

EUCLID'S ELEMENTS

BOOK XIII

The Five Regular Solids

Propositions 1–18

*The Golden Section, Regular Polygons,
and the Construction of the Five Regular Solids*

A classical illustrated edition

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Book XIII

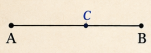
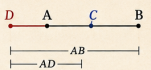
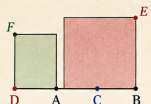
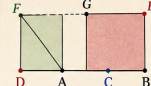
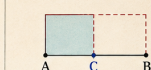
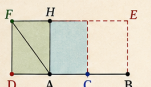
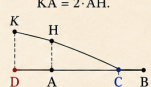
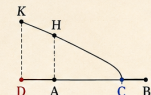
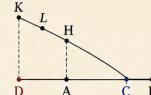
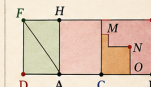
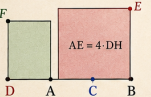
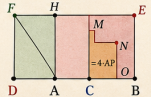
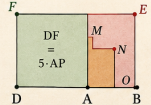
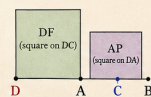
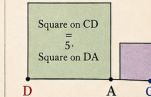
PROPOSITION 1

If a straight-line is cut in extreme and mean ratio, then the square on the greater piece, added to half of the whole, is five times the square on the half.

Proof.

1. Let AB be cut at C in extreme and mean ratio, with AC the greater piece.
2. Extend AD in a straight line with CA , making AD equal to half of AB .
3. Construct squares AE and DF on AB and DC respectively.
4. Draw the figure in DF and extend FC to G .
5. Since AB is cut in extreme and mean ratio at C , rectangle ABC equals the square on AC .
6. CE is rectangle ABC , and FH is the square on AC ; hence $CE = FH$.
7. Since $BA = 2 AD$, and $BA = KA$, $AD = AH$; therefore $KA = 2 AH$.
8. As $KA : AH = CK : CH$, it follows that $CK = 2 CH$.
9. Since $LH + HC = 2 CH$, we have $KC = LH + HC$.
10. CE was shown equal to HF , so square AE equals gnomon MNO .
11. Since $BA = 2 AD$, the square on BA is four times the square on AD , so $AE = 4 DH$.
12. Since AE equals gnomon MNO , gnomon $MNO = 4 AP$.
13. Therefore the whole $DF = 5 AP$.
14. DF is the square on DC , and AP is the square on DA .
15. Therefore the square on CD is five times the square on DA .

Therefore, the square on CD is five times the square on DA .

1 Let line AB be cut at C in extreme and mean ratio, with AC the greater piece. 	2 Extend AD in a straight line with CA, making AD equal to half of AB. 	3 Construct squares AE and DF on AB and DC respectively. 	4 Draw the figure in DF and extend FC to G. 	5 Since AB is cut in extreme and mean ratio at C, rectangle ABC equals square on AC. 
6 CE is rectangle ABC, FH is square on AC, so CE = FH. 	7 Since BA = 2 · AD, and BA = KA, AD = AH, then KA = 2 · AH. 	8 As KA:AH = CK:CH, so CK = 2 · CH. 	9 LH + HC = 2 · CH, so KC = LH + HC. 	10 CE was shown equal to HF, so square AE equals gnomon MNO. 
11 Since BA = 2 · AD, square BA = 4 · square AD, so AE = 4 · DH. 	12 AE = gnomon MNO, so gnomon MNO = 4 · AP. 	13 Therefore whole DF = 5 · AP. 	14 DF is square on DC, AP is square on DA. 	15 Therefore square on CD = 5 · square on DA. 

Proposition 1. *If a straight-line is cut in extreme and mean ratio, then the square on the greater piece, added to half of the whole, is five times the square on the half.*

PROPOSITION 2

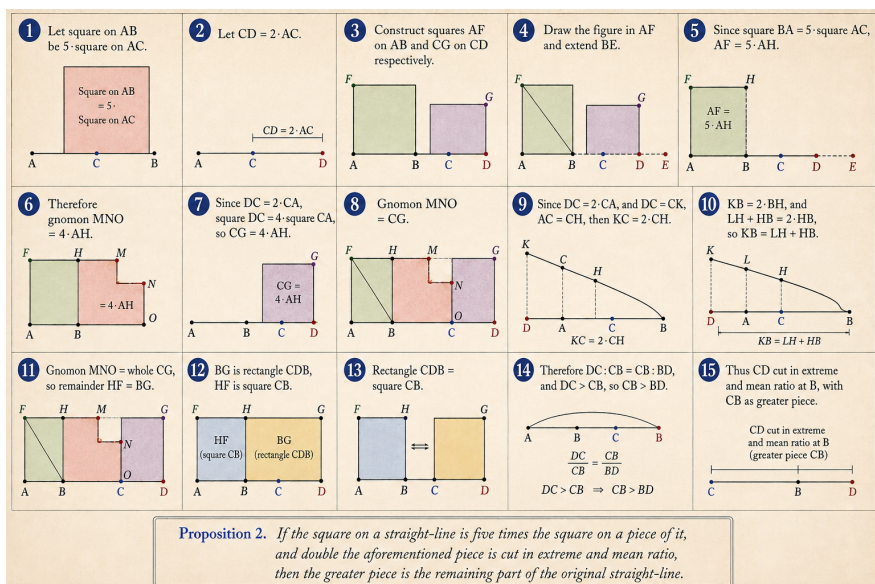
If the square on a straight-line is five times the square on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.

Proof.

1. Let the square on AB be five times the square on AC .
2. Let $CD = 2 AC$.
3. Construct squares AF and CG on AB and CD respectively.
4. Draw the figure in AF and extend BE .
5. Since the square on BA is five times the square on AC , $AF = 5 AH$.
6. Therefore gnomon $MNO = 4 AH$.
7. Since $DC = 2 CA$, the square on DC is four times the square on CA ; therefore $CG = 4 AH$.
8. Hence gnomon $MNO = CG$.
9. Since $DC = 2 CA$, and $DC = CK$, while $AC = CH$, it follows that $KC = 2 CH$.
10. $KB = 2 BH$, and $LH + HB = 2 HB$, so $KB = LH + HB$.
11. Since gnomon MNO equals the whole CG , the remainder $HF = BG$.
12. BG is rectangle CDB , while HF is the square on CB .
13. Therefore rectangle CDB equals the square on CB .
14. Hence

$$DC : CB = CB : BD,$$
 and since $DC > CB$, we have $CB > BD$.
15. Thus CD is cut in extreme and mean ratio at B , with CB as the greater piece.

Therefore, CD is cut in extreme and mean ratio at B , and CB is the greater piece.



Proposition 2. *If the square on a straight-line is five times the square on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.*

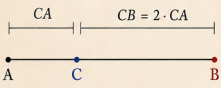
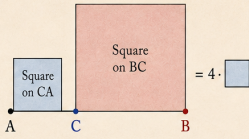
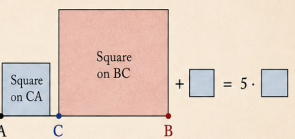
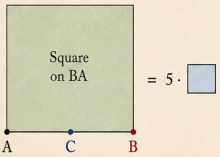
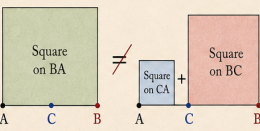
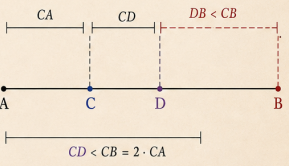
Lemma to Proposition 2

Double AC is greater than BC .

Proof.

1. Assume the contrary: $BC = 2 CA$.
2. Then the square on BC is four times the square on CA .
3. Therefore the squares on BC and CA together equal five times the square on CA .
4. But by hypothesis the square on BA is five times the square on CA .
5. Therefore the square on BA would equal the sum of the squares on BC and CA , which is impossible by Prop. II.4.
6. Similarly, no straight-line less than CB can be double AC .

Therefore, double AC is greater than BC .

1 Assume contrary: $BC = 2 \cdot CA$ 	2 Then square $BC = 4 \cdot \text{square } CA$. 	3 So squares on $BC + CA = 5 \cdot \text{square } CA$. 
4 But square $BA = 5 \cdot \text{square } CA$ by hypothesis. 	5 Therefore square $BA = \text{squares on } BC + CA$, impossible (by Prop. 2.4).  Impossible (by Proposition 2.4): the whole is equal to the sum of the parts.	6 Similarly, no line less than CB can be double AC. 

Lemma. Shows that double $AC > BC$.

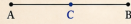
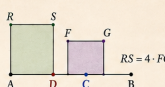
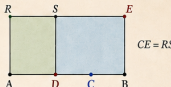
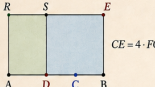
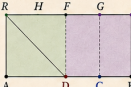
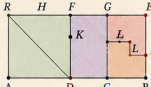
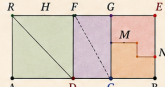
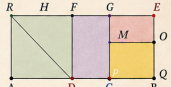
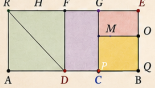
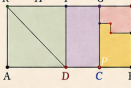
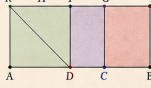
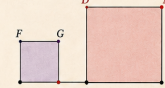
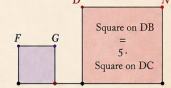
PROPOSITION 3

If a straight-line is cut in extreme and mean ratio, then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.

Proof.

1. Let AB be cut at C in extreme and mean ratio, with AC the greater piece.
2. Cut AC in half at D .
3. Construct square AE on AB and draw the figure double.
4. Since $AC = 2 DC$, the square on AC is four times the square on DC , so $RS = 4 FG$.
5. Rectangle ABC equals the square on AC , so $CE = RS$.
6. Since $RS = 4 FG$, therefore $CE = 4 FG$.
7. Since $AD = DC$, $HK = KF$.
8. The square GF equals the square HL , so $GK = KL$, and hence $MN = NE$.
9. Therefore $MF = FE$, and $MF = CG$; hence $CG = FE$.
10. Add CN to both: gnomon $OPQ = CE$.
11. Since $CE = 4 FG$, gnomon $OPQ = 4 FG$.
12. Gnomon OPQ together with square FG is therefore $5 FG$.
13. But gnomon OPQ together with square FG is square DN .
14. DN is the square on DB , while GF is the square on DC .
15. Therefore the square on DB is five times the square on DC .

Therefore, the square on DB is five times the square on DC .

1 Let AB be cut at C in extreme and mean ratio, with AC the greater piece. 	2 Cut AC in half at D. 	3 Construct square AE on AB and draw the figure double. 	4 Since $AC = 2 \cdot DC$, square $AC = 4 \cdot \text{square } DC$, so $RS = 4 \cdot FG$. 	5 Rectangle ABC = square AC, so $CE = RS$. 
6 $RS = 4 \cdot FG$, therefore $CE = 4 \cdot FG$. 	7 Since $AD = DC$, $HK = KF$. 	8 Square $GF = \text{square } HL$, so $GK = KL$, hence $MN = NE$. 	9 Therefore $MF = FE$, and $MF = CG$, so $CG = FE$. 	10 Add CN to both; gnomon OPQ = CE. 
11 $CE = 4 \cdot FG$, so gnomon $OPQ = 4 \cdot FG$. 	12 Gnomon OPQ + square $FG = 5 \cdot FG$. 	13 But gnomon OPQ + square $FG = \text{square } DN$. 	14 DN is square on DB, GF is square on DC. 	15 Therefore square on DB = $5 \cdot \text{square on DC}$. 

Proposition 3. If a straight-line is cut in extreme and mean ratio, then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.

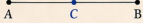
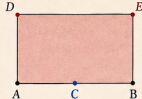
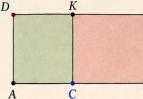
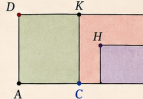
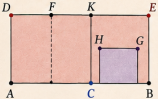
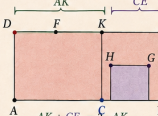
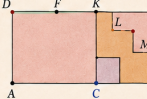
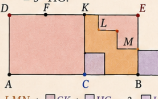
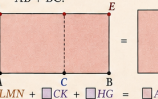
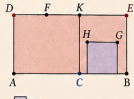
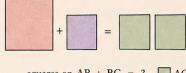
PROPOSITION 4

If a straight-line is cut in extreme and mean ratio, then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.

Proof.

1. Let AB be cut at C in extreme and mean ratio, with AC the greater piece.
2. Construct square $ADEB$ on AB and draw the figure.
3. Rectangle ABC equals the square on AC , since AB is cut in extreme and mean ratio.
4. AK is rectangle ABC , and HG is the square on AC ; therefore $AK = HG$.
5. $AF = FE$ by Prop. I.43. Add CK to both; hence $AK = CE$.
6. Therefore $AK + CE = 2 AK$.
7. But $AK + CE$ equals gnomon LMN together with square CK .
8. Therefore gnomon LMN together with square CK equals $2 AK$.
9. Since $AK = HG$, gnomon LMN together with the squares CK and HG equals $3 HG$.
10. Gnomon LMN together with the squares CK and HG equals square AE together with square CK , that is, the squares on AB and BC .
11. GH is the square on AC .
12. Therefore the squares on AB and BC together equal three times the square on AC .

Therefore, the squares on AB and BC together equal three times the square on AC .

1 Let AB be cut at C in extreme and mean ratio, with AC greater. 	2 Construct square ADEB on AB and draw the figure. 	3 Rectangle ABC = square AC (since AB cut in extreme ratio). 	4 AK is rectangle ABC, HG is square AC, so AK = HG. 
5 $AF = FE$ (Prop. 1.43), add CK to both, so $AK = CE$. 	6 $AK + CE = 2 \cdot AK$. 	7 $AK + CE =$ gnomon LMN + square CK. 	8 Therefore gnomon LMN + square CK = $2 \cdot AK$. 
9 $AK = HG$, so gnomon LMN + squares CK + HG = $3 \cdot HG$.  $LMN + \square CK + \square HG = 3 \cdot \square HG$	10 Gnomon LMN + squares CK + HG = square AE + CK = squares on AB + BC.  $LMN + \square CK + \square HG = \square AE + \square CK$ = squares on AB + BC	11 GH is square on AC.  $\square HG$ is square on AC	12 Therefore squares on AB + BC = $3 \cdot$ square on AC.  squares on AB + BC = $3 \cdot \square AC$

Proposition 4. If a straight-line is cut in extreme and mean ratio, then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.

PROPOSITION 5

If a straight-line is cut in extreme and mean ratio, and a straight-line equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.

Proof.

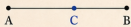
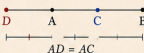
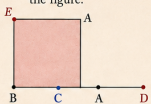
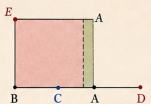
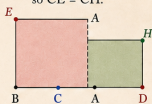
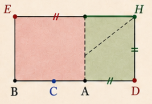
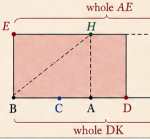
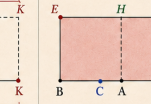
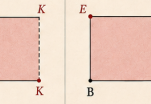
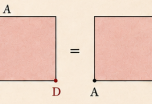
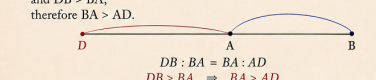
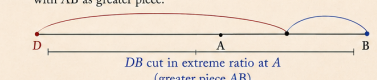
1. Let AB be cut at C in extreme and mean ratio, with AC the greater piece.
2. Make $AD = AC$.
3. Construct square AE on AB and draw the figure.
4. Rectangle ABC equals the square on AC .
5. CE is rectangle ABC , while CH is the square on AC ; therefore $CE = CH$.
6. $HE = CE$ by Prop. I.43, and $DH = HC$; hence $DH = HE$.
7. Add HB to both: the whole DK equals the whole AE .
8. DK is rectangle $BD \cdot DA$, since $AD = DL$, while AE is the square on AB .
9. Therefore rectangle BDA equals the square on AB .
10. Hence

$$DB : BA = BA : AD.$$

Since $DB > BA$, it follows that $BA > AD$.

11. Thus DB is cut in extreme and mean ratio at A , with AB as the greater piece.

Therefore, the whole DB is cut in extreme and mean ratio at A , and the original line AB is the greater piece.

1 Let AB be cut at C in extreme ratio, with AC greater. 	2 Make AD = AC. 	3 Construct square AE on AB and draw the figure. 	4 Rectangle ABC = square AC. 	5 CE is rectangle ABC, CH is square AC, so CE = CH. 
6 HE = CE (Prop. 1.43), DH = HC, so DH = HE. 	7 Add HB to both: whole DK = whole AE. 	8 DK is rectangle BD · DA (since AD = DL), AE is square AB. 	9 Therefore rectangle BDA = square AB. 	10 So DB : BA = BA : AD, and DB > BA, therefore BA > AD. 
11 Thus DB cut in extreme ratio at A, with AB as greater piece. 	12 Thus DB cut in extreme ratio at A, with AB as greater piece. 			

Proposition 5. If a straight-line is cut in extreme and mean ratio, and a straight-line equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.

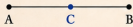
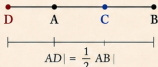
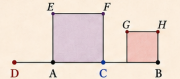
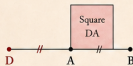
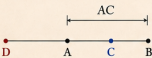
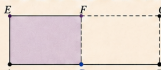
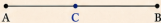
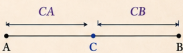
PROPOSITION 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is the irrational straight-line called an apotome.

Proof.

1. Let AB be rational, cut at C in extreme and mean ratio, with AC the greater piece.
2. Extend BA and make AD equal to half of BA .
3. Since AB is cut in extreme and mean ratio and AD , equal to half of AB , is added to AC , the square on CD is five times the square on DA by Prop. XIII.1.
4. Thus the square on CD is to the square on DA in a ratio of number to number.
5. Therefore the square on CD is commensurable with the square on DA .
6. The square on DA is rational, because DA is half of rational AB .
7. Therefore the square on CD is rational, and so CD is rational.
8. Since the ratio of the square on CD to the square on DA is not a square-number ratio, CD is incommensurable in length with DA .
9. Thus CD and DA are rational straight-lines commensurable in square only.
10. Therefore AC is an apotome by Prop. X.73.
11. Since rectangle ABC equals the square on the apotome AC , the square on apotome AC , applied to rational AB , makes BC as width.
12. Therefore CB is a first apotome by Prop. X.97.
13. Hence CA and CB are both apotomes.

Therefore, each piece CA and CB is an irrational straight-line called an apotome.

