL MULTIPLES OF C AND D.

Take another multiplier n .

$G = nC$ $H = nD$

By Proposition 15, $C : D = G : H$. But $C : D = E : F$, so by Proposition 11, $E : F = G : H$.

5 COMPARE THE NEW LEADING MULTIPLES.

From Proposition 14, corresponding terms of equal ratios have the same order.

Since $E : F = G : H$,

$E > G \Rightarrow F > H$

$E = G \Rightarrow F = H$

$E < G \Rightarrow F < H$

Thus whatever comparison holds between E and G holds between F and H .

6 TRANSLATE BACK TO A, C, B, D.

Remember:

$E = mA$
 $G = nC$
 $F = mB$
 $H = nD$

Hence for arbitrary m, n ,

$mA > nC \Leftrightarrow mB > nD$
 $mA = nC \Leftrightarrow mB = nD$
 $mA < nC \Leftrightarrow mB < nD$

The same comparison holds for the pairs (A, C) and for the pairs (B, D) .

7 CONCLUDE BY DEFINITION 5.

By Definition 5, if for arbitrary m, n , we have $mA = nC$ (always in the same category $>, =$ or $<$), and the same holds for mB and nD , then $A : C = B : D$.

Hence from $A : B = C : D$ we may pass alternately to $A : C = B : D$.

GENERAL RULE (Proposition 16)

If four magnitudes are proportional, then they will also be proportional alternately.

$A : B = C : D \Rightarrow A : C = B : D$

WE MAY CROSS THE COMPARISON IN CORRESPONDING ORDER.

BOOK V – PROPOSITION 16: If four magnitudes are proportional, then they will also be proportional alternately.

PROPOSITION 17

If composed magnitudes are proportional, then they will also be proportional when separated.

Proof.

1. Assume

$$AB : BE = CD : DF,$$

where

$$AB = AE + EB, \quad CD = CF + FD.$$

We seek to prove

$$AE : EB = CF : FD.$$

2. Take equal multiples

$$GH = mAE, \quad HK = mEB, \quad LM = mCF, \quad MN = mFD,$$

and other arbitrary equal multiples

$$KO = nEB, \quad NP = nFD.$$

3. Rebuild equal multiples of the wholes:

$$GK = GH + HK = mAB,$$

$$LN = LM + MN = mCD.$$

4. Build corresponding equal multiples of the following parts:

$$HO = HK + KO, \quad MP = MN + NP.$$

5. Since

$$AB : BE = CD : DF,$$

Definition V.5 gives

$$GK \gtrless HO \iff LN \gtrless MP.$$

6. Subtract the common parts HK and MN :

$$GH \gtrless KO \iff LM \gtrless NP.$$

7. Since

$$GH = mAE, \quad KO = nEB, \quad LM = mCF, \quad NP = nFD,$$

Definition V.5 yields

$$AE : EB = CF : FD.$$

Therefore. If composed magnitudes are proportional, the separated magnitudes are proportional:

$$(\alpha + \beta) : \beta = (\gamma + \delta) : \delta \implies \alpha : \beta = \gamma : \delta.$$

1

START WITH TWO COMPOSED RATIOS.

Suppose the whole to the part stands the same way in both pairs.

$AB : BE = CD : DF$

$A \xrightarrow{AE} E \xrightarrow{EB} B$
 $C \xrightarrow{CF} F \xrightarrow{FD} D$

We will prove: $AE : EB = CF : FD$

2

TAKE EQUAL MULTIPLES OF THE SEPARATED PARTS.

Take arbitrary equal multiples.

From AE $G \xrightarrow{H} K$
From EB $L \xrightarrow{M} N$
From CF $O \xrightarrow{P} Q$

Also take other equal multiples of EB and FD .

$K \xrightarrow{R} S$ ($= n \cdot EB$)
 $N \xrightarrow{T} U$ ($= n \cdot FD$)

(G, H from AE) (H, K from EB)
(L, M from CF) (M, N from FD)

3

REBUILD EQUAL MULTIPLES OF THE WHOLE.

