

EUCLID'S ELEMENTS

BOOK XII

Solid Geometry and the Method of Exhaustion

Propositions 1–18

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BOOK XII

PROPOSITION 1

Statement. Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.

Proof.

1. Let there be two circles, ABC and FGH , with similar polygons $ABCDE$ and $FGHKL$ inscribed in them respectively. Let BM and GN be the diameters of the circles.
2. Join BE , AM , GL , and FN .
3. Since the polygons are similar, $\angle BAE = \angle GFL$, and

$$\frac{BA}{AE} = \frac{GF}{FL}.$$

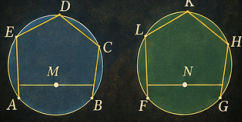
4. Therefore triangle ABE is equiangular with triangle FGL .
5. Thus $\angle AEB = \angle FLG$.
6. But $\angle AEB = \angle AMB$ (same circumference AB), and $\angle FLG = \angle FNG$ (same circumference FG). Hence $\angle AMB = \angle FNG$.
7. Also, $\angle BAM$ is a right angle, being in a semicircle, and equals $\angle GFN$, which is also a right angle.
8. Therefore triangle ABM is equiangular with triangle FGN .
9. Consequently,

$$\frac{BM}{GN} = \frac{BA}{GF}.$$

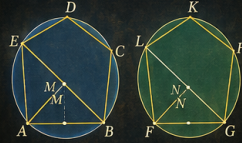
10. The ratio of the square on BM to the square on GN is the square of the ratio BM/GN .
11. The ratio of polygon $ABCDE$ to polygon $FGHKL$ is the square of the ratio BA/GF .
12. Therefore, as the square on BM is to the square on GN , so polygon $ABCDE$ is to polygon $FGHKL$.

Therefore. Similar polygons inscribed in circles are to one another as the squares on the diameters.

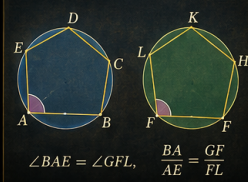
- 1** Let there be two circles, ABC and FGH, with similar polygons ABCDE and FGHL inscribed in them respectively. Let BM and GN be the diameters of the circles.



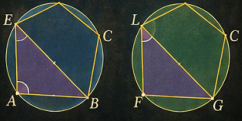
- 2** Join BE, AM, GL, and FN.



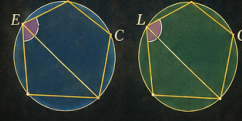
- 3** Since the polygons are similar, angle BAE equals angle GFL, and $\frac{BA}{AE} = \frac{GF}{FL}$.



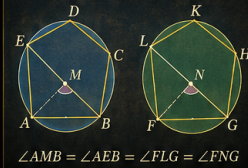
- 4** Therefore, triangle ABE is equiangular with triangle FGL.



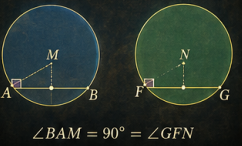
- 5** Thus, angle AEB is equal to angle FLG.



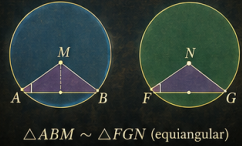
- 6** But angle AEB equals angle AMB (same circumference AB), and angle FLG equals angle FNG (same circumference FG). Hence, angle AMB equals angle FNG.



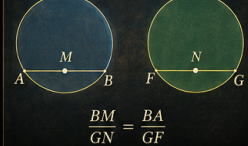
- 7** Also, angle BAM is a right angle (in a semicircle) and equals angle GFN, which is also a right angle.



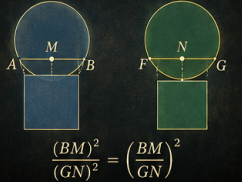
- 8** Therefore, triangle ABM is equiangular with triangle FGN.



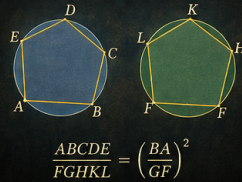
- 9** Consequently, $\frac{BM}{GN} = \frac{BA}{GF}$.