1 Let AB be rational, cut at C in extreme ratio, with AC greater.  AB is rational.	2 Extend BA and make AD = half of BA.  $AD = \frac{1}{2} AB$	3 Since AB cut in extreme ratio and AD = half of AB added to AC, square CD = 5 · square DA (Prop. 13.1).  Square CD = 5 · square DA	4 Square CD : square DA is a ratio of number to number.  Ratio of number to number.	5 Therefore square CD is commensurable with square DA.  Commensurable.
6 Square DA is rational (DA is half of rational AB).  Rational.	7 Therefore square CD is rational, so CD is rational.  CD is rational.	8 Since square CD : square DA is not a square number ratio, CD is incommensurable in length with DA.  CD is incommensurable in length with DA	9 Thus CD and DA are rational and commensurable in square only. CD, DA rational in length Square CD ~ Square DA (commensurable in square only)	
10 Therefore AC is an apotome (Prop. 10.73).  AC is an apotome.	11 Since rectangle ABC = square AC, the square on apotome AC applied to rational AB makes BC as width.  Rectangle ABC = square AC.	12 Therefore CB is a first apotome (Prop. 10.97).  CB is a first apotome	13 CA and CB are both apotomes.  Both CA and CB are apotomes.	

Proposition 6. *If a rational straight-line is cut in extreme and mean ratio then each of the pieces is the irrational straight-line called an apotome.*

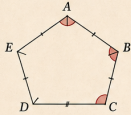
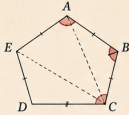
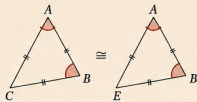
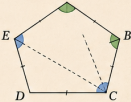
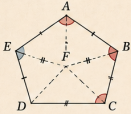
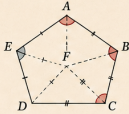
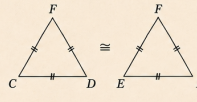
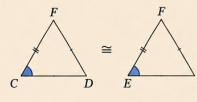
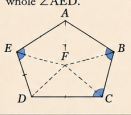
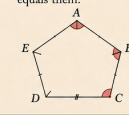
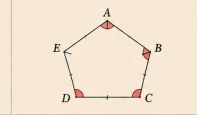
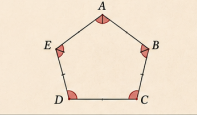
PROPOSITION 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal, then the pentagon will be equiangular.

Proof.

Consecutive case: the angles at A, B, C are equal.

1. Let pentagon $ABCDE$ be equilateral with the angles at A, B, C equal.
2. Join AC, BE , and FD .
3. Since $CB = BA = AE$, and $\angle CBA = \angle BAE$, triangles ABC and ABE are congruent by side-angle-side.
4. Therefore $AC = BE$, $\angle BCA = \angle BEA$, and $\angle ABE = \angle CAB$.
5. Thus $AF = BF$ by Prop. I.6.
6. Since $AC = BE$, it follows that $FC = FE$.
7. Since $CD = DE$, triangles FCD and FED are congruent by side-side-side.
8. Therefore $\angle FCD = \angle FED$.
9. Since $\angle BCA = \angle AEB$, the whole $\angle BCD$ equals the whole $\angle AED$.
10. But $\angle BCD$ was assumed equal to the angles at A and B ; therefore $\angle AED$ equals them.
11. Similarly, $\angle CDE$ equals them.
12. Therefore the pentagon is equiangular.

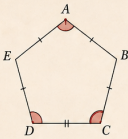
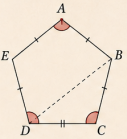
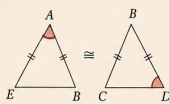
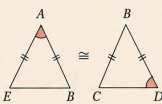
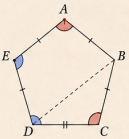
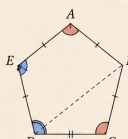
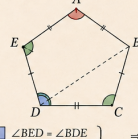
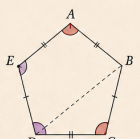
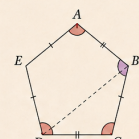
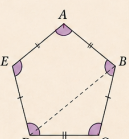
1 Let pentagon ABCDE be equilateral with angles at A, B, C equal. 	2 Join AC, BE, FD. 	3 Since $CB = BA = AE$, and $\angle CBA = \angle BAE$, triangles ABC and ABE are congruent (SAS). 	4 Therefore $AC = BE$, $\angle BCA = \angle BEA$, $\angle ABE = \angle CAB$. 
5 Thus $AF = BF$ (Prop. 1.6). 	6 $AC = BE$, so $FC = FE$. 	7 $CD = DE$, so triangles FCD and FED are congruent (SSS). 	8 Therefore $\angle FCD = \angle FED$. 
9 $\angle BCA = \angle AEB$, so whole $\angle BCD =$ whole $\angle AED$. 	10 $\angle BCD$ was assumed equal to angles at A and B, so $\angle AED$ equals them. 	11 Similarly, $\angle CDE$ equals them. 	12 Therefore pentagon is equiangular. 

Proposition 7. *If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.*

Non-consecutive case: the angles at A, C, D are equal.

1. Let the angles at A, C, D be equal.
2. Join BD .
3. $BA = BC, AE = CD$, and the contained angles are equal.
4. Therefore triangles ABE and BCD are congruent by side-angle-side.
5. Hence $BE = BD$ and $\angle AEB = \angle CDB$.
6. Since $BE = BD$, $\angle BED = \angle BDE$ by Prop. I.5.
7. Therefore the whole $\angle AED$ equals the whole $\angle CDE$.
8. Since $\angle CDE$ was assumed equal to the angles at A and C , $\angle AED$ equals them.
9. Similarly, $\angle ABC$ equals them.
10. Therefore the pentagon is equiangular.

Therefore, in either case, an equilateral pentagon with three equal angles is equiangular.

1 Let angles at A, C, D be equal. 	2 Join BD. 	3 $BA = BC$, $AE = CD$, and contained angles equal. 	4 Triangles ABE and BCD are congruent (SAS). 	5 Therefore $BE = BD$, $\angle AEB = \angle CDB$. 
6 Since $BE = BD$, $\angle BED = \angle BDE$ (Prop. 1.5). 	7 Therefore whole $\angle AED =$ whole $\angle CDE$.  $\angle BED = \angle BDE$ $\angle AEB = \angle CDB$ (from step 5) } $\Rightarrow$ whole angles equal	8 $\angle CDE$ assumed equal to angles at A and C, so $\angle AED$ equals them. 	9 Similarly, $\angle ABC$ equals them. 	10 Therefore pentagon is equiangular. 

Proposition 7 (Non-consecutive case). *If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.*

PROPOSITION 8

If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.

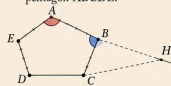
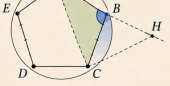
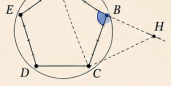
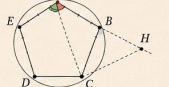
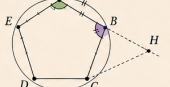
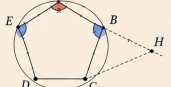
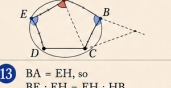
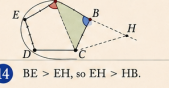
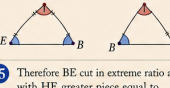
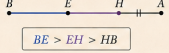
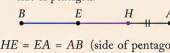
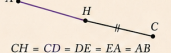
Proof.

1. Let AC and BE cut at H , subtending consecutive angles at A and B of pentagon $ABCDE$.
2. Circumscribe circle $ABCDE$ about the pentagon.
3. Triangles ABE and ABC are congruent by side-angle-side.
4. Therefore $\angle BAC = \angle ABE$.
5. $\angle AHE = 2 \angle BAH$ by Prop. I.32.
6. $\angle EAC = 2 \angle BAC$, since arc EDC is twice arc CB .
7. Therefore $\angle HAE = \angle AHE$, so $HE = EA = AB$.
8. Since $BA = AE$, $\angle ABE = \angle AEB$.
9. Since $\angle ABE = \angle BAH$, it follows that $\angle BEA = \angle BAH$.
10. $\angle ABE$ is common to triangles ABE and ABH .
11. Therefore triangle ABE is equiangular to triangle ABH .
12. Thus

$$EB : BA = AB : BH.$$
13. Since $BA = EH$,

$$BE : EH = EH : HB.$$
14. Since $BE > EH$, it follows that $EH > HB$.
15. Therefore BE is cut in extreme and mean ratio at H , with HE the greater piece equal to a side of the pentagon.
16. Similarly AC is cut in extreme and mean ratio at H , with CH equal to a side of the pentagon.

Therefore, the two subtending straight-lines cut one another in extreme and mean ratio, and their greater pieces equal the sides of the pentagon.

1 Let AC and BE cut at H, subtending consecutive angles at A and B of pentagon ABCDE. 	2 Circumscribe circle ABCDE about pentagon. 	3 Triangles ABE and ABC are congruent (SAS). 	4 Therefore $\angle BAC = \angle ABE$. 
5 $\angle AHE = 2 \cdot \angle BAH$ (Prop. 1.32). 	6 $\angle EAC = 2 \cdot \angle BAC$ (since arc EDC = 2 · arc CB). 	7 Therefore $\angle HAE = \angle AHE$, so $HE = EA = AB$. 	8 Since $BA = AE$, $\angle ABE = \angle AEB$. 
9 $\angle ABE = \angle BAH$, so $\angle BEA = \angle BAH$. 	10 $\angle ABE$ is common to triangles ABE and ABH. 	11 Therefore triangle ABE is equiangular to triangle ABH. 	12 Thus $EB : BA = AB : BH$. $\frac{EB}{BA} = \frac{AB}{BH}$
13 $BA = EH$, so $BE : EH = EH : HB$. $\frac{BE}{EH} = \frac{EH}{HB}$	14 $BE > EH$, so $EH > HB$. 	15 Therefore BE cut in extreme ratio at H, with HE greater piece equal to side of pentagon.  $HE = EA = AB$ (side of pentagon)	16 Similarly, AC cut in extreme ratio at H, with CH equal to side.  $CH = CD = DE = EA = AB$

Proposition 8. *If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.*

PROPOSITION 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together, the whole straight-line has been cut in extreme and mean ratio, and the greater piece is the side of the hexagon.

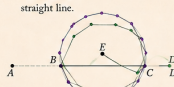
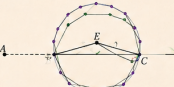
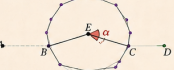
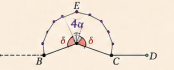
Proof.

1. Let BC be the side of a decagon and CD the side of a hexagon in circle ABC , placed on one straight line.
2. Find the center E ; join EB, EC, ED , and extend BE to A .
3. Since BC is the side of a decagon, arc ACB is five times arc BC .
4. Therefore arc AC is four times arc CB .
5. Thus $\angle AEC = 4 \angle ECB$.
6. Since $EB = EC$, $\angle EBC = \angle ECB$; therefore $\angle AEC = 2 \angle ECB$.
7. Since $EC = CD$, both being hexagon sides, $\angle CED = \angle CDE$.
8. Therefore $\angle ECB = 2 \angle EDC$.
9. Hence $\angle AEC = 2 \angle ECB = 4 \angle EDC$.
10. But $\angle AEC = 4 \angle BEC$; therefore $\angle EDC = \angle BEC$.
11. $\angle EBD$ is common to triangles BEC and BED .
12. Therefore triangle EBD is equiangular to triangle EBC .
13. Thus

$$DB : BE = EB : BC.$$
14. Since $EB = CD$,

$$BD : DC = DC : CB.$$
15. Since $BD > DC$, it follows that $DC > CB$.
16. Therefore BD is cut in extreme and mean ratio at C , with DC the greater piece.

Therefore, the sum of the hexagon and decagon sides is cut in extreme and mean ratio, and the greater piece is the hexagon side.

1 Let BC be side of decagon, CD side of hexagon in circle ABC, on a straight line.  BC = side of decagon CD = side of hexagon	2 Find center E, join EB, EC, ED, extend BE to A. 	3 Since BC is side of decagon, arc ACB = 5 · arc BC.  $x = \text{arc BC}$ $\text{arc ACB} = 5x$	4 Therefore arc AC = 4 · arc CB.  $\text{arc AC} = 4x$ $\text{arc CB} = x$
5 Thus $\angle AEC = 4 \cdot \angle ECB$. 	6 Since EB = EC, $\angle EBC = \angle ECB$, so $\angle AEC = 2 \cdot \angle ECB$. 	7 EC = CD (both hexagon sides), so $\angle CED = \angle CDE$. 	8 Therefore $\angle ECB = 2 \cdot \angle EDC$, (since $\angle CED = \angle CDE = \gamma$) 
9 $\angle AEC$ was shown = $2 \cdot \angle ECB = 4 \cdot \angle EDC$, (from steps 6 and 8) 	10 $\angle AEC = 4 \cdot \angle BEC$, so $\angle EDC = \angle BEC$. 	11 $\angle EBD$ is common to triangles BEC and BED. 	12 Therefore triangle EBD is equiangular to triangle EBC. 
13 Thus DB : BE = EB : BC. $\frac{DB}{BE} = \frac{EB}{BC}$	14 EB = CD, so BD : DC = DC : CB. $\frac{BD}{DC} = \frac{DC}{CB}$	15 BD > DC, so DC > CB.  $BD > DC > CB$	16 Therefore BD cut in extreme ratio at C, with DC as greater piece.  $DC \text{ is greater piece}$

Proposition 9. If the side of a hexagon and of a decagon inscribed in the same circle are added together, the whole straight-line has been cut in extreme and mean ratio, and the greater piece is the side of the hexagon.

PROPOSITION 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is equal to the sum of the squares on the sides of the hexagon and of the decagon inscribed in the same circle.

Proof.

1. Inscribe equilateral pentagon $ABCDE$ in circle $ABCDE$ with center F .
2. Join AF , extend it to G ; join FB ; draw $FH \perp AB$, and extend to K .
3. Join AK, KB ; draw $FL \perp AK$, extend to M ; join KN .
4. Since arc $ABCG$ equals arc $AEDG$, and arc ABC equals arc AED , arc CG equals arc GD .
5. CD is a side of the pentagon, so CG is a side of the decagon.
6. Since $FA = FB$ and $FH \perp AB$, $\angle AFK = \angle KFB$.
7. Therefore arc AK equals arc KB , so AK is a side of the decagon.
8. Similarly arc AK is twice arc KM .
9. Since arc AB is twice arc BK , and arc $CD = AB$, arc CD is twice arc BK .
10. Arc CD is twice arc CG ; therefore arc $CG = BK$.
11. $BK = 2 KM$, so $CG = 2 KM$.
12. Arc CB is twice arc BK , so arc GB is twice arc BM .
13. $\angle GFB = 2 \angle BFM$, and $\angle GFB = 2 \angle FAB$.
14. Therefore $\angle BFN = \angle FAB$; also $\angle ABF$ is common to triangles ABF and BFN .
15. Hence triangles ABF and BFN are equiangular.
16. Thus

$$AB : BF = FB : BN,$$

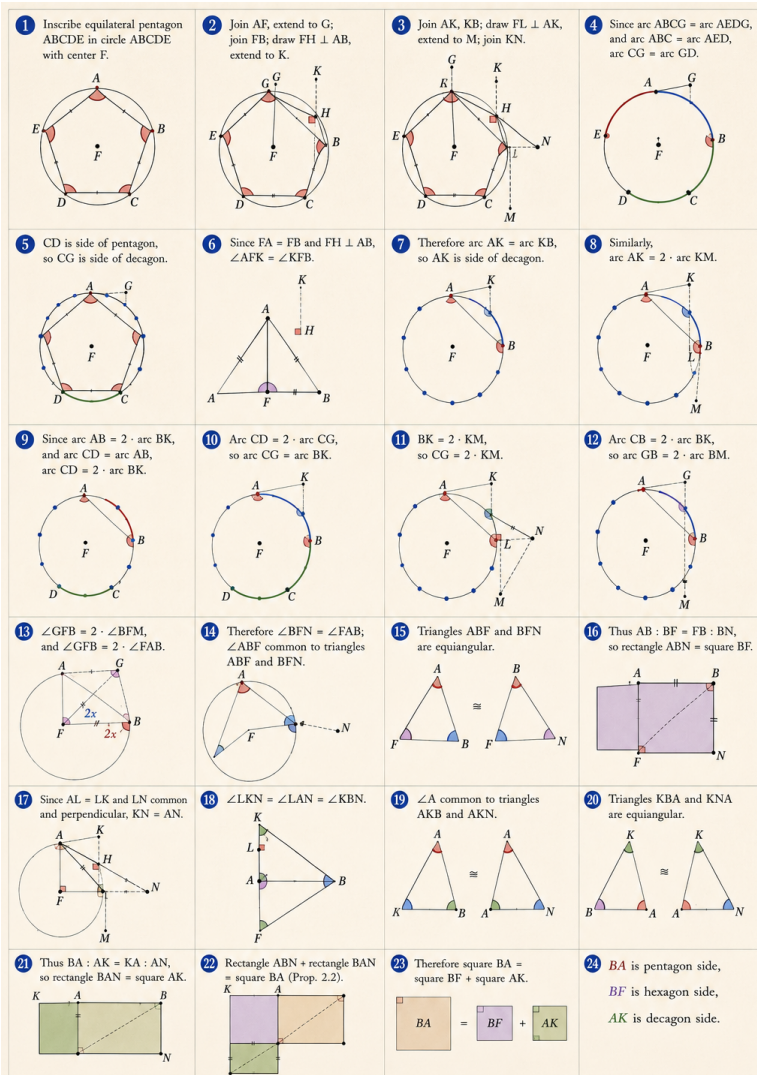
so rectangle ABN equals the square on BF .

17. Since $AL = LK$ and LN is common and perpendicular, $KN = AN$.
18. $\angle LKN = \angle LAN = \angle KBN$.
19. The angle at A is common to triangles AKB and AKN .
20. Therefore triangles KBA and KNA are equiangular.
21. Thus

$$BA : AK = KA : AN,$$
 so rectangle BAN equals the square on AK .
22. Rectangle ABN together with rectangle BAN equals the square on BA by Prop. II.2.
23. Therefore

$$BA^2 = BF^2 + AK^2.$$
24. BA is the pentagon side, BF the hexagon side, and AK the decagon side.

Therefore, the square on the pentagon side equals the sum of the squares on the hexagon and decagon sides in the same circle.



Proposition 10. *If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is equal to the sum of the squares of the sides of the hexagon and of the decagon inscribed in the same circle.*

PROPOSITION 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter, then the side of the pentagon is the irrational straight-line called minor.

Proof.