Since $AB = AE + EB$, by Prop. 1,
 $GK = GH + HK = m \cdot AB$

$G \xrightarrow{H} K \xrightarrow{R} S$

Since $CD = CF + FD$, by Prop. 1,
 $LN = LM + MN = m \cdot CD$

$L \xrightarrow{M} N \xrightarrow{T} U$

Thus GK and LN are equal multiples of the whole magnitudes AB and CD .

4

BUILD EQUAL MULTIPLES OF THE FOLLOWING PARTS.

Combine the multiples of the parts.

$H \xrightarrow{K} S \xrightarrow{R} T$ ($= m \cdot EB + n \cdot EB$)
 HO

$M \xrightarrow{N} U \xrightarrow{T} V$ ($= m \cdot FD + n \cdot FD$)
 MP

So HO and MP are equal multiples of the parts EB and FD .

5

USE THE ORIGINAL PROPORTIONALITY.

Since $AB : BE = CD : DF$, the same comparison holds for their equal multiples.

If $GK > HO$ then $LN > MP$
If $GK = HO$ then $LN = MP$
If $GK < HO$ then $LN < MP$

(Whole multiples compared to part multiples.)

6

SUBTRACT THE COMMON PARTS.

From $GK = GH + HK$ and $HO = HK + KO$

$G \xrightarrow{H} K \xrightarrow{R} S$
 $H \xrightarrow{K} S \xrightarrow{R} T$

Subtract the common HK .
So $GK \gtrless HO \Rightarrow GH \gtrless KO$

From $LN = LM + MN$ and $MP = MN + NP$

$L \xrightarrow{M} N \xrightarrow{T} U$
 $M \xrightarrow{N} U \xrightarrow{T} V$

Subtract the common MN .
So $LN \gtrless MP \Rightarrow LM \gtrless NP$

Therefore, $GH \gtrless KO$ exactly when $LM \gtrless NP$.

7

CONCLUDE BY DEFINITION 5.

Remember:
 $GH = m \cdot AE$
 $KO = n \cdot EB$
 $LM = m \cdot CF$
 $NP = n \cdot FD$

For arbitrary m, n ,
 $mAE \gtrless nEB$
iff
 $mCF \gtrless nFD$

By Definition 5,
 $AE : EB = CF : FD$

BOOK V – PROPOSITION 17
(Separation or Dividendo)

If composed magnitudes are proportional, then they will also be proportional when separated.

$AB : BE = CD : DF$
 $\Downarrow$
 $AE : EB = CF : FD$

Remove equal corresponding parts from proportional compositions.

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PROPOSITION 18

If separated magnitudes are proportional, then they will also be proportional when composed.

Proof.

1. Assume

$$AE : EB = CF : FD,$$

where

$$AB = AE + EB, \quad CD = CF + FD.$$

We seek to prove

$$AB : BE = CD : DF.$$

2. Suppose, for contradiction, that

$$AB : BE \neq CD : DF.$$

Then CD would have to stand to some magnitude DG , smaller or greater than DF , in the same ratio as AB stands to BE . Take first

$$DG < DF$$

and suppose

$$AB : BE = CD : DG.$$

3. By Prop. V.17, separating this hypothetical proportion gives

$$AE : EB = CG : GD.$$

4. But the original assumption gives

$$AE : EB = CF : FD.$$

Therefore, by Prop. V.11,

$$CG : GD = CF : FD.$$

5. Since

$$DG < DF,$$

the complementary magnitude satisfies

$$CG > CF.$$

By Prop. V.14, the equality

$$CG : GD = CF : FD$$

would force

$$GD > FD,$$

contradicting $GD < FD$.

6. The same reasoning rules out the alternative in which the hypothetical fourth magnitude is greater than DF .
7. Therefore the only possible fourth magnitude is DF , and

$$AB : BE = CD : DF.$$

Therefore. If separated magnitudes are proportional, they remain proportional when composed:

$$\alpha : \beta = \gamma : \delta \implies (\alpha + \beta) : \beta = (\gamma + \delta) : \delta.$$

1

START WITH THE SEPARATED RATIOS.