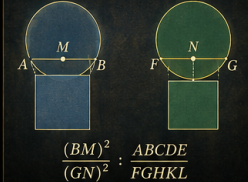
- 10** The ratio of the square on BM to the square on GN is the square of the ratio $\frac{BM}{GN}$.



- 11** The ratio of polygon ABCDE to polygon FGHL is the square of the ratio $\frac{BA}{GF}$.



- 12** Therefore, as the square on BM is to the square on GN, so polygon ABCDE is to polygon FGHL.



PROPOSITION 1. SIMILAR POLYGONS (INSCRIBED) IN CIRCLES ARE TO ONE ANOTHER AS THE SQUARES ON THE DIAMETERS (OF THE CIRCLES).

PROPOSITION 2

Statement. Circles are to one another as the squares on their diameters.

Proof.

1. Let $ABCD$ and $EFGH$ be two circles with diameters BD and FH .
2. Assume the contrary: as the square on BD is to the square on FH , so circle $ABCD$ is to some area S , where S is either less than or greater than circle $EFGH$. First, let S be a lesser area.
3. Inscribe square $EFGH$ in circle $EFGH$.
4. Establish that the inscribed square $EFGH$ is greater than half of circle $EFGH$.
5. Bisect the circumferences EF, FG, GH, HE at K, L, M, N , and join the chords.
6. Show that each of the triangles EKF, FLG, GMH, HNE is greater than half of its corresponding circular segment.
7. By repeatedly bisecting the remaining circumferences and inscribing triangles, leave some segments of circle $EFGH$ whose sum is less than the excess by which circle $EFGH$ exceeds area S .
8. Therefore the remaining polygon $EKFLGMHN$ is greater than area S .
9. Inscribe in circle $ABCD$ a polygon $AOBPCQDR$ similar to polygon $EKFLGMHN$.
10. By Proposition XII.1,

$$\frac{BD^2}{FH^2} = \frac{\text{polygon } AOBPCQDR}{\text{polygon } EKFLGMHN}.$$

11. By the assumption,

$$\frac{BD^2}{FH^2} = \frac{\text{circle } ABCD}{S}.$$

12. Therefore

$$\frac{\text{circle } ABCD}{S} = \frac{\text{polygon } AOBPCQDR}{\text{polygon } EKFLGMHN}.$$

13. Alternately,

$$\frac{\text{circle } ABCD}{\text{polygon } AOBPCQDR} = \frac{S}{\text{polygon } EKFLGMHN}.$$

14. But circle $ABCD$ is greater than the polygon within it, so S must be greater than polygon $EKFLGMHN$.

15. However, S is less than polygon $EKFLGMHN$, from steps 7 and 8. This is a contradiction.

16. Therefore the square on BD is not to the square on FH as circle $ABCD$ is to a lesser area than circle $EFGH$.

17. Similarly, the square on FH is not to the square on BD as circle $EFGH$ is to a lesser area than circle $ABCD$.

18. Now assume S is a greater area than circle $EFGH$.

19. Inversely,

$$\frac{FH^2}{DB^2} = \frac{S}{\text{circle } ABCD}.$$

20. By the lemma, as S is to circle $ABCD$, so circle $EFGH$ is to some area less than circle $ABCD$.

21. Thus

$$\frac{FH^2}{DB^2} = \frac{\text{circle } EFGH}{\text{a lesser area than circle } ABCD},$$

which contradicts what has already been shown.

22. Therefore

$$\frac{BD^2}{FH^2} = \frac{\text{circle } ABCD}{\text{circle } EFGH}.$$

Therefore. Circles are to one another as the squares on their diameters.

1 Let ABCD and EFGH be two circles with diameters BD and FH.

2 Assume the contrary: as the square on BD is to the square on FH, so circle ABCD is to some area S, where S is either less than or greater than circle EFGH. First, let S be a lesser area.