1. Inscribe equilateral pentagon $ABCDE$ in a circle with rational diameter and center F .
2. Join AF, FB ; extend to G, H ; join AC .
3. Let $FK = \frac{1}{4}AF$. Since AF is rational, FK is rational; BF is rational, so BK is rational.
4. Arc $ACG = ADG$, and arc $ABC = AED$; therefore arc $CG = GD$.
5. Join AD . The angles at L are right angles, and $CD = 2CL$.
6. Similarly the angles at M are right angles, and $AC = 2CM$.
7. $\angle ALC = \angle AMF$, while $\angle LAC$ is common; therefore triangles ACL and AMF are equiangular.
8. Thus $LC : CA = MF : FA$; hence double $LC : CA$ equals double $MF : FA$.
9. Double $MF : FA = MF : \frac{1}{2}FA$; therefore double $LC : \frac{1}{2}CA = MF : \frac{1}{4}FA$.
10. Since $DC = 2LC$, $CM = \frac{1}{2}CA$, and $FK = \frac{1}{4}FA$,
11. it follows that

$$DC : CM = MF : FK.$$

12. By composition,

$$(DC + CM) : CM = MK : KF.$$

13. Hence

$$(DC + CM)^2 : CM^2 = MK^2 : KF^2.$$

14. Since AC is cut in extreme and mean ratio with greater piece DC , the square on $DC + CM$ is five times the square on CM .

15. Therefore

$$MK^2 = 5 KF^2.$$

16. KF is rational, because the diameter is rational; therefore MK is rational in square only.

17. Since $BF = 4 FK$, we have $BK = 5 KF$, and hence $BK^2 = 25 KF^2$.

18. Since $MK^2 = 5 KF^2$, it follows that $BK^2 = 5 KM^2$.

19. The ratio $BK^2 : KM^2$ is not a square-number ratio, so BK is incommensurable in length with KM .

20. Thus BK and KM are rational straight-lines commensurable in square only.

21. Therefore MB is an apotome, and MK is its attachment.

22. To show it is a fourth apotome, let the square on N equal the excess of the square on BK over the square on KM .

23. Since KF is commensurable with FB , KB is commensurable with FB ; and BF is commensurable with BH , so BK is commensurable with BH .

24. Since $BK^2 : KM^2 = 5 : 1$, by conversion $BK^2 : N^2 = 5 : 4$, which is not a square-number ratio.

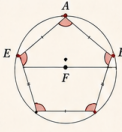
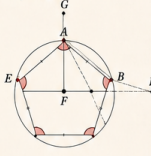
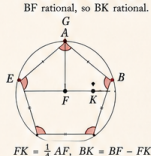
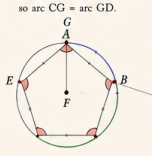
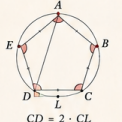
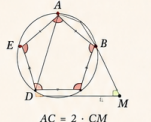
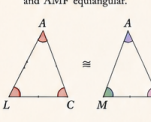
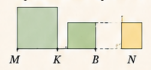
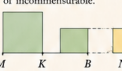
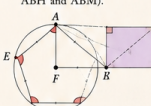
25. Therefore BK is incommensurable with N ; BK exceeds KM by the square of an incommensurable straight-line.

26. Thus MB is a fourth apotome; a rectangle contained by a rational straight-line and a fourth apotome produces a minor.

27. The square on AB equals rectangle HBM , from the similar triangles ABH and ABM .

28. Therefore AB is the irrational straight-line called minor.

Therefore, the side AB of the equilateral pentagon is the irrational straight-line called minor.

1 Inscribe equilateral pentagon ABCDE in a circle with rational diameter, center F. 	2 Join AF, FB; extend to G, H; join AC. 	3 Let $FK = \frac{1}{4} \cdot AF$. AF rational, so FK rational; BF rational, so BK rational.  $FK = \frac{1}{4} AF, BK = BF - FK$	4 Arc ACG = arc ADG, and arc ABC = arc AED, so arc CG = arc GD. 
5 Join AD: $\angle$ at L are right angles, $CD = 2 \cdot CL$.  $CD = 2 \cdot CL$	6 Similarly $\angle$ at M are right, $AC = 2 \cdot CM$.  $AC = 2 \cdot CM$	7 $\angle ALC = \angle AMF, \angle LAC$ common, so triangles ACL and AMF equiangular. 	8 Thus $LC : CA = MF : FA$; double $LC : CA =$ double $MF : FA$. $LC : CA = MF : FA$ $2LC : CA = 2MF : FA$
9 Double $MF : FA = MF : \frac{1}{2} FA$; so double $LC : \frac{1}{2} CA = MF : \frac{1}{2} FA$. $2MF : FA = MF : \frac{1}{2} FA$ $2LC : \frac{1}{2} CA = MF : \frac{1}{2} FA$	10 $DC = 2 \cdot LC, CM = \frac{1}{2} CA$, $FK = \frac{1}{4} FA$. $DC = 2LC$ $CM = \frac{1}{2} CA$ $FK = \frac{1}{4} FA$	11 Therefore $DC : CM = MF : FK$. $DC : CM = MF : FK$	12 By composition, $(DC + CM) : CM = MK : KF$. $DC + CM : CM = MK : KF$
13 Thus square on $(DC + CM)$: square on $CM =$ square MK : square KF. $(DC + CM)^2 : CM^2 = MK^2 : KF^2$	14 Since AC cut in extreme ratio, greater piece DC, square on $(DC + CM) = 5 \cdot$ square CM. $(DC + CM)^2 = 5 CM^2$	15 Therefore square $MK = 5$ square KF. $MK^2 = 5 KF^2$	16 KF rational (diameter rational), so MK rational [in square only]. $KF \text{ rational} \Rightarrow MK \text{ rational [in square only]}$
17 $BF = 4 \cdot FK$, so $BK = 5 \cdot KF$; square $BK = 25$ square KF. $BF = 4FK \quad BK = 5KF$ $BK^2 = 25 KF^2$	18 Square $MK = 5$ square KF, so square $BK = 5$ square KM. $MK^2 = 5 KF^2$ $\Rightarrow BK^2 = 5 KM^2$	19 Square BK : square KM not a square number ratio, so BK incommensurable with KM. $BK^2 : KM^2 = 5 : 1$ (not square number ratio) $\Rightarrow BK, KM \text{ incommensurable}$	20 BK and KM rational, commensurable in square only. $BK, KM \text{ rational and commensurable in square only}$
21 MB is an apotome, MK its attachment. $MB = \text{apotome}$ $MK = \text{attachment}$	22 Show it's a fourth apotome: Let square N equal excess of square BK over square KM.  $N^2 = BK^2 - KM^2$	23 Since KF commensurable with FB, KB commensurable with FB; BF commensurable with BH; so BK commensurable with BH. $F \quad K \quad B \quad H$ $BK \text{ commensurable with } BH$	24 Square BK : square $KM = 5 : 1$; by conversion, square BK : square $N = 5 : 4$ (not square number ratio). $BK^2 : KM^2 = 5 : 1$ $\Rightarrow BK^2 : N^2 = 5 : 4$ (not square number ratio)
25 BK incommensurable with N; BK greater than KM by square of incommensurable.  $BK > KM \text{ and } BK^2 - KM^2 = N^2$ (N not a square multiple of KM^2)	26 MB is fourth apotome; rectangle from rational and fourth apotome gives minor. $MB = \text{fourth apotome}$ $\text{Rectangle with rational and 4th apotome} \Rightarrow \text{minor}$	27 Square AB = rectangle HBM (from similar triangles ABH and ABM). 	28 Therefore AB is minor. $AB \text{ is minor (irrational straight-line called minor)}$

Proposition 11. If an equilateral pentagon is inscribed in a circle which has a rational diameter, then the side of the pentagon is the irrational straight-line called minor.

PROPOSITION 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the square on the radius of the circle.

Proof.

1. Let ABC be an equilateral triangle inscribed in a circle with center D .
2. Join AD , extend it to E , and join BE .
3. Since triangle ABC is equilateral, arc BEC is one third of the circumference.
4. Therefore arc BE is one sixth of the circumference, so BE is the side of the hexagon.
5. Thus $BE = DE$, both being radii of the circle.
6. Since $AE = 2 DE$,

$$AE^2 = 4 DE^2 = 4 BE^2.$$

7. The angle ABE is right because it stands in a semicircle; hence

$$AE^2 = AB^2 + BE^2.$$

8. Therefore

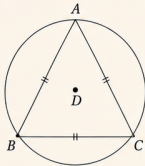
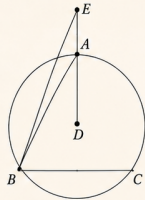
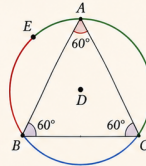
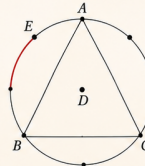
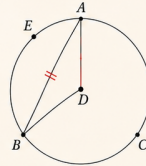
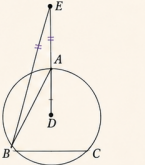
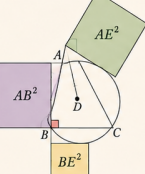
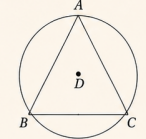
$$AB^2 + BE^2 = 4 BE^2.$$

9. Hence

$$AB^2 = 3 BE^2 = 3 DE^2.$$

10. Thus the square on the side of the equilateral triangle is three times the square on the radius.

Therefore, $AB^2 = 3 DE^2$.

1 Let ABC be equilateral triangle inscribed in circle with center D. 	2 Join AD, extend to E; join BE. 	3 Since triangle ABC equilateral, arc $BEC = \frac{1}{3}$ circumference. 	4 Therefore arc $BE = \frac{1}{6}$ circumference, so BE is side of hexagon. 	5 Thus $BE = DE$ (radius). 
6 $AE = 2 \cdot DE$, so square $AE = 4 \cdot$ square $DE = 4 \cdot$ square BE.  $AE^2 = 4 DE^2 = 4 BE^2$	7 Square $AE =$ square $AB +$ square BE (angle ABE right, from semicircle).  $AB^2 + BE^2 = 4 BE^2$	8 Therefore square $AB +$ square $BE = 4 \cdot$ square BE. AB^2 $+ BE^2$ $= 4 BE^2$ $AB^2 + BE^2 = 4 BE^2$	9 So square $AB = 3 \cdot$ square $BE = 3 \cdot$ square DE. AB^2 $= 3 BE^2$ $AB^2 = 3 BE^2 = 3 DE^2$	10 Thus side of triangle squared $= 3 \cdot$ radius squared.  $AB^2 = 3 DE^2$ (side² = 3 · radius²)

Proposition 12. *If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the square on the radius of the circle.*

PROPOSITION 13

To construct a regular tetrahedron, enclose it in a given sphere, and show that the square on the diameter of the sphere is one and a half times the square on the side of the tetrahedron.

Proof.

1. Let AB be the diameter of the given sphere, cut at C so that $AC = 2CB$.
2. Draw semicircle ADB on AB ; draw $CD \perp AB$ at C , and join DA .
3. Set circle EFG with radius DC ; inscribe equilateral triangle EFG , and let H be its center.
4. Join EH, FH, GH ; set $HK \perp$ plane EFG , with $HK = AC$.
5. Join KE, KF, KG .
6. Since $KH \perp$ plane EFG , $KH \perp HE, HF, HG$.
7. Since $AC = KH$, $CD = HE$, and the included angles are right, $DA = KE$.
8. Similarly $KF = KG = DA$.
9. Since $AC = 2CB$, $AB = 3BC$; by the lemma below, $AD^2 = 3DC^2$.
10. Since $FE^2 = 3EH^2$, and $DC = EH$, it follows that $DA = EF$.
11. Since $DA = KE = KF = KG$, we have

$$EF = FG = GE = KE = KF = KG.$$

12. Thus four equilateral triangles form a pyramid with base EFG and apex K .
13. Extend KH to L with $HL = CB$.
14. Since $AC : CD = CD : CB$, and $AC = KH$, $CD = HE$, $CB = HL$,

$$KH : HE = EH : HL.$$

15. Hence rectangle $KH \cdot HL$ equals the square on EH .
16. The semicircle on KL passes through E , because $\angle LEK$ is right.
17. Rotating the semicircle passes through F, G ; hence the pyramid is enclosed in a sphere.
18. $KL = AB$, the diameter of the given sphere.
19. Since $AC = 2CB$, $AB = 3BC$; by conversion,

$$BA = \frac{3}{2} AC.$$

20. Also

$$BA : AC = BA^2 : AD^2,$$

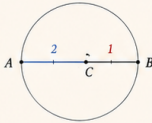
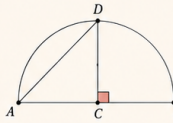
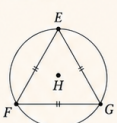
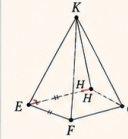
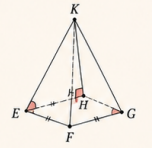
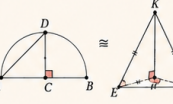
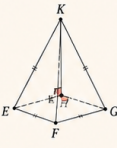
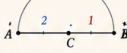
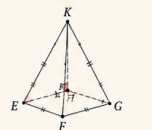
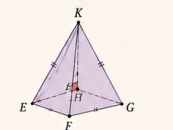
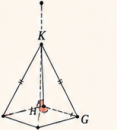
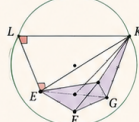
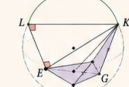
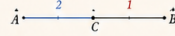
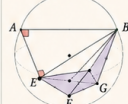
so

$$BA^2 = \frac{3}{2} AD^2.$$

21. BA is the sphere diameter, and AD is the side of the pyramid.
22. Therefore

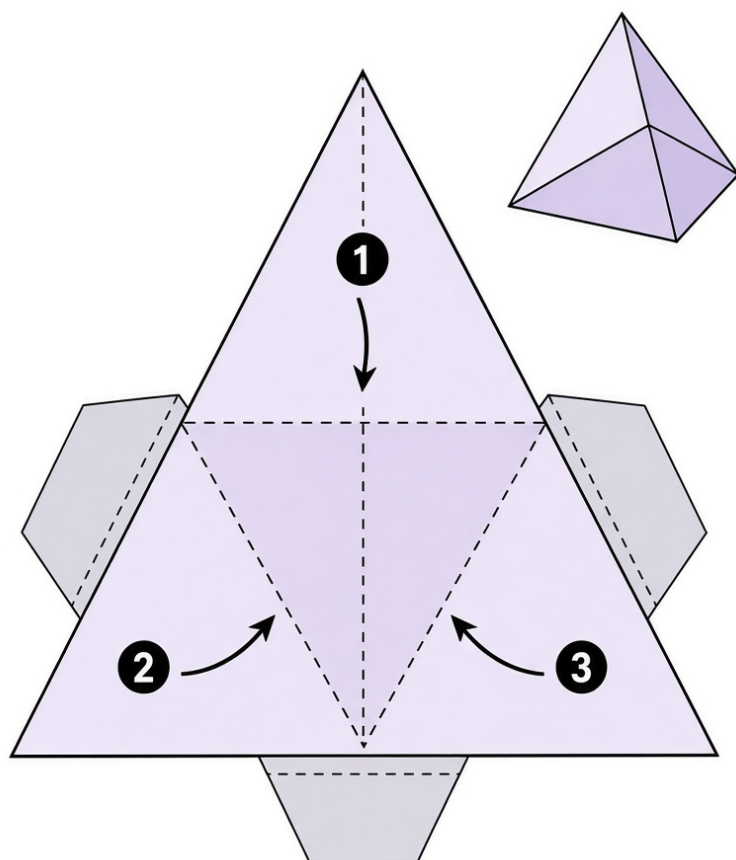
$$\text{diameter}^2 = \frac{3}{2} \text{side}^2.$$

Therefore, a regular tetrahedron has been constructed in the given sphere, and the square on the sphere's diameter is one and a half times the square on the tetrahedron's side.

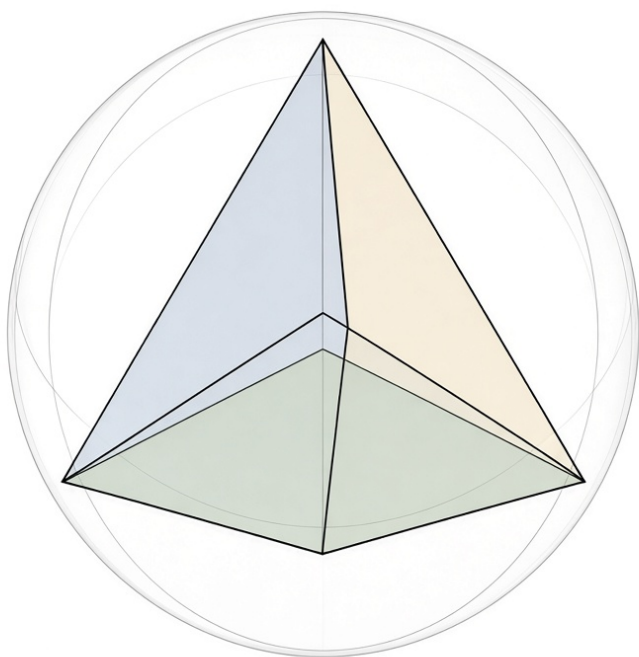
1 Let AB be diameter of given sphere, cut at C such that $AC = 2 \cdot CB$. 	2 Draw semicircle ADB on AB, draw $CD \perp AB$ at C, join DA. 	3 Set circle EFG with radius DC, inscribe equilateral triangle EFG, center H. 	4 Join EH, FH, GH; set $HK \perp$ plane EFG with $HK = AC$. 	5 Join KE, KF, KG. 
6 Since $KH \perp$ plane EFG, $KH \perp HE, HF, HG$. 	7 $AC = KH$, $CD = HE$, right angles, so $DA = KE$. 	8 Similarly $KF = KG = DA$. 	9 Since $AC = 2 \cdot CB$, $AB = 3 \cdot BC$; so square $AD = 3 \cdot$ square DC (by lemma).  $AD^2 = 3 \cdot DC^2$	10 Square $FE = 3 \cdot$ square EH, and $DC = EH$, so $DA = EF$.  $FE^2 = 3 \cdot EH^2$ $\Rightarrow FE = \sqrt{3} \cdot EH = \sqrt{3} \cdot DC$ and $DA = EF$
11 $DA = KE = KF = KG$, so $EF = FG = GE = KE = KF = KG$. 	12 Four equilateral triangles form pyramid with base EFG and apex K. 	13 Extend KH to L with $HL = CB$. 	14 Since $AC : CD = CD : CB$, and $AC = KH$, $CD = HE$, $CB = HL$. $KH : HE = EH : HL$	15 $KH : HE = EH : HL$, so rectangle $KH \cdot HL =$ square EH. $KH \cdot HL = EH^2$
16 Semicircle on KL passes through E (angle LEK right). 	17 Rotating semicircle passes through F, G; pyramid enclosed in sphere. 	18 $KL = AB =$ diameter of given sphere. 		
19 Since $AC = 2 \cdot CB$, $AB = 3 \cdot BC$; by conversion, $BA = 1\frac{1}{2} \cdot AC$.  $BA = 3 \cdot BC = 1\frac{1}{2} \cdot AC$	20 $BA : AC =$ square $BA : \text{square } AD$, so square $BA = 1\frac{1}{2} \cdot \text{square } AD$. $\frac{BA}{AC} = \frac{BA^2}{AD^2}$ $\Rightarrow BA^2 = 1\frac{1}{2} \cdot AD^2$	21 BA is sphere diameter, $AD =$ side of pyramid. 	22 Therefore diameter² = $1\frac{1}{2} \cdot \text{side}^2$. $AB^2 = \frac{3}{2} AD^2$ (i.e. square on diameter = one and a half times square on side)	

Proposition 13. To construct a regular tetrahedron, enclose it in a given sphere, and show that the square on the diameter of the sphere is one and a half times the square on the side of the tetrahedron.