Suppose the separated parts are proportional.

$$AE : EB = CF : FD$$

We will prove $AB : BE = CD : DF$.

2

SUPPOSE THE COMPOSED RATIOS ARE NOT EQUAL.

Assume, to the contrary, that $AB : BE \neq CD : DF$.

Then CD has the same ratio to BE as AB has to some magnitude less than DF .

Let that magnitude be DG , where $DG < DF$, and suppose $AB : BE = CD : DG$.

3

APPLY PROPOSITION 17 (SEPARATION).

From $AB : BE = CD : DG$, by Proposition 17, $AE : EB = CG : GD$.

So the new separated ratio is $CG : GD$.

4

COMPARE WITH THE GIVEN SEPARATED RATIO.

But we also have $AE : EB = CF : FD$. Therefore, by Proposition 11, $CG : GD = CF : FD$.

So the two ratios are equal.

5

USE PROPOSITION 14.

Since $CG : GD = CF : FD$ and $CG > CF$ (because $DG < DF$), by Proposition 14, we must have $GD > FD$.

But $GD > FD$ contradicts $GD < FD$.

6

RULE OUT THE OTHER POSSIBILITY.

If we had supposed CD to have the same ratio to BE as AB has to some magnitude greater than DF , the same reasoning leads to a contradiction.

Therefore CD cannot be proportional to $AB : BE$ with a denominator less than DF or greater than DF .

7

CONCLUDE THE PROPORTIONALITY.

The only magnitude that works is DF itself. Therefore, $AB : BE = CD : DF$.

Joining each part to its companion preserves the proportion.

BOOK V – PROPOSITION 18
(Composition or Componendo)

GENERAL RULE (Proposition 18)

If separated magnitudes are proportional, then the corresponding composed magnitudes are proportional.

$$AE : EB = CF : FD$$

$$\downarrow$$

$$AB : BE = CD : DF$$

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PROPOSITION 19

If as the whole is to the whole, so the part taken away is to the part taken away, then the remainder is to the remainder as the whole is to the whole.

Proof.

1. Assume

$$AB : CD = AE : CF,$$

with

$$AB = AE + EB, \quad CD = CF + FD.$$

2. By Prop. V.16, alternando,

$$AB : AE = CD : CF.$$

3. Since these are composed magnitudes, Prop. V.17 gives

$$EB : AE = FD : CF.$$

4. Alternating again,

$$EB : FD = AE : CF.$$

5. Recall the original proportion:

$$AE : CF = AB : CD.$$

6. By Prop. V.11,

$$EB : FD = AB : CD.$$

Thus

$$AB : CD = AE : CF = EB : FD.$$

7. Hence the remainder to the remainder is as the whole to the whole.

Therefore.

$$AB : CD = AE : CF \implies EB : FD = AB : CD.$$

1

START WITH TWO WHOLE
AND PROPORTIONAL REMOVED PARTS.

The wholes have the same ratio as the parts taken away.
What then of what remains?

Assume $AB : CD = AE : CF$.

2

ALTERNATE THE GIVEN PROPORTION
(Proposition 16).

From $AB : CD = AE : CF$,
by alternation:
 $AB : AE = CD : CF$.

The whole-to-removed-part ratios are equal.

3

SEPARATE THE COMPOSED MAGNITUDES
(Proposition 17).

Since $AB = AE + EB$ and $CD = CF + FD$,
the ratios $AB : AE$ and $CD : CF$ are composed.
By separation:
 $EB : AE = FD : CF$.

The ratios of the remainders to the removed parts are equal.

4

ALTERNATE AGAIN (Proposition 16).

From $EB : AE = FD : CF$,
by alternation:
 $EB : FD = AE : CF$.

The ratios of the remainders are equal to the ratios of the removed parts.

5

RECALL THE ORIGINAL PROPORTION.