3 Inscribe square EFGH in the circle EFGH.

4 Establish that the inscribed square EFGH is greater than half of the circle EFGH.

$\text{Area}(\text{square EFGH}) > \frac{1}{2} \text{Area}(\text{circle EFGH})$

5 Bisect the circumferences EF, FG, GH, and HE at K, L, M, and N, and join the chords.

6 Show that each of the triangles EKF, FLG, GMH, and HNE is greater than half of its corresponding circular segment. Each triangle $> \frac{1}{2}$ corresponding circular segment.

7 By repeatedly bisecting the remaining circumferences and inscribing triangles, leave some segments of the circle EFGH whose sum is less than the excess by which the circle EFGH exceeds area S.

8 Therefore, the remaining polygon EKFLGMHN is greater than area S.

$\text{Polygon EKFLGMHN} > S$

9 Inscribe in circle ABCD a polygon AOBPCQDR similar to polygon EKFLGMHN.

10 By Proposition 12.1,

$$\frac{(BD)^2}{(FH)^2} = \frac{\text{Area}(\text{AOBPCQDR})}{\text{Area}(\text{EKFLGMHN})}$$

11 By the assumption,

$$\frac{(BD)^2}{(FH)^2} = \frac{\text{Area}(\text{circle ABCD})}{S}$$

12 Therefore,

$$\frac{\text{Area}(\text{circle ABCD})}{S} = \frac{\text{Area}(\text{AOBPCQDR})}{\text{Area}(\text{EKFLGMHN})}$$

13 Alternating,

$$\frac{\text{Area}(\text{circle ABCD})}{\text{Area}(\text{AOBPCQDR})} = \frac{S}{\text{Area}(\text{EKFLGMHN})}$$

14 But circle ABCD is greater than the polygon within it, so S must be greater than polygon EKFLGMHN.
 $S > \text{Area}(\text{EKFLGMHN})$

15 However, S is less than polygon EKFLGMHN (from steps 7 and 8). This is a contradiction.
 $S < \text{Area}(\text{EKFLGMHN})$
 Contradiction!

16 Therefore, the square on BD is not to the square on FH as circle ABCD is to a lesser area than circle EFGH.
 Contradiction to assumption.

17 Similarly, it can be shown that the square on FH is not to the square on BD as circle EFGH is to a lesser area than circle ABCD.
 Contradiction.

18 Now, assume S is a greater area than circle EFGH.

19 Inversely,

$$\frac{(FH)^2}{(DB)^2} = \frac{S}{\text{Area}(\text{circle ABCD})}$$

20 By the lemma, as $\frac{S}{\text{Area}(\text{circle ABCD})}$, so circle EFGH is to some area less than circle ABCD.
 $S' < \text{Area}(\text{circle ABCD})$

21 Thus,

$$\frac{(FH)^2}{(DB)^2} = \frac{\text{Area}(\text{circle EFGH})}{S'}$$

This contradicts what was shown.

22 Therefore, as the square on BD is to the square on FH, so circle ABCD is to circle EFGH.

$\frac{\text{Square on BD}}{\text{Square on FH}} = \frac{\text{Circle ABCD}}{\text{Circle EFGH}}$

PROPOSITION 2.
 CIRCLES ARE TO ONE ANOTHER AS THE SQUARES ON (THEIR) DIAMETERS.

LEMMA TO PROPOSITION 2

Used in the converse part of Proposition XII.2

Statement. If area S is greater than circle $EFGH$, then as area S is to circle $ABCD$, so circle $EFGH$ is to some area less than circle $ABCD$.

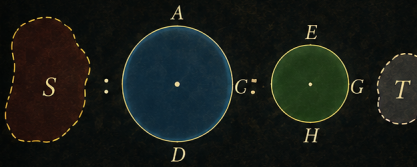
Proof.

1. Let it be contrived that as area S is to circle $ABCD$, so circle $EFGH$ is to area T .
2. Alternately,

$$\frac{S}{\text{circle } EFGH} = \frac{\text{circle } ABCD}{T}.$$
3. Since S is greater than circle $EFGH$, circle $ABCD$ is greater than T .
4. Hence, as area S is to circle $ABCD$, so circle $EFGH$ is to a lesser area than circle $ABCD$.

Therefore. The required lesser fourth proportional exists.