TETRAHEDRON — 4 equilateral triangles



Cut solid lines • Fold dashed lines • Glue tabs



Lemma to Proposition 13

In the semicircle figure, $AB : BC$ equals the square on AD to the square on DC .

Proof.

1. In the semicircle figure, join DB ; construct square EC on AC ; complete parallelogram FB .
2. Triangles DAB and DAC are equiangular, so

$$BA : AD = DA : AC.$$

3. Therefore rectangle $BA \cdot AC$ equals the square on AD .
4. Since $AB : BC = EB : BF$ by Prop. VI.1,
5. and $EB = BA \cdot AC$, while $BF = AC \cdot CB$,
6. we have

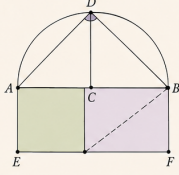
$$AB : BC = BA \cdot AC : AC \cdot CB.$$

7. But $BA \cdot AC = AD^2$, and $AC \cdot CB = DC^2$, because DC is a mean proportional.
8. Therefore

$$AB : BC = AD^2 : DC^2.$$

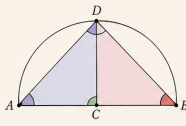
Therefore, $AB : BC = AD^2 : DC^2$.

- 1 In semicircle figure, join DB ; construct square EC on AC ; complete parallelogram FB .



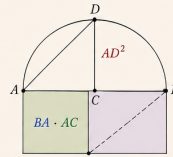
- 2 Triangles DAB and DAC equiangular, so

$$BA : AD = DA : AC$$



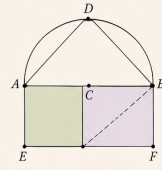
- 3 Therefore

$$\text{rectangle } BA \cdot AC = \text{square } AD$$

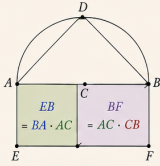


- 4 Since

$$AB : BC = EB : BF \quad (\text{Prop. 6.1})$$

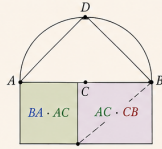


- 5 $EB = BA \cdot AC$ (since $EA = AC$),
 $BF = AC \cdot CB$.

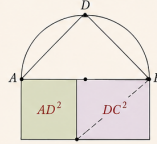


- 6 Therefore

$$AB : BC = BA \cdot AC : AC \cdot CB$$

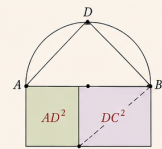


- 7 But $BA \cdot AC = \text{square } AD$ (from step 3), and
 $AC \cdot CB = \text{square } DC$ (since DC is mean proportional).



- 8 Therefore

$$AB : BC = AD^2 : DC^2$$



LEMMA. In the semicircle figure described, $AB : BC = \text{square on } AD : \text{square on } DC$.

PROPOSITION 14

To construct an octahedron, enclose it in a given sphere, and show that the square on the diameter of the sphere is double the square on the side of the octahedron.

Proof.

1. Let AB be the diameter, cut in half at C .
2. Draw semicircle ADB ; draw $CD \perp AB$ at C , and join DB .
3. Construct square $EFGH$ with side equal to DB , and let K be its center.
4. Join HF and EG ; set $KL \perp$ plane $EFGH$, and extend to KM on the other side.
5. Let

$$KL = KM = EK = FK = GK = HK.$$

6. Join $LE, LF, LG, LH, ME, MF, MG, MH$.
7. Since $KE = KH$ and $\angle EKH$ is right,

$$HE^2 = 2 EK^2.$$

8. Since $LK = KE$ and $\angle LKE$ is right,

$$EL^2 = 2 EK^2.$$

9. Hence $HE^2 = EL^2$, so $LE = EH$; similarly $LH = HE$.
10. Triangle LEH is equilateral; similarly all eight triangles are equilateral.
11. Therefore the octahedron is formed from eight equilateral triangles.
12. Since $LK = KM = KE$, the semicircle on LM passes through E .
13. Rotating it passes through F, G, H ; therefore the octahedron is enclosed by a sphere.

14. Since $LK = KM = KE$, and KE is common with right angles, $LE = EM$.

15. $\angle LEM$ is right, since it stands in a semicircle; hence

$$LM^2 = 2 LE^2.$$

16. Since $AC = CB$, $AB = 2 BC$; and by the lemma,

$$AB : BC = AB^2 : BD^2.$$

17. Therefore

$$AB^2 = 2 BD^2.$$

18. Since $BD^2 = LE^2$, because $EH = DB$,

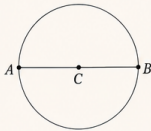
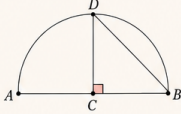
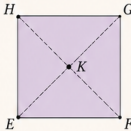
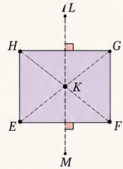
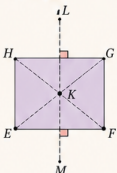
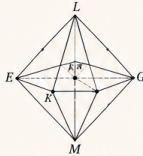
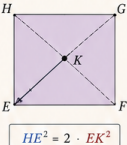
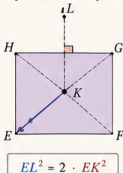
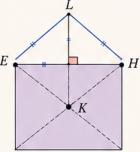
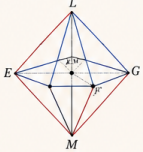
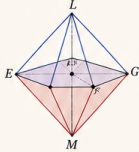
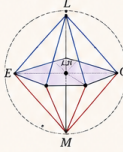
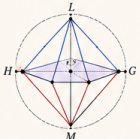
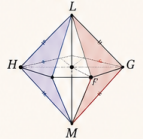
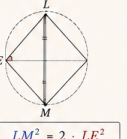
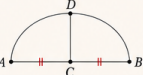
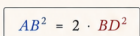
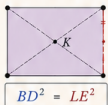
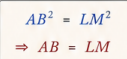
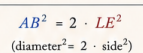
$$AB^2 = LM^2.$$

19. Therefore $AB = LM$, the diameter of the given sphere.

20. Thus

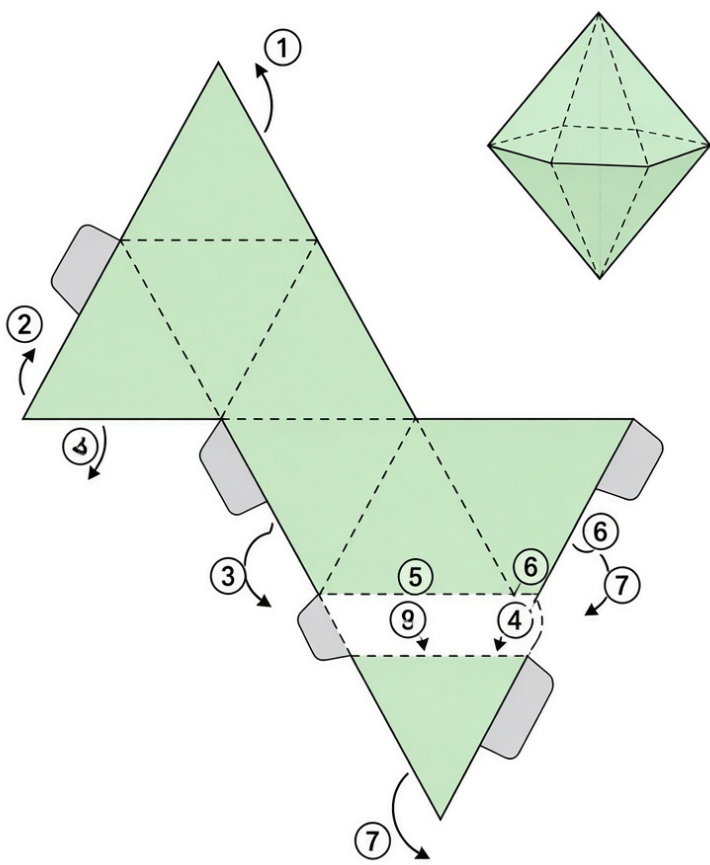
$$\text{diameter}^2 = 2 \text{ side}^2.$$

Therefore, the octahedron is enclosed in the given sphere, and the square on the diameter is double the square on its side.

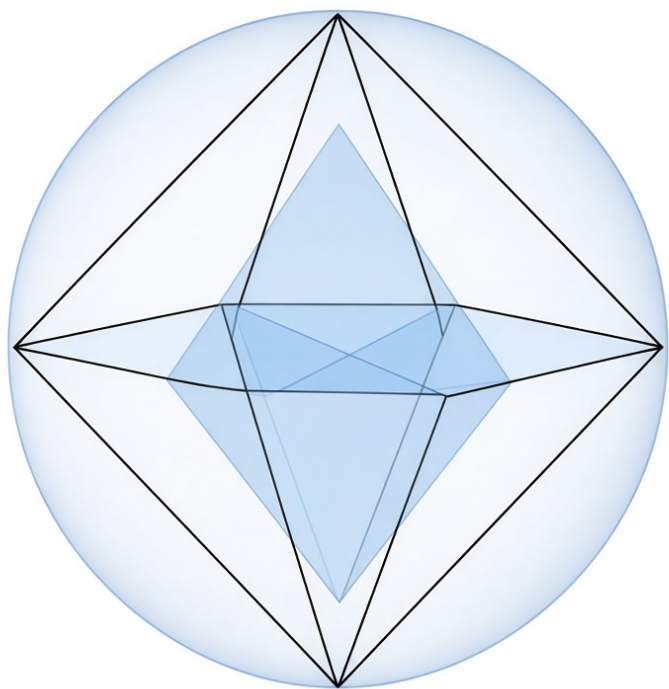
1 Let AB be diameter, cut in half at C. 	2 Draw semicircle ADB, $CD \perp AB$ at C, join DB. 	3 Construct square $EFGH$ with side $= DB$, center K. 	4 Join HF, EG; set $KL \perp$ plane $EFGH$, extend to KM on other side. 
5 Let $KL = KM = EK = FK = GK = HK$. 	6 Join $LE, LF, LG, LH, ME, MF, MG, MH$. 	7 Since $KE = KH$ and $\angle EKH$ right, square $HE = 2 \cdot \text{square } EK$. 	8 $LK = KE$ and $\angle LKE$ right, square $EL = 2 \cdot \text{square } EK$. 
9 Square $HE = \text{square } EL$, so $LE = EH$; similarly $LH = HE$. 	10 Triangle LEH equilateral; similarly all eight triangles equilateral. 	11 Therefore octahedron formed from eight equilateral triangles. 	12 Since $LK = KM = KE$, semicircle on LM passes through E. 
13 Rotating semicircle passes through F, G, H; octahedron enclosed by sphere. 	14 Since $LK = KM = KE$ and KE common, right angles, so $LE = EM$. 	15 $\angle LEM$ right (in semicircle), so square $LM = 2 \cdot \text{square } LE$. 	16 $AC = CB$, so $AB = 2 \cdot BC$; $AB : BC = \text{square } AB : \text{square } BD$ (lemma). 
17 Therefore square $AB = 2 \cdot \text{square } BD$. 	18 Square $BD = \text{square } LE$ (since $EH = DB$, both sides of square $EFGH$). 	19 Therefore square $AB = 2 \cdot \text{square } LE = \text{square } LM$ (from step 15). 	20 Thus diameter² = $2 \cdot \text{side}^2$. 

Proposition 14. To construct an octahedron, enclose it in a given sphere, and show that the square on the diameter of the sphere is double the square on the side of the octahedron.

OCTAHEDRON — 8 equilateral triangles



Cut solid lines • Fold dashed lines • Glue tabs



PROPOSITION 15

To construct a cube, enclose it in a sphere, and show that the square on the diameter of the sphere is three times the square on the side of the cube.

Proof.

1. Let AB be the diameter of the given sphere, cut at C so that $AC = 2CB$.
2. Draw semicircle ADB , draw $CD \perp AB$ at C , and join DB .
3. Construct square $EFGH$ with side equal to DB .
4. From E, F, G, H draw EK, FL, GM, HN perpendicular to plane $EFGH$.
5. Make

$$EK = FL = GM = HN = EF = FG = GH = HE.$$

6. Join KL, LM, MN, NK .
7. A cube is thereby formed from six equal squares.
8. Join KG, EG . Since $\angle KEG$ is right, $KE \perp EG$, and the semicircle on KG passes through E .
9. Since $GF \perp FL$ and $GF \perp FE$, GF is perpendicular to plane FK ; hence $GF \perp FK$.
10. The semicircle on GK therefore passes through F , and similarly through all vertices.
11. Rotating the semicircle encloses the cube in a sphere.
12. Since $GF = FE$ and $\angle F$ is right,

$$EG^2 = 2EF^2.$$

13. Since $EF = EK$,

$$EG^2 = 2EK^2.$$

14. Therefore

$$GK^2 = GE^2 + EK^2 = 3EK^2.$$

15. Since $AB = 3 BC$, by the lemma

$$AB^2 = 3 BD^2.$$

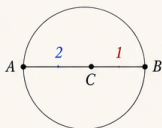
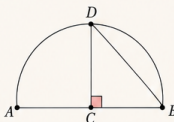
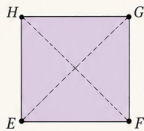
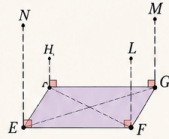
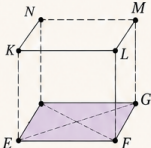
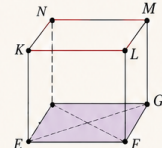
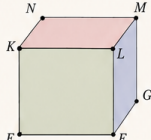
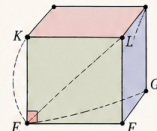
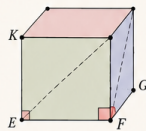
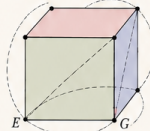
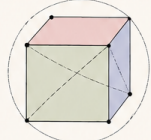
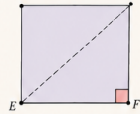
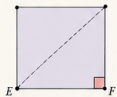
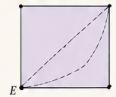
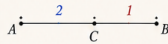
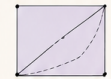
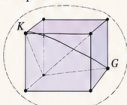
16. Since $GK^2 = 3 KE^2$, and $KE = DB$, it follows that $GK = AB$.

17. Thus KG equals the diameter of the given sphere.

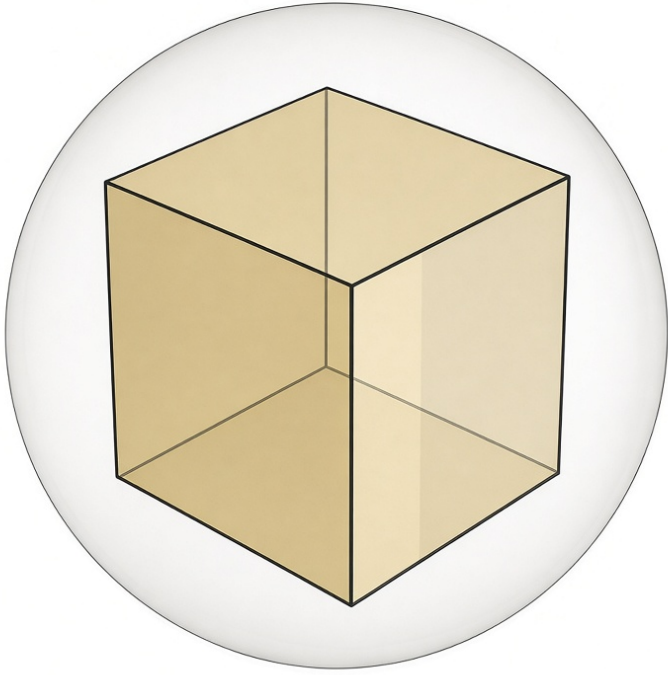
18. Therefore

$$\text{diameter}^2 = 3 \text{ side}^2.$$

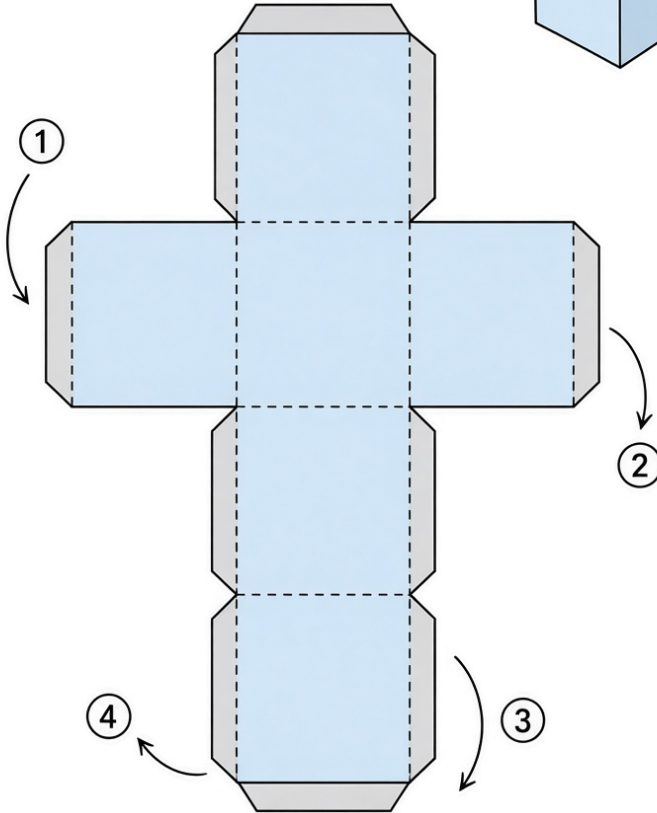
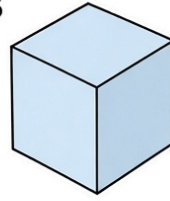
Therefore, the cube is enclosed in the given sphere, and the square on the diameter is three times the square on the cube's side.