We assumed $AE : CF = AB : CD$.
Therefore, by Proposition 11,
 $EB : FD = AB : CD$.

The remainders have the same ratio as the wholes.

6

SEE THE STRUCTURE.

Whole to whole
 $AB : CD$
↓
Removed to removed
 $AE : CF$
↓
Remainder to remainder
 $EB : FD$

The same ratio is preserved at all three levels.

7

STATE THE GENERAL RULE.

If the whole is to the whole as the part taken away is to the part taken away,
then the remainder is to the remainder as the whole is to the whole.

$AB : CD = AE : CF$
↓
 $EB : FD = AB : CD$

Modern notation:

$\alpha : \beta :: \gamma : \delta$
↓
 $(\alpha - \gamma) : (\beta - \delta) :: \alpha : \beta$

Equal proportions remain after equal corresponding parts are removed.

BOOK V – PROPOSITION 19:

If as the whole is to the whole, so the part taken away is to the part taken away,
then the remainder is to the remainder as the whole is to the whole.

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PROPOSITION 20

If there are three magnitudes and another equal number, taken two by two in the same ratio, then the comparison between the first and third of one set is the same as the comparison between the first and third of the other.

Proof.

1. Take the parallel chains

$$A : B = D : E, \quad B : C = E : F.$$

2. Suppose first that

$$A > C.$$

By Prop. V.8,

$$A : B > C : B.$$

3. Since

$$A : B = D : E,$$

replace $A : B$ by $D : E$:

$$D : E > C : B.$$

4. From

$$B : C = E : F,$$

invert by the corollary to Prop. V.7:

$$C : B = F : E.$$

Thus

$$D : E > F : E.$$

5. By Prop. V.10,

$$D > F.$$

6. The same reasoning gives

$$A = C \implies D = F,$$

and

$$A < C \implies D < F.$$

7. Therefore

$$A \gtrless C \iff D \gtrless F.$$

Therefore. Parallel proportional chains preserve the comparison of their extremes.

1 BUILD TWO PARALLEL CHAINS.
Take three magnitudes A, B, C and another three D, E, F .

Assume
 $A : B = D : E$
 $B : C = E : F$

These two chains change in exactly the same proportional way.

2 SUPPOSE THE FIRST EXCEEDS THE THIRD.
Suppose $A > C$.
Take B as the common comparison.
By Proposition 8,
 $A : B > C : B$

$A > C \Rightarrow A : B > C : B$

3 TRANSFER $A : B$ INTO THE SECOND CHAIN.
We know $A : B = D : E$.
Replace $A : B$ by $D : E$.
 $D : E > C : B$

$D : E > C : B$

4 REVERSE THE SECOND GIVEN PROPORTION.
We know $B : C = E : F$.
By the inverse ratio (Prop. 7),
 $C : B = F : E$.
Replace $C : B$ by $F : E$.
 $D : E > F : E$

Both ratios now have the same following magnitude E .

5 USE PROPOSITION 10.
From $D : E > F : E$, when two magnitudes have ratios to the same magnitude E , the one having the greater ratio is itself greater.
Therefore, $D > F$.

6 HANDLE THE OTHER TWO CASES.

If $A = C$, then
 $A : B = C : B$
 $\Rightarrow D : E = F : E$
 $\Rightarrow D = F$

If $A < C$, then
 $A : B < C : B$
 $\Rightarrow D : E < F : E$
 $\Rightarrow D < F$

Thus:
 $A > C \Rightarrow D > F$
 $A = C \Rightarrow D = F$
 $A < C \Rightarrow D < F$

Each relation between the first and third of the first chain matches the relation between the first and third of the second chain.

7 STATE THE GENERAL RULE.

Whenever
 $A : B = D : E$
and
 $B : C = E : F$

then
 $A : C \Leftrightarrow D : F$

SAME ORDER ALWAYS.

**BOOK V
PROPOSITION 20**

If there are three magnitudes and another equal number of magnitudes, taken two by two in the same ratio, then whatever relation holds between the first and third of the one chain holds between the first and third of the other.