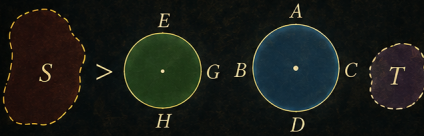
- 1 Let it be contrived that as area S is to circle $ABCD$, so circle $EFGH$ is to area T .



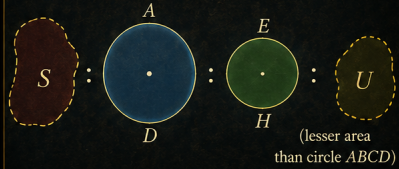
- 2 Alternately,

$$\frac{S}{(\text{circle } EFGH)} = \frac{(\text{circle } ABCD)}{T}$$

- 3 Since S is greater than circle $EFGH$, circle $ABCD$ is greater than T .



- 4 Hence, as area S is to circle $ABCD$, so circle $EFGH$ is to a lesser area than circle $ABCD$.



LEMMA

IF AREA S IS GREATER THAN CIRCLE $EFGH$, THEN AS AREA S IS TO CIRCLE $ABCD$, SO CIRCLE $EFGH$ IS TO SOME AREA LESS THAN CIRCLE $ABCD$.

PROPOSITION 3

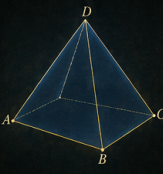
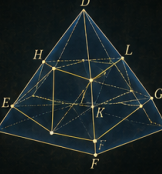
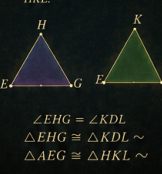
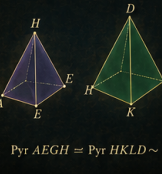
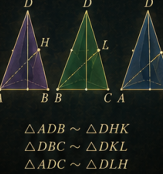
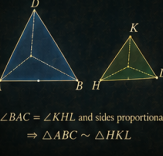
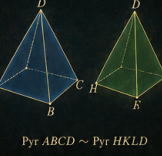
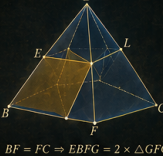
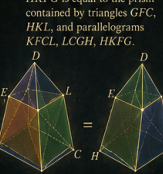
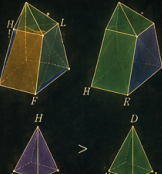
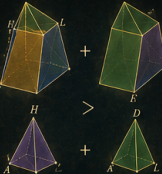
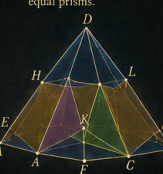
Statement. Any pyramid having a triangular base is divided into two pyramids having triangular bases, equal and similar to one another and similar to the whole, and into two equal prisms; and the sum of the two prisms is greater than half of the whole pyramid.

Proof.

1. Let the base of the pyramid be triangle ABC , and its apex be point D .
2. Bisect the edges AB, BC, CA, AD, DB, DC at points E, F, G, H, K, L respectively.
3. Join $HE, EG, GH, HK, KL, LH, KF, FG$.
4. Show $EH \parallel DB$ and $HK \parallel AB$, making $HEBK$ a parallelogram. Therefore $AE = HK$.
5. Conclude triangle AEH is equal and similar to triangle HKD , and triangle AHG is equal and similar to triangle HLD .
6. Show $\angle EHG = \angle KDL$, and triangle EHG is equal and similar to triangle KDL . Similarly, triangle AEG is equal and similar to triangle HKL .
7. Therefore pyramid $AEHG$, with base AEG and apex H , is equal and similar to pyramid $HKLD$, with base HKL and apex D .
8. Show triangle ADB is similar to triangle DHK , DBC to DKL , and ADC to DLH .
9. Show $\angle BAC = \angle KHL$ and that the corresponding sides are proportional; hence triangle ABC is similar to triangle HKL .
10. Therefore the pyramid whose base is ABC and apex D is similar to pyramid $HKLD$.
11. Since pyramid $HKLD$ is similar to pyramid $AEHG$, both pyramids $AEHG$ and $HKLD$ are similar to the whole pyramid $ABCD$.
12. Since $BF = FC$, parallelogram $EBFG$ is double triangle GFC .