1 Let AB be diameter of given sphere, cut at C such that $AC = 2 \cdot CB$. 	2 Draw semicircle ADB, $CD \perp AB$ at C, join DB. 	3 Construct square $EFGH$ with side $= DB$. 	4 From E, F, G, H draw $EK, FL, GM, HN \perp$ plane $EFGH$. 
5 Make $EK = FL = GM = HN = EF = FG = GH = HE$. 	6 Join KL, LM, MN, NK. 	7 Cube formed from six equal squares. 	8 Join KG, EG; $\angle KEG$ right ($KE \perp EG$), so semicircle on KG passes through E. 
9 $GF \perp FL$ and FE, so $GF \perp$ plane FK; thus $GF \perp FK$. 	10 Semicircle on GK passes through F; similarly through all vertices. 	11 Rotating semicircle encloses cube in sphere. 	12 $GF = FE$, $\angle F$ right, so square $EG = 2 \cdot$ square EF.  $EG^2 = 2 \cdot EF^2$
13 $EF = EK$, so square $EG = 2 \cdot$ square EK.  $EG^2 = 2 \cdot EK^2$	14 Square $GK =$ square $GE +$ square $EK = 3 \cdot$ square EK.  $GK^2 = 3 \cdot EK^2$	15 $AB = 3 \cdot BC$, so $AB : BC =$ square $AB : \text{square } BD$ (lemma).  $AB^2 = 3 \cdot BD^2$	16 Square $GK = 3 \cdot$ square KE, and $KE = DB$, so $GK = AB$.  $GK^2 = 3 \cdot KE^2 = 3 \cdot DB^2 = AB^2 \Rightarrow GK = AB$
17 KG equals diameter of given sphere. 	18 Therefore $\text{diameter}^2 = 3 \cdot \text{side}^2$. $AB^2 = 3 \cdot KE^2$ ($\text{diameter}^2 = 3 \cdot \text{side}^2$)		

Proposition 15. To construct a cube, enclose it in a given sphere, and show that the square on the diameter of the sphere is three times the square on the side of the cube.



CUBE — 6 squares



Cut solid lines • Fold dashed lines • Glue tabs

PROPOSITION 16

To construct an icosahedron, enclose it in a sphere, and show that the side of the icosahedron is the irrational straight-line called minor.

Proof.

Panel 1 — Actions 1–11.

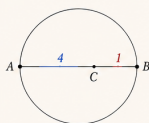
1. Let AB be the diameter, cut at C so that $AC = 4CB$.
2. Draw semicircle ADB , draw $CD \perp AB$ at C , and join DB .
3. Construct circle $EFGHK$ with radius DB ; inscribe an equilateral and equiangular pentagon $EFGHK$.
4. Bisect the arcs at L, M, N, O, P ; join LM, MN, NO, OP, PL, EP .
5. Pentagon $LMNOP$ is equilateral; EP is a side of the decagon.
6. Draw EQ, FR, GS, HT, KU perpendicular to the plane of the circle, each equal to the radius.
7. Join

$QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, PQ.$

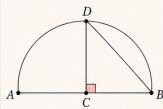
8. Since EQ and KU are perpendicular to the same plane, they are parallel and equal; therefore $QU \parallel EK$ and $QU = EK$.
9. EK is a pentagon side, so QU is a pentagon side; similarly QR, RS, ST, TU are pentagon sides.
10. Therefore pentagon $QRSTU$ is equilateral.
11. QE is a hexagon side, EP a decagon side, and $\angle QEP$ is right; hence QP is a pentagon side by Prop. XIII.10.

PROPOSITION 16 — Construct an icosahedron, enclose it in a sphere, and show that the side of the icosahedron is the irrational straight-line called minor.

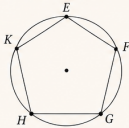
- 1 Let AB be diameter, cut at C such that $AC = 4 \cdot CB$.



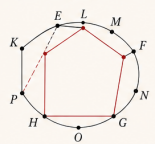
- 2 Draw semicircle ADB , $CD \perp AB$ at C , join DB .



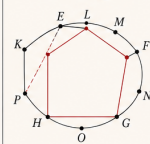
- 3 Construct circle $EFGHK$ with radius DB ; inscribe equilateral/ equiangular pentagon $EFGHK$.



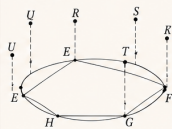
- 4 Bisect arcs at L, M, N, O, P ; join LM, MN, NO, OP, PL, EP .



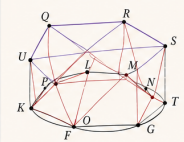
- 5 Pentagon $LMNOP$ equilateral; EP is side of decagon.



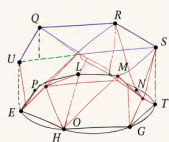
- 6 Draw $EQ, FR, GS, HT, KU \perp$ plane circle, each equal to radius.



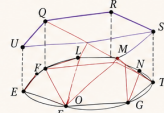
- 7 Join $QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, PQ$.



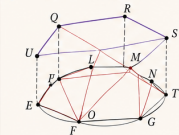
- 8 EQ and $KU \perp$ same plane, so $EQ \parallel KU$ and equal; thus $QU \parallel EK$ and equal.



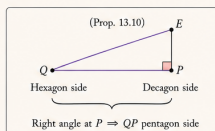
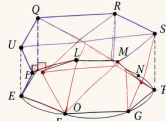
- 9 EK is side of pentagon, so QU is pentagon side; similarly QR, RS, ST, TU are pentagon sides.



- 10 Pentagon $QRSTU$ equilateral.



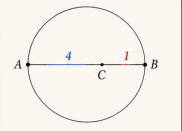
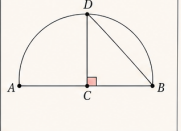
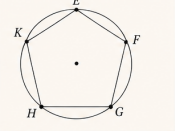
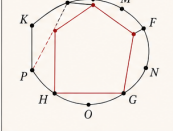
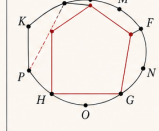
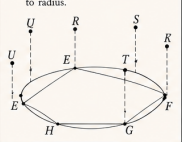
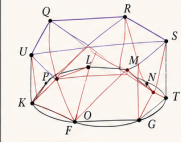
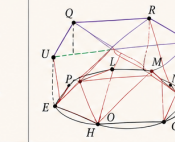
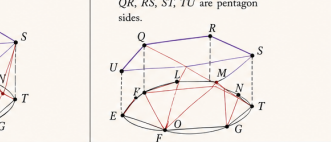
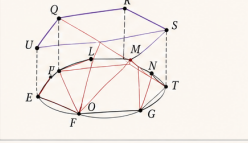
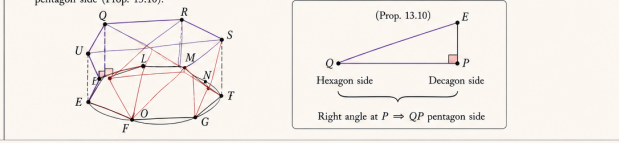
- 11 QE is hexagon side, EP decagon side, $\angle QEP$ right, so QP is pentagon side (Prop. 13.10).



Panel 2 — Actions 12–22.

12. Similarly PU is a pentagon side; QU is a pentagon side, so triangle QPU is equilateral.
13. Similarly triangles QLR , RMS , SNT , TOU are equilateral.
14. Since QL and QP are pentagon sides and LP is a pentagon side, triangle QLP is equilateral.
15. Similarly triangles LRM , MSN , NTO , OUP are equilateral.
16. Find the center V of the circle; set $VZ \perp$ the plane and produce it to X .
17. Cut VW equal to a hexagon side, and $VX = WZ$ equal to a decagon side.
18. Join QZ , QW , UZ , EV , LV , LX , XM .
19. Since VW and QE are perpendicular to the same plane, $VW \parallel QE$ and they are equal; likewise $EV \parallel QW$ and $EV = QW$.
20. EV is a hexagon side, so QW is a hexagon side.
21. QW is a hexagon side, WZ a decagon side, and $\angle QWZ$ is right; hence QZ is a pentagon side by Prop. XIII.10.
22. Similarly UZ is a pentagon side; since QU is a pentagon side, triangle QUZ is equilateral.

PROPOSITION 16 — Construct an icosahedron, enclose it in a sphere, and show that the side of the icosahedron is the irrational straight-line called minor.

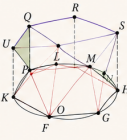
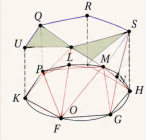
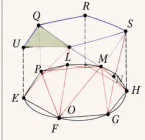
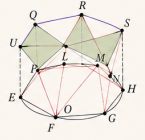
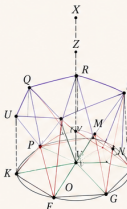
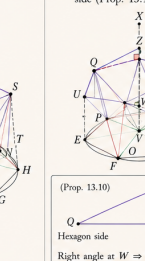
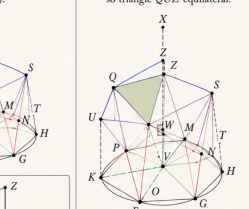
1 Let AB be diameter, cut at C such that $AC = 4 \cdot CB$. 	2 Draw semicircle ADB, $CD \perp AB$ at C, join DB. 	3 Construct circle $EFGHK$ with radius DB; inscribe equilateral/ equiangular pentagon $EFGHK$. 	4 Bisect arcs at L, M, N, O, P; join LM, MN, NO, OP, PL, EP. 	5 Pentagon $LMNOP$ equilateral; EP is side of decagon. 
6 Draw $EQ, FR, GS, HT, KU \perp$ plane circle, each equal to radius. 	7 Join $QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, PQ$. 	8 EQ and $KU \perp$ same plane, so $EQ \parallel KU$ and equal; thus $QU \parallel EK$ and equal. 	9 EK is side of pentagon, so QU is pentagon side; similarly QR, RS, ST, TU are pentagon sides. 	
10 Pentagon $QRSTU$ equilateral. 	11 QE is hexagon side, EP decagon side, $\angle QEP$ right, so QP is pentagon side (Prop. 13.10). 			

Panel 3 — Actions 23–33.

23. Similarly the remaining triangles with apex Z are equilateral.
24. VL is a hexagon side, VX a decagon side, and $\angle LVX$ is right; hence LX is a pentagon side.
25. Similarly MX is a pentagon side; since LM is a pentagon side, triangle LMX is equilateral.
26. Similarly the remaining triangles with apex X are equilateral.
27. Thus twenty equilateral triangles form the icosahedron.
28. Since VW is a hexagon side and WZ a decagon side, VZ is cut in extreme and mean ratio at W by Prop. XIII.9.
29. Therefore

$$VZ : VW = VW : WZ,$$
 while $VW = VE$ and $WZ = VX$.
30. Hence

$$ZV : VE = EV : VX,$$
 and $\angle ZVE, \angle EVX$ are right.
31. Join EZ ; then $\angle XEZ$ is right, from the similar triangles XEZ and VEZ .
32. Similarly, join QX ; the angle at Q is right.
33. Therefore the semicircle on XZ passes through Q and all the vertices of the icosahedron; the sphere encloses it.

12 Similarly PU is pentagon side; QU pentagon side, so triangle QPU equilateral. 	13 Similarly triangles QLR, RMS, SNT, TOU equilateral. 	14 QL and QP pentagon sides, LP pentagon side, so triangle QLP equilateral. 	15 Similarly triangles LRM, MSN, NTO, OUP equilateral. 	16 Find center V of circle; set $VZ \perp$ plane, and produce to X. 	17 Cut VW = hexagon side, $VX = WZ$ = decagon side. 
18 Join QZ, QW, UZ, EV, LV, LX, XM. 	19 VW and $QE \perp$ plane, so $VW \parallel QE$ and equal; thus $EV \parallel QW$ and equal. 	20 EV is hexagon side, so QW is hexagon side. 	21 QW hexagon, WZ decagon, $\angle QWZ$ right, so QZ pentagon side (Prop. 13.10). 	22 Similarly UZ pentagon side; QU pentagon side, so triangle QUZ equilateral. 	

Panel 4 — Actions 34–44.

34. Cut VW in half at a . Since VZ is cut in extreme and mean ratio at W , ZW is the lesser piece.

35. By Prop. XIII.3, the square on ZW together with the square on half the greater piece is five times the square on half the greater piece.

36. Therefore

$$Za^2 = 5aW^2,$$

while $XZ = 2Za$ and $VW = 2aW$.

37. Hence

$$XZ^2 = 5WV^2.$$

38. Since $AC = 4CB$, $AB = 5BC$; by the lemma,

$$AB^2 = 5BD^2.$$

39. Since $XZ^2 = 5WV^2$, and $DB = WV$, both being radii, it follows that $AB = XZ$.

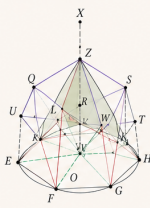
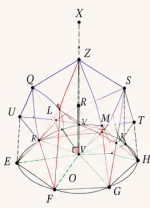
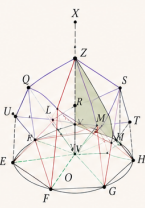
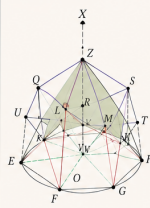
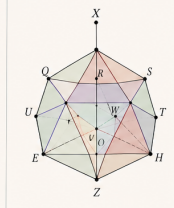
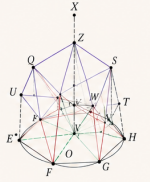
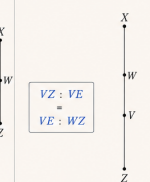
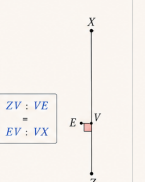
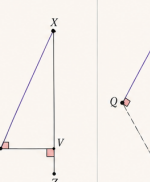
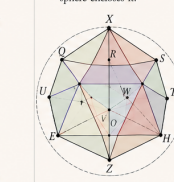
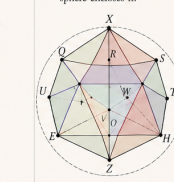
40. Thus XZ equals the diameter of the given sphere.

41. The diameter is rational, and its square is five times the square on the radius of the circle; therefore that radius is rational.

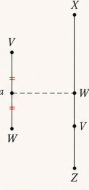
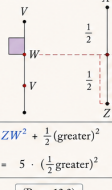
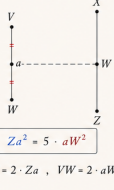
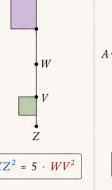
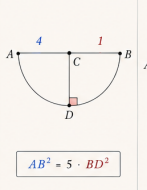
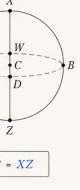
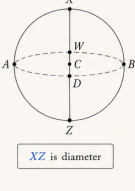
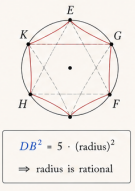
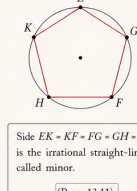
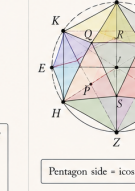
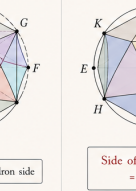
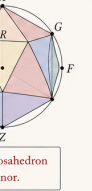
42. An equilateral pentagon inscribed in a circle with rational diameter has its side equal to the irrational straight-line called minor, by Prop. XIII.11.

43. The side of pentagon $EFGHK$ is a side of the icosahedron.

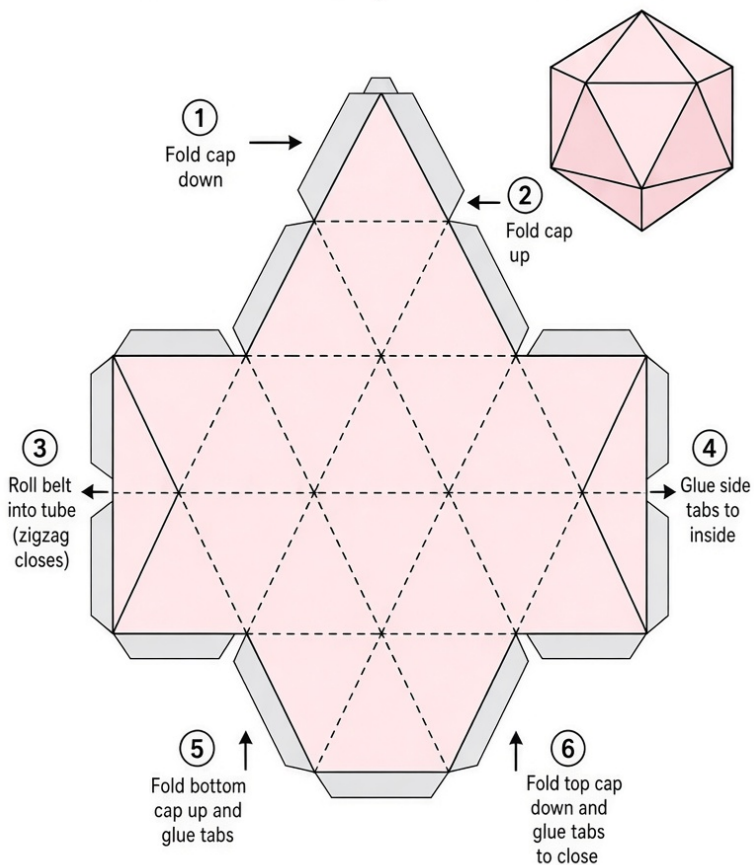
44. Therefore the side of the icosahedron is the irrational straight-line called minor.