$A : B = D : E$
 $B : C = E : F$

$\Rightarrow A : C \Leftrightarrow D : F$

PROPOSITION 21

If there are three magnitudes and another equal number, taken two by two in the same ratio, but the proportion is perturbed, then the comparison between the first and third of one set is the same as the comparison between the first and third of the other.

Proof.

1. Take the perturbed arrangement

$$A : B = E : F, \quad B : C = D : E.$$

2. Suppose first that

$$A > C.$$

By Prop. V.8,

$$A : B > C : B.$$

3. Since

$$A : B = E : F,$$

replace $A : B$ by $E : F$:

$$E : F > C : B.$$

4. From

$$B : C = D : E,$$

invert:

$$C : B = E : D.$$

Thus

$$E : F > E : D.$$

5. By Prop. V.10,

$$F < D,$$

so

$$D > F.$$

6. Similarly,

$$A = C \implies D = F,$$

and

$$A < C \implies D < F.$$

7. Therefore

$$A \gtrless C \iff D \gtrless F.$$

Therefore. Even in a perturbed proportion, the comparison of the extremes is preserved.

1

BUILD THE PERTURBED CHAINS.

Take three magnitudes A, B, C and another three D, E, F , with the ratios crossed:

$$A:B = E:F$$

$$B:C = D:E$$

The chains are crossed (perturbed).

2

SUPPOSE THE FIRST EXCEEDS THE THIRD.

Suppose $A > C$.
Take B as the common comparison.
By Proposition 8,

$$A:B > C:B$$

$A > C \Rightarrow A:B > C:B$

3

TRANSFER USING THE FIRST GIVEN RATIO.

We know $A:B = E:F$.
Replace $A:B$ by $E:F$.

$$E:F > C:B$$

4

REVERSE THE SECOND GIVEN RATIO.

We know $B:C = D:E$.
By the inverse ratio (Prop. 7),
 $C:B = E:D$.
Replace $C:B$ by $E:D$.

$$E:F > E:D$$

Now both ratios have the same first magnitude E .

5

APPLY PROPOSITION 10.

From $E:F > E:D$,
when two ratios have the same first magnitude, the greater ratio is the one with the smaller second magnitude (Prop. 10).

$$F < D$$

$$D > F$$

Thus $A > C \Rightarrow D > F$.

6

HANDLE THE OTHER TWO CASES.

If $A = C$,
then
 $A:B = C:B$
 $\downarrow$
 $E:F = E:D$
 $\downarrow$
 $D = F$

If $A < C$,
then
 $A:B < C:B$
 $\downarrow$
 $E:F < E:D$
 $\downarrow$
 $F > D$

Summary:
 $A > C \Rightarrow D > F$
 $A = C \Rightarrow D = F$
 $A < C \Rightarrow D < F$

In every case, the order is preserved.

7

STATE THE GENERAL RULE.

Whenever
 $A:B = E:F$
and
 $B:C = D:E$,
then
 $A ? C \Leftrightarrow D ? F$

Even when the chains are crossed, the first-to-last comparison survives.

BOOK V – PROPOSITION 21

"Perturbed Proportions."

If there are three magnitudes and another equal number of magnitudes, taken two by two in the same ratio, but their proportion is perturbed, then whatever relation holds between the first and third of the first set also holds between the first and third of the second set.

Modern notation:

$$\left. \begin{array}{l} \alpha : \beta :: \epsilon : \zeta \\ \beta : \gamma :: \delta : \epsilon \end{array} \right\} \Rightarrow \alpha : \gamma \Leftrightarrow \delta : \zeta$$

"The proportional chains may be crossed, but truth keeps the order."
— Aristotle

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PROPOSITION 22

If any number of magnitudes and another equal number of magnitudes are in the same ratio taken two by two, then they are also in the same ratio via equality.

Proof.