13. Conclude the prism contained by triangles BKF, EHG , and parallelograms $EBFG, EBKH, HKFG$ is equal to the prism contained by triangles GFC, HKL , and parallelograms $KFCL, LCGH, HKFG$.
14. Show each of these prisms is greater than each of the pyramids $AEHG$ and $HKLD$.
15. Therefore the sum of the two prisms is greater than the sum of the two pyramids.
16. Hence the whole pyramid is divided into two equal and similar pyramids and two equal prisms.

Therefore. The required division and inequality have been established.

1 Let the base of the pyramid be triangle ABC, and its apex be point D. 	2 Bisect the edges AB, BC, CA, AD, DB, and DC at points E, F, G, H, K, and L respectively. 	3 Join HE, EG, GH, HK, KL, LH, KF, and FG. 	4 Show EH is parallel to DB, and HK is parallel to AB, making $HEBK$ a parallelogram. Therefore, $AE = HK$.  $HE \parallel DB, \quad HK \parallel AB$ $\Rightarrow HEBK$ is a parallelogram $\Rightarrow AE = HK$
5 Conclude triangle AEH is equal and similar to triangle HKD, and triangle AHG is equal and similar to triangle HLD.  $\triangle AEH \cong \triangle HKD \sim$ $\triangle AHG \cong \triangle HLD \sim$	6 Show angle EHG equals angle KDL, and triangle EHG is equal and similar to triangle KDL. Similarly, triangle AEG is equal and similar to triangle HKL.  $\angle EHG = \angle KDL$ $\triangle EHG \cong \triangle KDL \sim$ $\triangle AEG \cong \triangle HKL \sim$	7 Therefore, pyramid $AEGH$ (base AEG, apex H) is equal and similar to pyramid $HKLD$ (base HKL, apex D).  $\text{Pyt } AEGH \sim \text{Pyt } HKLD \sim$	8 Show triangle ADB is similar to triangle DHK, DBC to DKL, and ADC to DLH.  $\triangle ADB \sim \triangle DHK$ $\triangle DBC \sim \triangle DKL$ $\triangle ADC \sim \triangle DLH$
9 Show angle BAC equals angle KHL and sides are proportional, so triangle ABC is similar to triangle HKL.  $\angle BAC = \angle KHL$ and sides proportional $\Rightarrow \triangle ABC \sim \triangle HKL$	10 Therefore, the pyramid whose base is ABC and apex D is similar to pyramid $HKLD$.  $\text{Pyt } ABCD \sim \text{Pyt } HKLD$	11 Since pyramid $HKLD$ is similar to pyramid $AEGH$, both pyramids $AEGH$ and $HKLD$ are similar to the whole pyramid $ABCD$.  $\text{Pyt } AEGH \sim \text{Pyt } HKLD \sim \text{Pyt } ABCD$	12 Since $BF = FC$, parallelogram $EBFG$ is double triangle GFC.  $BF = FC \Rightarrow EBFG = 2 \times \triangle GFC$
13 Conclude the prism contained by triangles BKF, EHG, and parallelograms $EBFG$, $EBKH$, $HKFG$ is equal to the prism contained by triangles GFC, HKL, and parallelograms $KFCL$, $LCGH$, $HKFG$. 	14 Show each of these prisms is greater than each of the pyramids $AEGH$ and $HKLD$. 	15 Therefore, the sum of the two prisms is greater than the sum of the two pyramids. 	16 Hence, the whole pyramid is divided into two equal and similar pyramids and two equal prisms. 

PROPOSITION 3.

ANY PYRAMID HAVING A TRIANGULAR BASE IS DIVIDED INTO TWO PYRAMIDS HAVING TRIANGULAR BASES (WHICH ARE) EQUAL, SIMILAR TO ONE ANOTHER, AND [SIMILAR] TO THE WHOLE, AND INTO TWO EQUAL PRISMS. AND THE (SUM OF THE) TWO PRISMS IS GREATER THAN HALF OF THE WHOLE PYRAMID.