23 Similarly, the remaining triangles with apex Z are equilateral. 	24 VL is hexagon side, VX is decagon side, $\angle LVX$ right, so LX is pentagon side. 	25 Similarly MX is pentagon side; LM is pentagon side, so triangle LMX equilateral. 	26 Similarly, the remaining triangles with apex X are equilateral. 	27 Twenty equilateral triangles form the icosahedron. 	
28 VW is hexagon side, WZ is decagon side, so VZ is cut in extreme ratio at W (Prop. 13.9).  (Prop. 13.9)	29 VZ : VW = VW : WZ; but VW = VE, and WZ = VX.  $VZ : VE = VE : WZ$	30 ZV : VE = EV : VX; $\angle ZVE$ and $\angle EVX$ are right angles.  $ZV : VE = EV : VX$	31 Join EZ: $\angle XEZ$ is right (similar triangles XEZ and VEZ). 	32 Similarly, join QX: $\angle XQZ$ is right. 	33 Semicircle on XZ passes through Q and all vertices of icosahedron; thus the sphere encloses it. 

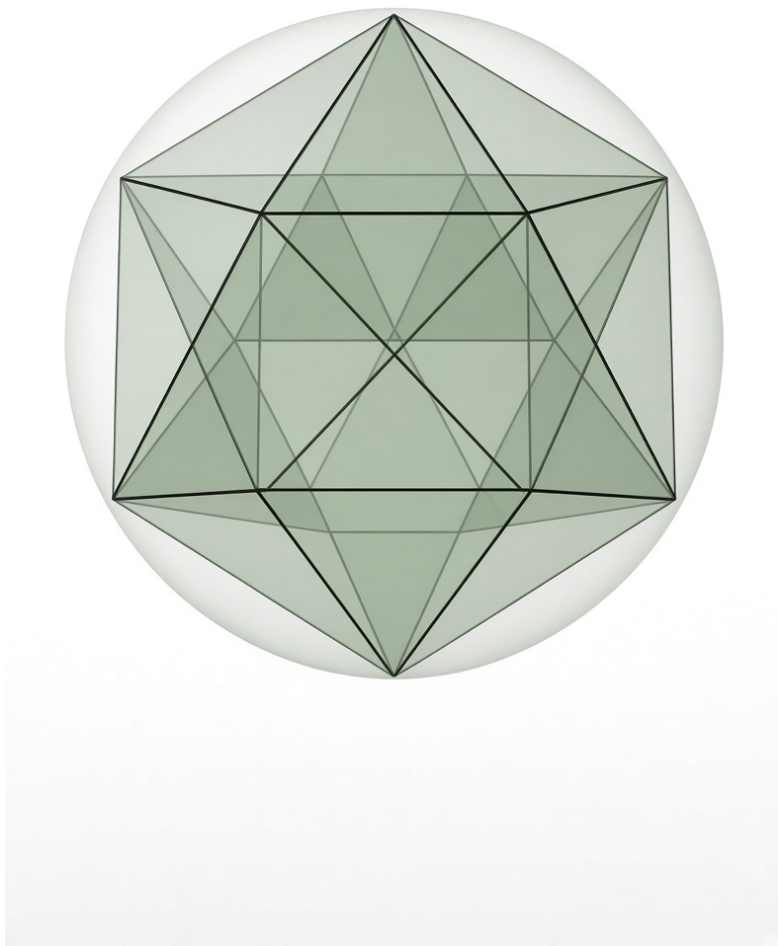
Additional Illustration.

34 Cut VW in half at a; VZ is cut in extreme ratio at W with ZW lesser.  $ZW^2 + \frac{1}{2}(\text{greater})^2 = 5 \cdot (\frac{1}{2}\text{greater})^2$ (Prop. 13.3)	35 Square on ZW plus half of greater piece = 5 times square on half greater (Prop. 13.3).  $Za^2 = 5 \cdot aW^2$ $XZ = 2 \cdot Za, VW = 2 \cdot aW$	36 Therefore square $Za = 5 \cdot \text{square } aW$; also $XZ = 2 \cdot Za$, $VW = 2 \cdot aW$.  $XZ^2 = 5 \cdot WV^2$	37 So square $XZ = 5 \cdot \text{square } WV$.  $AB^2 = 5 \cdot BD^2$	38 $AC = 4 \cdot CB$, so $AB = 5 \cdot BC$; square $AB = 5 \cdot \text{square } BD$.  $AB = XZ$	39 Square $XZ = 5 \cdot \text{square } WV$, and $DB = WV$ (both radii); therefore $AB = XZ$. 
40 Thus XZ equals the diameter of the given sphere.  XZ is diameter	41 Diameter is rational; square diameter = $5 \cdot \text{radius}^2$ of circle. Hence radius is rational.  $DB^2 = 5 \cdot (\text{radius})^2$ $\Rightarrow$ radius is rational	42 An equilateral pentagon inscribed in a circle with rational diameter has side minor (Prop. 13.11).  Side $EK = KF = FG = GH = HE$ is the irrational straight-line called minor. (Prop. 13.11)	43 Each side of the pentagon $EFGHK$ is a side of the constructed icosahedron.  Pentagon side = icosahedron side	44 Therefore the side of the icosahedron is the irrational straight-line called minor.  Side of icosahedron = minor.	45 Therefore the side of the icosahedron is the irrational straight-line called minor. 

ICOSAHEDRON — 20 equilateral triangles



Cut solid lines • Fold dashed lines • Glue tabs



Therefore, the icosahedron is enclosed in the given sphere, and its side is the irrational straight-line called minor.

Corollary to Proposition 16

Relations for the sphere and the auxiliary circle used in the construction of the icosahedron.

1. The square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron is described:

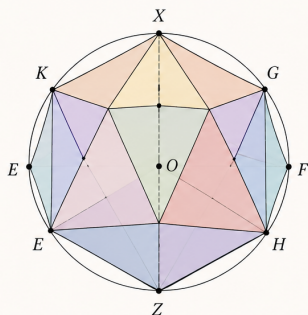
$$\text{diameter}^2 = 5 \text{ radius}^2.$$

2. The diameter equals the side of the hexagon together with twice the side of the decagon inscribed in the same circle:

$$\text{diameter} = \text{hexagon side} + 2 (\text{decagon side}).$$

COROLLARY. *From the circle from which the icosahedron is described.*

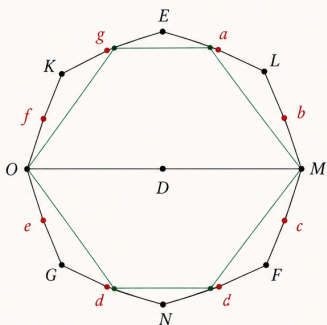
- ① Diameter² = 5 · radius² of circle from which icosahedron described.



$$XZ^2 = 5 \cdot OE^2$$

(diameter² = 5 · radius²)

- ② Diameter = hexagon side + 2 · decagon sides inscribed in same circle.



- EK = KL = LM = MF = FN = NG = GO = OK = hexagon side
 — a = b = c = d = e = f = g = decagon side

$$\text{Diameter } OM = EK + 2 \cdot a$$

(diameter = hexagon side + 2 · decagon sides)

PROPOSITION 17

To construct a dodecahedron, enclose it in a sphere, and show that the side of the dodecahedron is the irrational straight-line called an apotome.

Proof.

Panel 1 — Actions 1–8: the framework and the equilateral pentagon $UBWCV$.

1. Take two perpendicular planes of the cube of Prop. XIII.15: $ABCD$ and $CBEF$.
2. Bisect sides $AB, BC, CD, DA, EF, EB, FC$ at G, H, K, L, M, N, O .
3. Join GK, HL, MH, NO .
4. Cut NP, PO, HQ in extreme and mean ratio at R, S, T , with greater pieces RP, PS, TQ .
5. Draw RU, SV, TW on the exterior sides at right angles to the cube planes.
6. Make

$$RU = RP, \quad SV = PS, \quad TW = TQ.$$

7. Join UB, BW, WC, CV, VU .
8. To show pentagon $UBWCV$ is equilateral:

a. Join RB, SB, VB .

b. Since NP is cut in extreme and mean ratio at R , with RP greater,

$$PN^2 + NR^2 = 3 RP^2$$

by Prop. XIII.4.

c. Since $PN = NB$ and $PR = RU$,

$$BN^2 + NR^2 = 3 RU^2.$$

d. Since $BR^2 = BN^2 + NR^2$,

$$BR^2 = 3 RU^2.$$

e. Therefore

$$BU^2 = BR^2 + RU^2 = 4RU^2,$$

so $BU = 2RU$.

f. Also $VU = 2UR$, because $SR = 2PR = 2RU$; hence $BU = VU$.

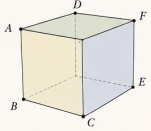
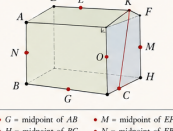
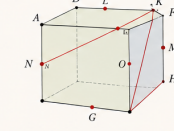
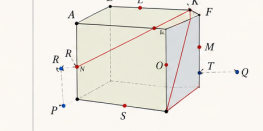
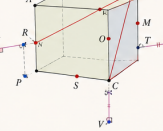
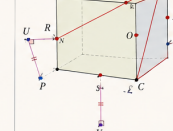
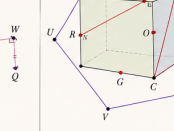
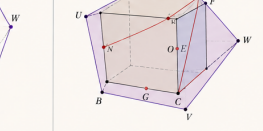
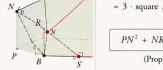
g. Similarly,

$$BW = WC = CV = BU = VU.$$

Thus pentagon $UBWCV$ is equilateral.

PROPOSITION 17 — *To construct a dodecahedron, enclose it in a sphere, and show that the side of the dodecahedron is the irrational straight-line called an apotome.*

PANEL 1: Actions 1–8 — The framework and the equilateral pentagon $UBWCV$

1 Take two perpendicular planes of the cube (Prop. 13.15): $ABCD$ and $CBEF$.  Planes $ABCD$ and $CBEF$ are perpendicular.	2 Bisect the sides AB, BC, CD, DA, EF, EB, FC at G, H, K, L, M, N, O.  G = midpoint of AB H = midpoint of BC K = midpoint of CD L = midpoint of DA M = midpoint of EF N = midpoint of EB O = midpoint of FC	3 Join GK, HL, MH, NO. 	4 Cut NP, PO, HQ in extreme ratio at R, S, T; greater pieces RP, PS, TQ.  $RP > RN$, $PS > PO$, $TQ > TH$.
5 Draw RU, SV, TW on exterior sides at right-angles to the cube planes. 	6 Make $RU = RP$, $SV = PS$, $TW = TQ$. 	7 Join UB, BW, WC, CV, VU. 	8 Show that pentagon $UBWCV$ is equilateral (all five sides equal). 
(a) Join RS, SR, VR. 	(b) NP cut in extreme ratio at R with RP greater: squares on $PN + NR = 3 \cdot \text{square } RP$ (Prop. 13.4). $PN^2 + NR^2 = 3 \cdot RP^2$ (Prop. 13.4)	(c) $PN + NR$, $PR = RU$, so squares on $BN + NR = 3 \cdot \text{square } RU$. $BN^2 + NR^2 = 3 \cdot RU^2$	(d) Square $BR = \text{squares } BN + NR$, so square $BR = 3 \cdot \text{square } RU$. $BR^2 = 3 \cdot RU^2$
(e) Square $BU = \text{square } BR + \text{square } RU = 4 \cdot \text{square } RU$, so $BU = 2 \cdot RU$. $BU^2 = 4 \cdot RU^2 \Rightarrow BU = 2 \cdot RU$	(f) $VU = 2 \cdot UR$ (because $SR = 2 \cdot PR = 2 \cdot RU$). So $BU = VU$.	(g) Similarly $BW = WC = CV = BU = VU$. Hence pentagon $UBWCV$ is equilateral. $UB = BW = WC = CV = VU$	(h) Similarly $BW = WC = CV = VU$. Hence pentagon $UBWCV$ is equilateral.

Panel 2 — Actions 9a–9d: planarity of pentagon $UBWCV$.

9. To show the pentagon is planar:

- a. Draw $PX \parallel RU$ and SV ; join XH, HW .
- b. Since HQ is cut in extreme and mean ratio at T ,

$$HQ : QT = QT : TH.$$

Also $HQ = HP$, and $QT = TW = PX$.

c. Hence

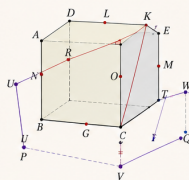
$$HP : PX = WT : TH,$$

while $HP \parallel TW$ and $TH \parallel PX$.

- d. Therefore triangles XPB and HTW have proportional sides with corresponding sides parallel; by Prop. VI.32, XB and HT are on the same straight line. Hence the five points U, B, W, C, V lie in one plane.

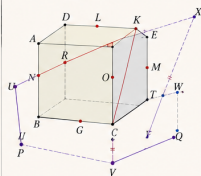
PROPOSITION 17 – Panel 2: Actions 9a–9d — Planarity of the pentagon UBWCV

- 1 Draw $PX \parallel RU$ and SV ;
join XH , HW .



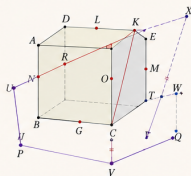
--- $PX \parallel RU \parallel SV$
— XH, HW joined

- 2 HQ cut in extreme ratio at T :
 $HQ : QT = QT : TH$.
Also $HQ = HP$, $QT = TW = PX$.



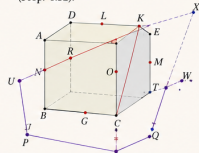
$HQ = HP$
 $QT = TW = PX$
 $HQ : QT = QT : TH$

- 3 Hence $HP : PX = WT : TH$.
Moreover $HP \parallel TW$ and
 $TH \parallel PX$.



$HP : PX = WT : TH$
 $HP \parallel TW$
 $TH \parallel PX$

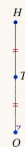
- 4 Thus triangles XPH and HTW
have proportional sides with
corresponding sides parallel;
therefore XH and HW are
on the same straight line
(Prop. 6.32).



X, H, W are collinear.
Hence pentagon $UBWCV$ is
planar.

- (a) HQ cut in extreme ratio at T :
 $HQ : QT = QT : TH$.

But $HQ = HP$ (opposite sides
of square $CDEF$),
and $QT = TW$ (given),
and $TW = PX$ (by Step 1).



$HQ : QT = QT : TH$
 $HP = HQ$
 $QT = TW = PX$

- (b) From $HQ : QT = QT : TH$,
substitute HP for HQ
and PX for QT , we get
 $HP : PX = PX : TH$.
But in extreme ratio,
if $a : b = b : c$ then $a : b = b : c$,
hence $HP : PX = WT : TH$.
Also $HP \parallel TW$ (both horizontal
edges), and $TH \parallel PX$ (drawn so).



$HP : PX = WT : TH$
 $HP \parallel TW, TH \parallel PX$

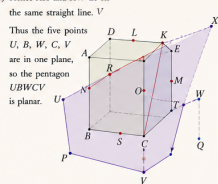
- (c) Triangles XPH and HTW :
 HP corresponds to WT ,
 PX corresponds to TH ,
and XH corresponds to HW .
We have

$HP : PX = WT : TH$
 $HP \parallel WT, PX \parallel TH$

Therefore by Prop. 6.32
(sides proportional and
parallel), the third sides
are in the same straight line.



- (d) Hence XH and HW lie on
the same straight line. V
Thus the five points
 U, B, W, C, V
are in one plane,
so the pentagon
 $UBWCV$
is planar.



Panel 3 — Actions 10a–10h: equiangularity of pentagon $UBWCV$.

10. To show the pentagon is equiangular:

a. Since NP is cut in extreme and mean ratio at R ,

$$SN : NP = NP : PS$$

by Prop. XIII.5.

b. Therefore

$$NS^2 + SP^2 = 3NP^2$$

by Prop. XIII.4.

c. Since $NP = NB$ and $PS = SV$,

$$NS^2 + SV^2 = 3NB^2.$$

d. Since $VB^2 = VS^2 + SB^2$, it follows that

$$VB^2 = 4NB^2,$$

so $VB = 2NB$.

e. But $BC = 2NB$; hence $VB = BC$.

f. From $BU = UV = BW = WC$ and $BV = BC$, triangles BUV and BWC are congruent by side-side-side; therefore

$$\angle BUV = \angle BWC.$$

g. Similarly,

$$\angle UVC = \angle BWC.$$

Thus three angles are equal.

h. An equilateral pentagon with three equal angles is equiangular by Prop. XIII.7.

PROPOSITION 17 – Panel 3: Actions 10a-10h — Equiangularity of the pentagon UBWCV

10a NP is cut in extreme ratio at R: $SV : NP = NP : PS$ (by Prop. 13.5). $SN : NP = NP : PS$	10b Squares on $NS + SP$ $= 3 \cdot \text{square } NP$ (Prop. 13.4). $NS^2 + SP^2 = 3 \cdot NP^2$	10c $NP = NB$, $PS = SV$; therefore squares $NS + SV = 3 \cdot \text{square } NB$. $NS^2 + SV^2 = 3 \cdot NB^2$	10d Square $VB = \text{square } VS$ $+ \text{square } SB$ $= 4 \cdot \text{square } NB$. $VB^2 = 4 \cdot NB^2$	10e $BC = 2 \cdot NB$; therefore $VB = BC$. $VB = BC$	10f $BU = UV = BW = WC$; (see Panel 1, Step 8), and $SV = BC$; thus $\angle BUV = \angle BWC$ (SSS). $\angle BUV = \angle BWC$	10g Similarly, $\angle UVC = \angle BWC$; thus $\angle BUV = \angle UVC = \angle BWC$. $\angle BUV = \angle UVC = \angle BWC$	10h An equilateral pentagon in which three angles are equal is equiangular (Prop. 13.7). Therefore pentagon $UBWCV$ is equiangular. Pentagon $UBWCV$ is equiangular.
(a) NP cut in extreme ratio at R with greater piece RP. By Prop. 13.5: $SN : NP = NP : PS$. $\frac{SN}{NP} = \frac{NP}{PS}$	(b) From Prop. 13.4, if $a : b = b : c$ (extreme ratio, b greater), then square on a + square on c $= 3 \cdot \text{square on } b$. $NS^2 + SP^2 = 3 \cdot NP^2$ (by Prop. 13.4)	(c) But $NP = NB$ (edges of cube), and $PS = SV$ (Step 6). Substitute to get $NS^2 + SV^2 = 3 \cdot NB^2$. From (c), $VS^2 + SB^2 = 3 \cdot NB^2$ and $SB = NB$. Hence $VB^2 = 4 \cdot NB^2$. $NS^2 + SV^2 = 3 \cdot NB^2$	(d) In right triangle VS $(\angle VSB = \angle VSK = 90^\circ)$: $VB^2 = VS^2 + SB^2$. From (c), $VS^2 + SB^2 = 3 \cdot NB^2$ and $SB = NB$. Hence $VB^2 = 4 \cdot NB^2$. $VB^2 = 4 \cdot NB^2$	(e) In the cube, G is midpoint of AB, so $AG = GB = \frac{1}{2}AB$. Also $NB = \frac{1}{2}AB$. But $BC = AB$, hence $BC = 2 \cdot NB$. From (d), $VB = 2 \cdot NB$, so $VB = BC$. $VB = BC$	(f) From Panel 1 (Step 8f-g) we have $BU = UV = BW = WC$. Also $SV = BC$ (Step 10e). Thus in triangles BUV and BWC the three sides are respectively equal (SSS). Therefore $\angle BUV = \angle BWC$. $\angle BUV = \angle BWC$	(g) Similarly, computing triangles CVU and CWB: $CV = WC$, $VU = WB$, $VC = CB$ (= BC). Hence $\angle UVC = \angle BWC$. Thus the three angles at U, V, W are equal. $\angle BUV = \angle UVC = \angle BWC$	(h) By Prop. 13.7: In an equilateral polygon, if three angles are equal, then all are equal. Therefore the equilateral pentagon $UBWCV$ is equiangular. Pentagon $UBWCV$ is equiangular.