1. Begin with

$$A : B = D : E, \quad B : C = E : F.$$

We seek to prove

$$A : C = D : F.$$

2. Take arbitrary equal multiples of the first terms:

$$G = mA, \quad H = mD.$$

3. Take arbitrary equal multiples of the middle terms:

$$K = nB, \quad L = nE.$$

By Prop. V.4,

$$G : K = H : L.$$

4. Take arbitrary equal multiples of the last terms:

$$M = pC, \quad N = pF.$$

Again by Prop. V.4,

$$K : M = L : N.$$

5. Apply Prop. V.20 to the triples G, K, M and H, L, N :

$$G \gtrless M \iff H \gtrless N.$$

6. Since

$$G = mA, \quad M = pC, \quad H = mD, \quad N = pF,$$

for arbitrary m, p ,

$$mA \gtrless pC \iff mD \gtrless pF.$$

7. By Definition V.5,

$$A : C = D : F.$$

Therefore. The middle terms may be removed via equality:

$$A : B = D : E, \quad B : C = E : F \implies A : C = D : F.$$

PROPOSITION 23

If there are three magnitudes and another equal number, taken two by two in the same ratio, but their proportion is perturbed, then they are also in the same ratio via equality.

Proof.

1. Begin with

$$A : B = E : F, \quad B : C = D : E.$$

We seek to prove

$$A : C = D : F.$$

2. Take equal multiples of A, B, D :

$$G = mA, \quad H = mB, \quad K = mD.$$

3. Take equal multiples of C, E, F :

$$L = nC, \quad M = nE, \quad N = nF.$$

4. From

$$A : B = E : F,$$

and Prop. V.15,

$$A : B = G : H, \quad E : F = M : N.$$

By Prop. V.11,

$$G : H = M : N.$$

5. From

$$B : C = D : E,$$

Prop. V.16 gives

$$B : D = C : E.$$

Using Prop. V.15,

$$B : D = H : K, \quad C : E = L : M.$$

Hence

$$H : K = L : M,$$

and alternando,

$$H : L = K : M.$$

6. The triples G, H, L and K, M, N now form the perturbed arrangement of Prop. V.21. Therefore

$$G \gtrless L \iff K \gtrless N.$$

7. Since

$$G = mA, \quad L = nC, \quad K = mD, \quad N = nF,$$

Definition V.5 gives

$$A : C = D : F.$$

Therefore. Ex aequali remains valid in perturbed proportion:

$$A : B = E : F, \quad B : C = D : E \implies A : C = D : F.$$

1 BUILD THE PERTURBED CHAINS.

Take three magnitudes A, B, C and another three D, E, F with the crossed ratios:

$$A : B = E : F$$

$$B : C = D : E$$

The chains are crossed (perturbed).
We want to prove: $A : C = D : F$.

2 TAKE EQUAL MULTIPLES OF A, B, D .

Take any positive integer m .
Form equal multiples:

$$G = mA$$

$$H = mB$$

$$K = mD$$

G, H, K are equal multiples of A, B, D .

3 TAKE EQUAL MULTIPLES OF C, E, F .

Take any positive integer n .
Form equal multiples:

$$L = nC$$

$$M = nE$$

$$N = nF$$

Now we have two triples:
 G, H, L and K, M, N

4 TRANSFER THE FIRST GIVEN RATIO.

Since $A : B = E : F$,
by Prop. 15 we have
 $A : B = G : H$
and
 $E : F = M : N$.
Therefore, by Prop. 11,
 $G : H = M : N$.

The first crossed proportion holds among the multiples.

5 TRANSFER THE SECOND GIVEN RATIO.

From $B : C = D : E$, by Prop. 16,
 $B : D = C : E$.
By Prop. 15:
 $B : D = H : K$
and
 $C : E = L : M$.
Thus, by Prop. 11,
 $H : K = L : M$.
By Prop. 16 (alternando),
 $H : L = K : M$.

The second crossed proportion also holds among the multiples.

6 APPLY PROPOSITION 21 TO THE MULTIPLES.

We have the perturbed arrangement:

$$G : H = M : N$$

$$H : L = K : M$$

By Prop. 21:

If $G > L$ then $K > N$
If $G = L$ then $K = N$
If $G < L$ then $K < N$

The comparison of the extremes of the first triple matches that of the second triple.

7 TRANSLATE BACK AND CONCLUDE.

Recall:
 $G = mA, L = nC$
 $K = mD, N = nF$
So for any positive integers m, n :
 $mA \gtrless nC \Leftrightarrow mD \gtrless nF$.
By Definition 5 (equality of ratios):
 $A : C = D : F$

Thus the extremes are in the same ratio via equality.

BOOK V – PROPOSITION 23

"Ex Aequali for Perturbed Proportion"

If there are three magnitudes and another equal number of magnitudes, taken two by two in the same ratio, but with the proportion perturbed, then they are also in the same ratio via equality.

$$A : B = E : F$$

$$B : C = D : E$$

$$\Rightarrow A : C = D : F$$

The middle terms may be removed, but truth keeps the proportion.

PROPOSITION 24

If a first magnitude has to a second the same ratio that a third has to a fourth, and a fifth also has to the second the same ratio that a sixth has to the fourth, then the sum of the first and fifth has to the second the same ratio that the sum of the third and sixth has to the fourth.

Proof.

1. Assume

$$A : B = C : D$$

and

$$E : B = F : D.$$

We seek to prove

$$(A + E) : B = (C + F) : D.$$

2. Invert the second proportion:

$$B : E = D : F.$$

3. Combine

$$A : B = C : D$$

with

$$B : E = D : F.$$

By Prop. V.22,

$$A : E = C : F.$$

4. Apply Prop. V.18, composition:

$$(A + E) : E = (C + F) : F.$$

5. Recall

$$E : B = F : D.$$

6. Apply Prop. V.22 again:

$$(A + E) : B = (C + F) : D.$$

7. Thus

$$A : B = C : D, \quad E : B = F : D$$

implies

$$(A + E) : B = (C + F) : D.$$

Therefore. Corresponding proportional magnitudes may be added while preserving the common ratio.

1

START WITH TWO PROPORTIONS.

Let six magnitudes be given:

$$A : B = C : D$$

$$E : B = F : D$$

A and E are both compared with B, C and F with D.

We aim to prove: $(A + E) : B = (C + F) : D$

2

INVERT THE SECOND PROPOSITION.

From $E : B = F : D$,
by the inverse ratio (Prop. 7):
 $B : E = D : F$

We now have: $B : E = D : F$

3

CHAIN THE RATIOS VIA EQUALITY.

We have $A : B = C : D$
and $B : E = D : F$.
By Proposition 22 (ex aequali):
 $A : E = C : F$

So the parts to be added are in the same ratio.

4

COMPOSE THOSE RATIOS.

From $A : E = C : F$,
by Composition (Prop. 18):
 $(A + E) : E = (C + F) : F$

The sums stand to the added parts in the same ratio.

5

REINTRODUCE THE ORIGINAL COMPARISON.

We still know
 $E : B = F : D$.

So we have two proportional links:
 $(A + E) : E = (C + F) : F$ and $E : B = F : D$

6

APPLY EX AEQUALI AGAIN.

From $(A + E) : E = (C + F) : F$
and $E : B = F : D$,
by Proposition 22 (ex aequali):
 $(A + E) : B = (C + F) : D$

The middle terms are removed.
The extremes remain in the same ratio.

7

STATE THE GENERAL RULE.

Therefore:
If $A : B = C : D$ and $E : B = F : D$,
then
 $(A + E) : B = (C + F) : D$

Adding corresponding magnitudes preserves the proportional ratio.

BOOK V

PROPOSITION 24

"Addition of Proportional Magnitudes"

If a first magnitude has to a second the same ratio that a third has to a fourth, and a fifth also has to the second the same ratio that a sixth has to the fourth, then the sum of the first and fifth has to the second the same ratio that the sum of the third and sixth has to the fourth.

$(A + E) : B = (C + F) : D$

"When we add corresponding parts of proportional magnitudes, the ratio is preserved."
— Aristotle

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PROPOSITION 25

If four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two magnitudes.