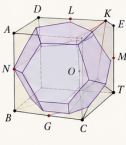
Panel 4 — Actions 11–12h: completion of the dodecahedron, enclosure in the sphere, and the apotome.

11. Construct the same figure on all twelve sides of the cube; the dodecahedron is thereby formed by twelve equilateral and equiangular pentagons.
12. Enclose it in the given sphere:
 - a. Extend PX to meet the cube diameter at Ω ; Ω is the center of the enclosing sphere.
 - b. ΩP is half of the cube's side.
 - c. Join $U\Omega$.
 - d. Since NP is cut in extreme and mean ratio at P with greater piece NP , NS is also cut in extreme and mean ratio at P , with greater piece NP , by Prop. XIII.5.
 - e. Hence UV is the greater piece of NS cut in extreme and mean ratio at P .
 - f. The diameter of the sphere is rational, and its square is three times the square on the cube side; hence NO is rational.
 - g. If a rational straight-line is cut in extreme and mean ratio, each piece is an apotome by Prop. XIII.6.
 - h. Therefore UV , the side of the dodecahedron, is an apotome.

Therefore, the dodecahedron is enclosed in the given sphere, and its side UV is an apotome.

PROPOSITION 17 – Panel 4: Actions 11–12b — Construction of the dodecahedron and enclosure in a sphere; the side is an apotome.

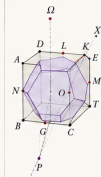
- 11 Construct the same figure on all 12 sides of the cube. The dodecahedron is formed by 12 equilateral and equiangular pentagons.



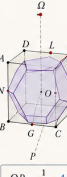
Dodecahedron with 12 equilateral, equiangular pentagons.

- 12 Enclose it in the given sphere and show that the side is an apotome.

1. Extend PX to meet the cube diameter BD produced at Ω . Ω is the center of the enclosing sphere.

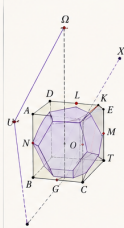


2. Since Ω is the center of the sphere, $\Omega P = \frac{1}{2}$ of the side of the cube.

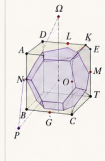


$$\Omega P = \frac{1}{2} \cdot AB$$

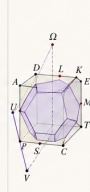
3. Join $U\Omega$.



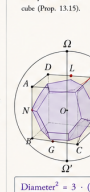
4. NP cut in extreme ratio at P with greater piece NP . Therefore NS is also cut in extreme ratio at P with greater NP (Prop. 13.5).



5. Thus UV is the greater piece of NS cut in extreme ratio at P .

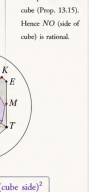


6. Diameter of the sphere (rational line). Square on diameter = 3 · square on side of cube (Prop. 13.15).

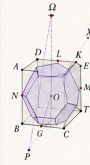


$$\text{Diameter}^2 = 3 \cdot (\text{cube side})^2 \Rightarrow NO \text{ is rational.}$$

7. Diameter of the sphere (rational line). Square on diameter = 3 · square on side of cube (Prop. 13.15). Hence NO (side of cube) is rational.

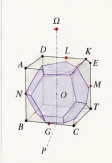


- (a) Extend PX to meet BD produced at Ω .



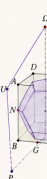
Ω is the center of the sphere.

- (b) Ω is the center of the sphere. Therefore $\Omega P = \frac{1}{2} AB$.

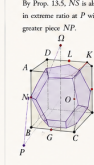


$$\Omega P = \frac{1}{2} \cdot AB$$

- (c) Join $U\Omega$.

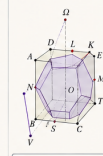


- (d) NP cut in extreme ratio at P with greater piece NP . By Prop. 13.5, NS is also cut in extreme ratio at P with greater piece NP .



$$SN : NP = NP : PS$$

- (e) Therefore UV is the greater piece of NS cut in extreme ratio at P .



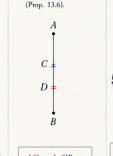
$$UV = \text{greater piece of } NS \text{ in extreme ratio at } P.$$

- (f) Diameter of sphere rational. By Prop. 13.15, $(\text{diameter})^2 = 3 \cdot (\text{cube side})^2$. Hence cube side NO is rational.



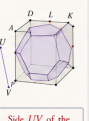
$$\text{Diameter}^2 = 3 \cdot (\text{cube side})^2 \Rightarrow NO \text{ is rational.}$$

- (g) If a rational straight line is cut in extreme ratio, then each piece is an apotome. (Prop. 13.6).

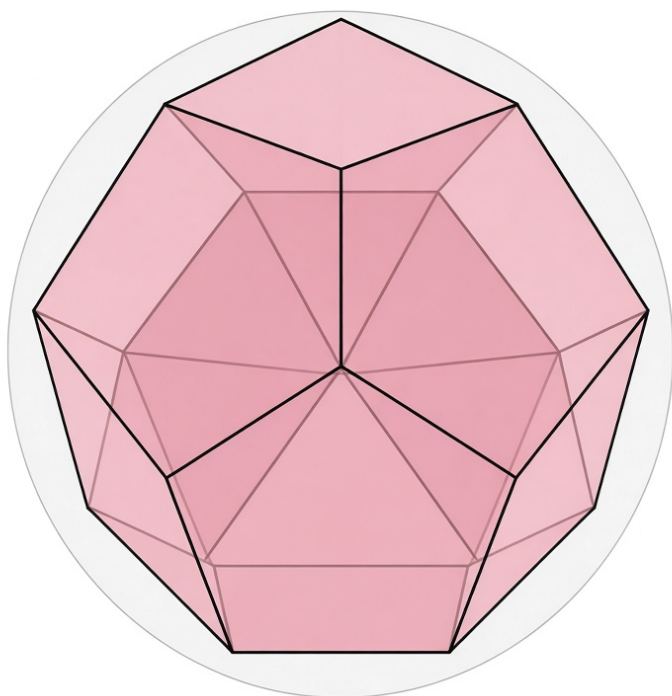


AC and CB are apotomes.

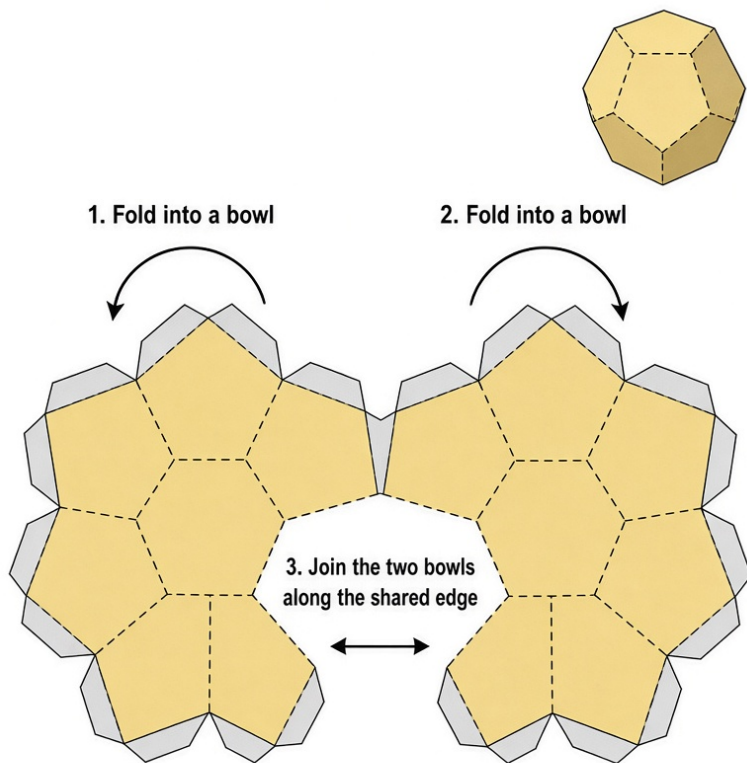
- (h) Since NS (a rational line) is cut in extreme ratio at P , therefore each piece NP and PS is an apotome. In particular, UV (equal to NP) is an apotome.



Side UV of the dodecahedron is an apotome.



DODECAHEDRON — 12 regular pentagons



Cut solid lines • Fold dashed lines • Glue tabs

Corollary to Proposition 17

The side of the dodecahedron is the greater piece of the cube's side when that side is cut in extreme and mean ratio.

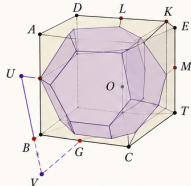
Since NS is cut in extreme and mean ratio at P , with NP the greater piece, and the constructed side of the dodecahedron satisfies $UV = NP$, the side UV is exactly the greater piece of the cube's side under the corresponding extreme-and-mean-ratio division.

$$NS : UV = UV : PS.$$

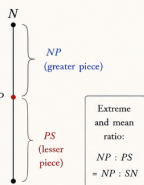
Therefore, the side of the dodecahedron is the greater piece of the cube's side when the latter is cut in extreme and mean ratio.

COROLLARY. *Side of the dodecabedron is the greater piece of the cube's side when cut in extreme and mean ratio.*

- 1 From Prop. 17 (Panel 4, Step 12c): side UV of the dodecabedron is the greater piece of NS cut in extreme and mean ratio at P .

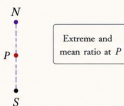


- 2 Thus NS is cut in extreme and mean ratio at P with greater piece NP .

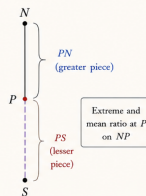


- 3 By Prop. 13.5: If a straight line is cut in extreme and mean ratio at a point, each of the pieces is also cut in extreme and mean ratio at that point.

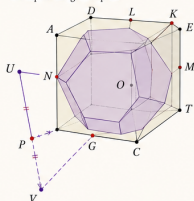
Therefore NS cut in extreme and mean ratio at P implies NP cut in extreme and mean ratio at P .



- 4 Hence NP is cut in extreme and mean ratio at P , with PN the greater piece.

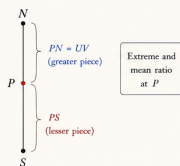


- 5 But $UV = PN$ (Step 6 of Panel 4): the constructed perpendicular on the exterior side equals the greater piece.



$$UV = PN \\ \text{(greater piece of } NP) \\ = NP$$

- 6 Therefore the side UV of the dodecabedron is the greater piece of the cube's side NS when NS is cut in extreme and mean ratio at P .



COROLLARY.

The side of the dodecabedron is the **greater piece** of the cube's side when that side is cut in **extreme and mean ratio**.

(Where "extreme and mean ratio" means: a straight line l is cut at P so that the whole is to the greater piece as the greater piece is to the lesser piece: $l : PN = PN : PS$.)

PROPOSITION 18

To set out the sides of the five figures, and to compare them with one another.

Proof.

Panel 1 — Actions 1–7: tetrahedron, cube, and octahedron.

1. Let AB be the diameter of the given sphere.
2. Cut at C so that $AC = CB$, and at D so that $AD = 2 DB$.
3. Draw semicircle AEB ; draw CE and DF perpendicular to AB .
4. Join AF, FB, EB .
5. Tetrahedron side AF :

- a. Since $AD = 2 DB$, $AB = 3 BD$; by conversion, $BA = \frac{3}{2} AD$.
- b. From the similar triangles AFB and AFD ,

$$BA : AD = BA^2 : AF^2.$$

- c. Therefore

$$BA^2 = \frac{3}{2} AF^2.$$

- d. By Prop. XIII.13, diameter squared is one and a half times tetrahedron side squared; hence AF is the tetrahedron side.

6. Cube side BF :

- a. Since $AD = 2 DB$, $AB = 3 BD$.
- b. Hence

$$AB : BD = AB^2 : BF^2.$$

- c. Therefore

$$AB^2 = 3 BF^2.$$

- d. By Prop. XIII.15, diameter squared is three times cube side squared; hence BF is the cube side.

7. Octahedron side BE :

- a. Since $AC = CB$, $AB = 2 BC$.

b. Hence

$$AB : BC = AB^2 : BE^2.$$

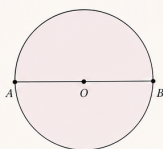
c. Therefore

$$AB^2 = 2 BE^2.$$

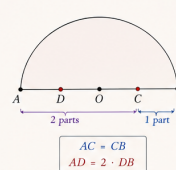
d. By Prop. XIII.14, diameter squared is twice octahedron side squared; hence BE is the octahedron side.

PROPOSITION 18 – Panel 1: Actions 1–7. Setting out the sides of the tetrahedron, cube, and octahedron on a given sphere.

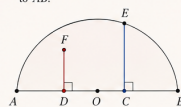
- 1 Let AB be the diameter of the given sphere.



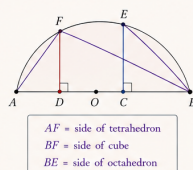
- 2 Cut at C such that $AC = CB$; at D such that $AD = 2 \cdot DB$.



- 3 Draw the semicircle AEB . Draw CE and DF perpendicular to AB .



- 4 Join AF , FB , and EB .



5–7. Identification of the sides of the three solids.

- 5 TETRAHEDRON SIDE AF .

- (a) $AD = 2 \cdot DB$, hence $AB = 3 \cdot BD$.
By conversion, $BA = 1\frac{1}{2} \cdot AD$.
(b) In similar triangles AFB and AED ,
 $BA : AD = \text{square } AB : \text{square } AF$.
(c) Therefore square $BA = 1\frac{1}{2} \cdot \text{square } AF$.
(d) But for a tetrahedron,
 $\text{diameter}^2 = 1\frac{1}{2} \cdot \text{side}^2$
(Prop. 13.13).
Hence AF is the side of the tetrahedron.

TETRAHEDRON:

$$\text{Diameter}^2 = 1\frac{1}{2} \cdot \text{side}^2 \Rightarrow AF = \text{side}.$$

- 6 CUBE SIDE BF .

- (a) $AD = 2 \cdot DB$, hence $AB = 3 \cdot BD$.
(b) In similar triangles FDB and FAB ,
 $AB : BD = \text{square } AB : \text{square } BF$.
(c) Therefore square $AB = 3 \cdot \text{square } BF$.
(d) But for a cube,
 $\text{diameter}^2 = 3 \cdot \text{side}^2$
(Prop. 13.15).
Hence BF is the side of the cube.

CUBE:

$$\text{Diameter}^2 = 3 \cdot \text{side}^2 \Rightarrow BF = \text{side}.$$

- 7 OCTAHEDRON SIDE BE .

- (a) $AC = CB$, hence $AB = 2 \cdot BC$.
(b) In similar triangles CEB and CBA ,
 $AB : BC = \text{square } AB : \text{square } BE$.
(c) Therefore square $AB = 2 \cdot \text{square } BE$.
(d) But for an octahedron,
 $\text{diameter}^2 = 2 \cdot \text{side}^2$
(Prop. 13.14).
Hence BE is the side of the octahedron.

OCTAHEDRON:

$$\text{Diameter}^2 = 2 \cdot \text{side}^2 \Rightarrow BE = \text{side}.$$

On the given semicircle AEB with $AD = 2 \cdot DB$ and $AC = CB$, we have set out:

AF = side of tetrahedron, BF = side of cube, BE = side of octahedron.

Panel 2 — Action 8a–8n: construction of the icosahedron side MB .

8. Icosahedron side MB :

a. Draw $AG \perp AB$ with $AG = AB$; join GC .

b. From H draw $HK \perp AB$.

c. Since $GA = 2 AC$, and by Prop. VI.4

$$GA : AC = HK : KC,$$

we obtain $HK = 2 KC$.

d. Therefore $HK^2 = 4 KC^2$, and

$$HC^2 = HK^2 + KC^2 = 5 KC^2.$$

e. Since $HC = CB$,

$$BC^2 = 5 CK^2.$$

f. Since $AB = 2 BC$ and $AD = 2 DB$, we have $BD = 2 DC$, and hence $BC = 3 CD$.

g. Therefore

$$BC^2 = 9 CD^2.$$

h. But $BC^2 = 5 CK^2$; hence $CK > CD$.

i. Let $CL = CK$; draw $LM \perp AB$, and join MB .

j. Since $BC^2 = 5 CK^2$, $AB = 2 BC$, and $KL = 2 CK$,

$$AB^2 = 5 KL^2.$$

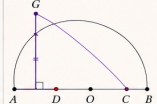
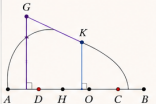
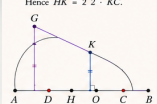
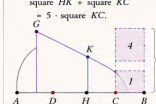
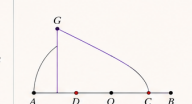
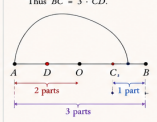
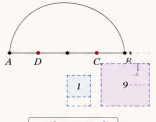
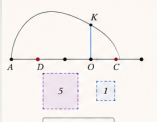
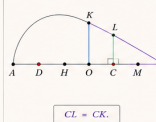
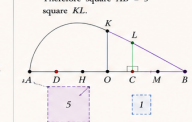
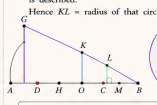
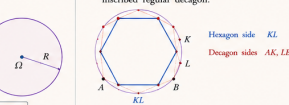
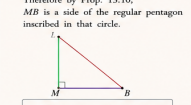
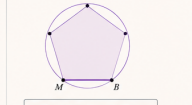
k. By the corollary to Prop. XIII.16, diameter squared is five times the square on the radius of the circle from which the icosahedron is described; hence KL is that radius.

l. KL is the hexagon side, while AK and LB are decagon sides.

m. Since LB is a decagon side, ML a hexagon side, and the angle at L is right, MB is a pentagon side by Prop. XIII.10.

n. Therefore MB is the side of the icosahedron.