Proof.

1. Let

$$AB : CD = E : F,$$

with AB the greatest and F the least. We seek to prove

$$AB + F > CD + E.$$

2. On AB , mark AG equal to E :

$$AG = E.$$

On CD , mark CH equal to F :

$$CH = F.$$

Hence

$$AB : CD = AG : CH.$$

3. Remove the proportional parts. The remainders are GB and HD .
By Prop. V.19,

$$GB : HD = AB : CD.$$

4. Since

$$AB > CD,$$

Prop. V.14 gives

$$GB > HD.$$

5. Because

$$AG = E, \quad CH = F,$$

we have

$$AG + F = CH + E.$$

6. Add these equal quantities to the unequal remainders:

$$GB + (AG + F) > HD + (CH + E).$$

Since

$$GB + AG = AB, \quad HD + CH = CD,$$

we obtain

$$AB + F > CD + E.$$

7. Therefore, for proportional magnitudes

$$A : B = C : D$$

with A greatest and D least,

$$A + D > B + C.$$

Therefore. Among four proportional magnitudes, the greatest plus the least exceeds the sum of the remaining two.

1

START WITH FOUR PROPORTIONAL MAGNITUDES.

Let four magnitudes be proportional:
 $AB : CD = E : F$.
Let AB be the greatest and F the least.

We want to prove:
 $AB + F > CD + E$

2

COPY THE THIRD AND FOURTH MAGNITUDES INSIDE THE FIRST TWO.

On AB mark AG equal to E .
On CD mark CH equal to F .

Since $E = AG$ and $F = CH$,
we have $AB : CD = AG : CH$.

3

REMOVE THOSE PROPORTIONAL PARTS (PROPOSITION 19).

Remove AG from AB leaving GB .
Remove CH from CD leaving HD .

By Proposition 19:
 $GB : HD = AB : CD$

4

USE THAT AB IS THE GREATEST.

Since AB is the greatest,
 $AB > CD$.
But $GB : HD = AB : CD$.
By Proposition 14 (same order):
 $GB > HD$

So the remainder GB is greater than the remainder HD .

5

BUILD TWO EQUAL PACKAGES.

Since $AG = E$ and $CH = F$,
 $AG + F = E + F$

Therefore,
 $AG + F = CH + E$.

6

ADD EQUALS TO THE UNEQUAL REMAINDERS.

We know $GB > HD$.
Add the equal amount $(AG + F)$ to GB .

Add the equal amount $(CH + E)$ to HD .

Adding equal magnitudes to unequal magnitudes preserves the inequality.

$GB + (AG + F) > HD + (CH + E)$
But $GB + AG = AB$ and $HD + CH = CD$.
Hence,
 $AB + F > CD + E$.

7

STATE THE GENERAL RULE.

Therefore, if four magnitudes are proportional and A is the greatest and D the least, in $A : B = C : D$, then
 $A + D > B + C$.

The sum of the greatest and the least is greater than the sum of the remaining two magnitudes.

BOOK V
PROPOSITION 25
"Inequality in Four Proportional Magnitudes"

If four magnitudes are proportional,
then the sum of the greatest and the least
is greater than the sum of the remaining
two magnitudes.

If $A : B = C : D$ and A is greatest, D is least,
 $A + D > B + C$

"The extremes prevail:
add the greatest to the least,
they outweigh the middle pair."
— Aristotle

CONCLUSION

Book V establishes a general theory of ratio and proportion applicable to magnitudes whether commensurable or incommensurable. Its central principle is Definition V.5: equality of ratios is tested by the behavior of arbitrary equal multiples. From this foundation Euclid develops the arithmetic of multiples, equality and inequality of ratios, alternation, separation, composition, proportional remainders, ordinary and perturbed chains, and finally the rules of *ex aequali* and proportional addition.

Part and multiple $\longrightarrow$ ratio $\longrightarrow$ proportion $\longrightarrow$ <i>ex aequali</i>
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