PROPOSITION 18 – Panel 2: Action 8a–8n. Determination of MB , the side of the icosahedron.

8a Draw $AG \perp AB$ with $AG = AB$. Join GC.  $AG = AB$	8b From H (the midpoint of AC), draw $HK \perp AB$. 	8c Since $GA = 2 \cdot AC$ ($GA = AB$ and $AC = \frac{1}{2} AB$), by Prop. 6.4, $GA : AC = HK : KC$. Hence $HK = 2 \cdot KC$.  $HK = 2 \cdot KC$	8d Square on HK is 4 times square on KC. Square on HC is square HK + square KC = 5 · square KC.  $HC^2 = 5 \cdot KC^2$	8e But $HC = CB$ (since H is midpoint of AC and $AC = CB$). Hence square $BC^2 = 5 \cdot \text{square } KC$.  $BC^2 = 5 \cdot KC^2$
8f Since $AB = 2 \cdot BC$ (as $AC = CB$) and $AD = 2 \cdot DB$, we have $BD = 2 \cdot DC$. Thus $BC = 3 \cdot CD$.  $BC^2 = 9 \cdot CD^2$	8g Therefore square $BC^2 = 9 \cdot \text{square } CD$.  $BC^2 = 9 \cdot CD^2$	8h But square $BC^2 = 5 \cdot \text{square } KC$. Therefore square $KC < \text{square } CD$ and hence $KC > CD$.  $KC > CD$	8i Let $CL = CK$. From L draw $LM \perp AB$. Join MB.  $CL = CK$	8j Square $BC^2 = 5 \cdot \text{square } KC$. But $AB = 2 \cdot BC$ and $KL = 2 \cdot CK$ (since $CL = CK$). Therefore square $AB^2 = 5 \cdot \text{square } KL$.  $AB^2 = 5 \cdot KL^2$
8k By the Corollary to Prop. 13.16, diameter² = 5 · radius² of the circle from which the icosahedron is described. Hence $KL = R$ (radius of circle of icosahedron).  $KL = R$ (radius of circle of icosahedron).	8l In that circle, KL is the side of the inscribed regular hexagon (one side); and AK and LB are sides of the inscribed regular decagon.  Hexagon side KL Decagon sides AK, LB	8m Since $LM \perp AB$ (hexagon side) and LB is a decagon side, the triangle LBM is right-angled with ML a hexagon side. Therefore by Prop. 13.10, MB is a side of the regular pentagon inscribed in that circle.  MB = side of regular pentagon.	8n Since the pentagon LBM is inscribed in the same circle as the icosahedron, MB is a side of the icosahedron.  MB = side of icosahedron.	

Panel 3 — Actions 9–10: dodecahedron side and rational comparisons.

9. Dodecahedron side NB :

- a. Cut FB , the cube side, in extreme and mean ratio at N , with NB the greater piece.
- b. By the corollary to Prop. XIII.17, NB is the side of the dodecahedron.

10. Comparisons:

- a. Let the square on the diameter be six equal parts. Then

$$\text{tetrahedron side}^2 = 4,$$

$$\text{octahedron side}^2 = 3,$$

$$\text{cube side}^2 = 2.$$

- b. Therefore

$$\text{tetrahedron}^2 : \text{octahedron}^2 = 4 : 3,$$

$$\text{tetrahedron}^2 : \text{cube}^2 = 4 : 2 = 2 : 1,$$

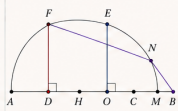
and

$$\text{octahedron}^2 : \text{cube}^2 = 3 : 2.$$

- c. Thus these three sides are in rational ratios to one another.
- d. The icosahedron and dodecahedron sides are irrational straight-lines, respectively a minor and an apotome, and therefore are not in rational ratios with the other three.

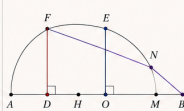
PROPOSITION 18 – Panel 3: Actions 9 and 10. Dodecahedron side NB and comparison of the sides.

9a Cut FB (the side of the cube) in extreme ratio at N , the greater piece being NB .



Extreme ratio at N :
 $FN : NB = NB : BF$
(NB is the greater piece.)

9b By the Corollary to Prop. 13.17, NB is the side of the dodecahedron.



$NB = \text{side of dodecahedron.}$

10 Comparisons of the three rational sides.

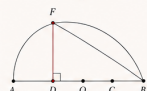
10a Let diameter² be 6 parts.

Diameter² = 6 parts.

10b (i) Tetrahedron.

Diameter² = $1\frac{1}{2} \cdot \text{side}^2$.

Hence side² = $\frac{2}{3} \cdot 6 = 4$ parts.

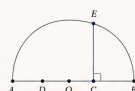


Tetrahedron side² = 4 parts.

10c (ii) Octahedron.

Diameter² = $2 \cdot \text{side}^2$.

Hence side² = $\frac{1}{2} \cdot 6 = 3$ parts.

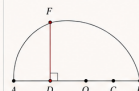


Octahedron side² = 3 parts.

10d (iii) Cube.

Diameter² = $3 \cdot \text{side}^2$.

Hence side² = $\frac{1}{3} \cdot 6 = 2$ parts.



Cube side² = 2 parts.

10e Therefore,

10e (i) Tetrahedron : Octahedron

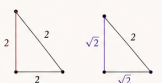
$4 : 3 = 1\frac{1}{3}$.



Tetrahedron side² : Octahedron side² = $4 : 3$.
(Side ratio = $1\frac{1}{3}$.)

10e (ii) Tetrahedron : Cube

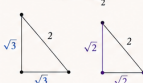
$4 : 2 = 2 : 1$.



Tetrahedron side² : Cube side² = $4 : 2 = 2 : 1$.
(Side ratio = $2 : 1$.)

10e (iii) Octahedron : Cube

$3 : 2 = 1\frac{1}{2}$.



Octahedron side² : Cube side² = $3 : 2$.
(Side ratio = $1\frac{1}{2}$.)

10f Thus the sides of the tetrahedron, octahedron, and cube are in rational ratios to one another.

10g But the side MB of the icosahedron (minor) and the side NB of the dodecahedron (apotome) are irrational straight-lines (Prop. 13.6 and Prop. 13.11 Cor.). Therefore they are not in rational ratio to the others, nor to one another.

MB = minor (irrational).
 NB = apotome (irrational).
Irrational sides: not in rational ratio to the rational sides, nor to one another.

SUMMARY OF SQUARED SIDES
(when diameter² = 6 parts)

Tetrahedron side² = 4 parts

Octahedron side² = 3 parts

Cube side² = 2 parts

RATIOS OF SQUARED SIDES

Tetra : Octa = $4 : 3 = 1\frac{1}{3}$

Tetra : Cube = $4 : 2 = 2 : 1$

Octa : Cube = $3 : 2 = 1\frac{1}{2}$

Irrational sides: MB (minor)
 NB (apotome)

Not in rational ratio to the others.

Panel 4 — Actions 11a–11g: first comparison of the remaining two sides.

11. To show that the icosahedron side MB is greater than the dodecahedron side NB :

a. Triangles FDB and FAB are equiangular by Prop. VI.8.

b. Therefore, by Prop. VI.4,

$$DB : BF = BF : BA.$$

c. For three proportionals,

$$DB : BA = DB^2 : BF^2.$$

d. By inversion,

$$AB : BD = FB^2 : BD^2.$$

e. Since $AB = 3 BD$,

$$FB^2 = 3 BD^2.$$

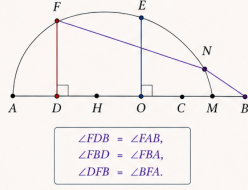
f. Since $AD = 2 DB$,

$$AD^2 = 4 DB^2.$$

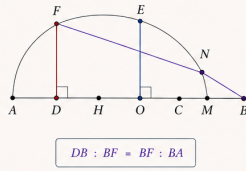
g. Therefore $AD^2 > FB^2$, so $AD > FB$, and much more $AL > FB$.

PROPOSITION 18 – Panel 4: Action 11a–11g. From similar triangles to $AD > FB$ and $AL > FB$.

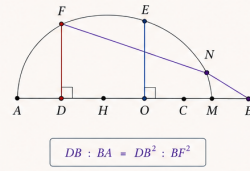
- 11a Triangles FDB and EAB are equiangular (Prop. 6.8).



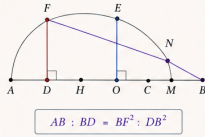
- 11b Therefore by Prop. 6.4 (similar triangles), $DB : BF = BF : BA$.



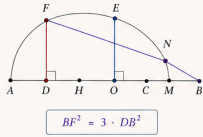
- 11c Therefore, by the property of three proportionals, $DB : BA = \text{square } DB : \text{square } BF$.



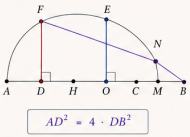
- 11d By inversion of the proportion (Prop. 5.16), $AB : BD = \text{square } BF : \text{square } DB$.



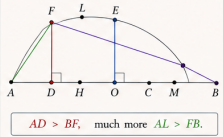
- 11e But $AB = 3 \cdot BD$ (since $AD = 2 \cdot DB$, so $AB = AD + DB = 3 \cdot DB$). Hence square $BF = 3 \cdot \text{square } DB$.



- 11f Also $AD = 2 \cdot DB$. Therefore square $AD = 4 \cdot \text{square } DB$.



- 11g Since $AD^2 = 4 \cdot DB^2$ and $BF^2 = 3 \cdot DB^2$, we have $AD^2 > BF^2$. Therefore $AD > BF$. And much more, $AL > FB$.



From similarity we obtain that $AD > BF$, and because L lies on the arc on the side of A , the chord AL (from A to L) is much longer than FB . This prepares the comparison of the greater pieces in extreme ratio (in the next panel).

Panel 5 — Actions 11h–11l: comparison of the greater pieces.

Continuing Action 11:

- h. If AL is cut in extreme and mean ratio, KL is the greater piece, with KL the hexagon side and KA the decagon side.
- i. If FB is cut in extreme and mean ratio, NB is the greater piece.
- j. Since $AL > FB$, it follows that $KL > NB$. But $KL = LM$, so $LM > NB$.
- k. Since $MB > LM$, we have

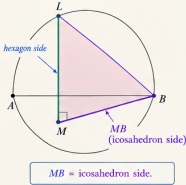
$$MB > NB.$$

- l. Thus the side of the icosahedron is greater than the side of the dodecahedron.

Therefore, the five sides have been set out and compared; in particular, the icosahedron side is greater than the dodecahedron side.

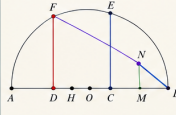
PROPOSITION 18 – Panel 5: Action 11h–11l. Showing that the icosahedron side MB is greater than the dodecahedron side NB .

8h Since $LM \parallel AB$ (hexagon side) and LB is a decagon side, the triangle LBM is right-angled with ML a hexagon side. Therefore by Prop. 13.10, MB is a side of the regular pentagon inscribed in that circle.



MB = icosahedron side.

9a Cut FB (the side of the cube) in extremest ratio at N , the greater piece being NB .



NB = dodecahedron side.
(Prop. 13.17 cor.)

10a Comparisons of the three rational sides. Let diameter² be 6 parts.

Tetrahedron side² = 4 parts
(since diameter² = $1\frac{1}{2}$ · side²)
Octahedron side² = 3 parts
(diameter² = 2 · side²)
Cube side² = 2 parts
(diameter² = 3 · side²)

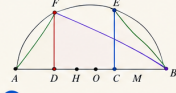
10b Therefore:

Tetrahedron² : Octahedron² = $4 : 3$ ($1\frac{1}{2}$)
Tetrahedron² : Cube² = $4 : 2 = 2 : 1$
Octahedron² : Cube² = $3 : 2$ ($1\frac{1}{2}$)

10c These three sides are in rational ratios.

10d Icosahedron and dodecahedron sides are irrational (minor and apotome), so not in rational ratios.

11a Triangles FDB and FAB are equiangular (Prop. 6.8).



11b $DB : BF = BF : BA$ (Prop. 6.4).

11c $DB : BA = \text{square } DB : \text{square } BF$.

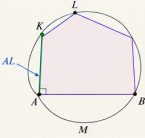
11d By inversion: $AB : BD = \text{square } FB : \text{square } BD$.

11e $AB = 3 \cdot BD$, so square $FB = 3 \cdot \text{square } BD$.

11f Square $AD = 4 \cdot \text{square } DB$ ($AD = 2 \cdot DB$).

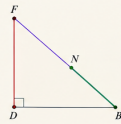
11g Therefore square $AD > \text{square } FB$, so $AD > FB$, much more $AL > FB$.

11h If AL is cut in extreme ratio, KL is the greater piece.



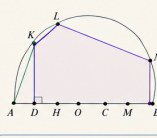
KL = hexagon side
 KA = decagon side.

11i If FB is cut in extreme ratio, NB is the greater piece.



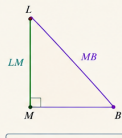
NB = greater piece.
(Prop. 13.17 cor.)

11j Therefore $KL > NB$; $KL = LM$, $LM > NB$.



$KL > NB$ and $LM = KL \Rightarrow LM > NB$.

11k $MB > LM$, so $MB > NB$.



$MB > LM > NB$.

11l Thus the side of the icosahedron is greater than the side of the dodecahedron.

Therefore,
 $MB > NB$
side of icosahedron
is greater than
side of dodecahedron.

FINAL PROPOSITION — CLASSIFICATION OF THE REGULAR SOLIDS

No other solid figure can be constructed besides the five regular solids.

Proof.

A solid angle requires at least three plane angles whose sum is less than four right angles, that is, less than 360° .

Equilateral triangles. Each plane angle is 60° . Three, four, or five may meet:

$$3 \times 60^\circ = 180^\circ < 360^\circ,$$

giving the tetrahedron;

$$4 \times 60^\circ = 240^\circ < 360^\circ,$$

giving the octahedron; and

$$5 \times 60^\circ = 300^\circ < 360^\circ,$$

giving the icosahedron. Six cannot meet to form a solid angle, because

$$6 \times 60^\circ = 360^\circ.$$

Squares. Each plane angle is 90° . Three may meet:

$$3 \times 90^\circ = 270^\circ < 360^\circ,$$

giving the cube. Four cannot form a solid angle, because

$$4 \times 90^\circ = 360^\circ.$$

Regular pentagons. Each plane angle is 108° . Three may meet:

$$3 \times 108^\circ = 324^\circ < 360^\circ,$$

giving the dodecahedron.

Regular hexagons and higher polygons. Each hexagon angle is 120° , so three give

$$3 \times 120^\circ = 360^\circ,$$

and no solid angle is formed. Every regular polygon with more than six sides has an interior angle greater than 120° ; hence three such angles have sum greater than 360° , and no solid angle can be formed.

Therefore there exist exactly five regular solids:

1. tetrahedron,
2. cube,
3. octahedron,
4. dodecahedron,
5. icosahedron.

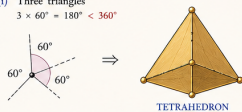
Therefore, no other regular solid figure can be constructed besides these five.

FINAL PROPOSITION (SUMMARY) – NO OTHER SOLID FIGURE CAN BE CONSTRUCTED BESIDES THE FIVE REGULAR SOLIDS

In every solid angle at a vertex, let plane angles meet. The sum of the angles must be less than four right angles (360°).

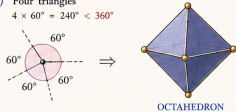
1. EQUILATERAL TRIANGLES (each angle = 60°) At a vertex they can meet 3, 4, or 5.

(i) Three triangles
 $3 \times 60^\circ = 180^\circ < 360^\circ$



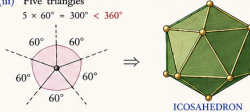
TETRAHEDRON

(ii) Four triangles
 $4 \times 60^\circ = 240^\circ < 360^\circ$



OCTAHEDRON

(iii) Five triangles
 $5 \times 60^\circ = 300^\circ < 360^\circ$



ICOSAHEDRON

2. SQUARES (each angle = 90°)

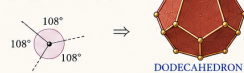
Three squares
 $3 \times 90^\circ = 270^\circ < 360^\circ$



CUBE

3. PENTAGONS (each angle = 108°)

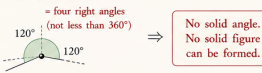
Three pentagons
 $3 \times 108^\circ = 324^\circ < 360^\circ$



DODECAHEDRON

4. HEXAGONS (each angle = 120°)

Three hexagons
 $3 \times 120^\circ = 360^\circ$



5. POLYGONS WITH MORE THAN SIX SIDES

The angle of a regular polygon increases as the number of sides increases.

Heptagon (7 sides)
each angle = $128\frac{1}{2}^\circ$



$$3 \times 128\frac{1}{2}^\circ = 385\frac{1}{2}^\circ > 360^\circ$$

Octagon (8 sides)
each angle = 135°



$$3 \times 135^\circ = 405^\circ > 360^\circ$$

Nonagon (9 sides)
each angle = 140°



$$3 \times 140^\circ = 420^\circ > 360^\circ$$

Decagon (10 sides)
each angle = 144°



$$3 \times 144^\circ = 432^\circ > 360^\circ$$

For any regular polygon with n sides
($n \geq 7$), each angle is

$$\frac{(n-2) \cdot 180^\circ}{n} > 120^\circ.$$

Therefore, three or more such angles
have sum $\geq 360^\circ$,
so no solid angle can be formed.

CONCLUSION

The only regular polygons
that can form a solid angle
(sum of angles $< 360^\circ$) are:

- Equilateral triangle
- Square
- Regular pentagon

Therefore, there exist
exactly five regular
solids.

- Tetrahedron (4 faces)
- Cube (6 faces)
- Octahedron (8 faces)
- Dodecahedron (12 faces)
- Icosahedron (20 faces)



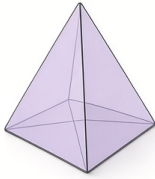
No other solid figure can be constructed besides the five regular solids.



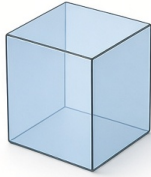
EUCLID ELEMENTS • BOOK XIII



THE FIVE PLATONIC SOLIDS



1. TETRAHEDRON
4 TRIANGLES



2. CUBE
6 SQUARES



3. OCTAHEDRON
8 TRIANGLES



4. DODECAHEDRON
12 PENTAGONS



5. ICOSAHEDRON
20 TRIANGLES

End of Book XIII