PROPOSITION 4

Statement. If there are two pyramids with the same height, having triangular bases, and each of them is divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then as the base of one pyramid is to the base of the other, so the sum of all the prisms in one pyramid is to the sum of the equal number of prisms in the other pyramid.

Proof.

1. Let there be two pyramids of equal height, $ABCG$ and $DEFH$, with bases ABC and DEF . Divide each similarly as in Proposition XII.3.
2. Since $BO = OC$ and $AL = LC$, $LO \parallel AB$, so triangle ABC is similar to triangle LOC . Similarly, triangle DEF is similar to triangle RVF .
3. Since $BC = 2CO$ and $EF = 2FV$,

$$\frac{BC}{CO} = \frac{EF}{FV}.$$

4. Since similar figures ABC and LOC are described on BC and CO , and DEF and RVF on EF and FV ,

$$\frac{\triangle ABC}{\triangle LOC} = \frac{\triangle DEF}{\triangle RVF}.$$

5. Alternately,

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\triangle LOC}{\triangle RVF}.$$

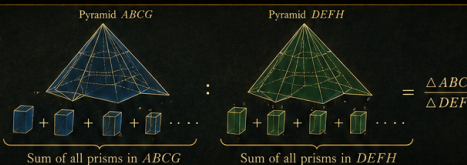
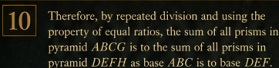
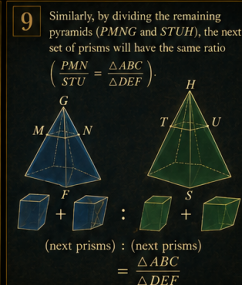
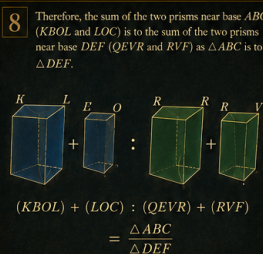
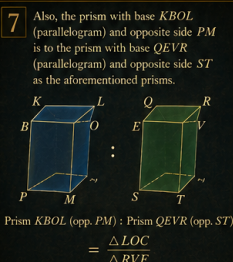
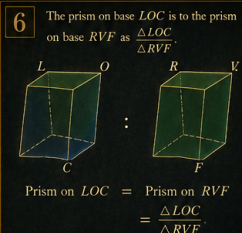
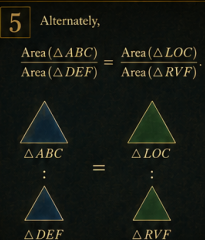
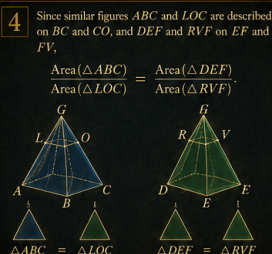
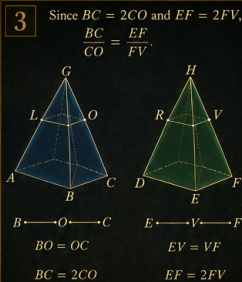
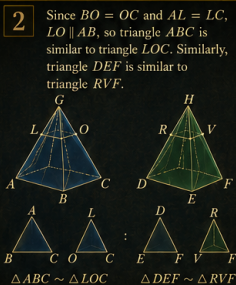
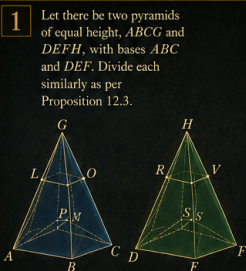
6. The prism on base LOC is to the prism on base RVF as triangle LOC is to triangle RVF .
7. Also, the prism with base $KBOL$ and opposite side PM is to the prism with base $QEV R$ and opposite side ST as the aforementioned prisms.
8. Therefore, the sum of the two prisms near base ABC is to the sum of the two prisms near base DEF as triangle ABC is to triangle DEF .

9. Similarly, by dividing the remaining pyramids $PMNG$ and $STUH$, the next set of prisms has the same ratio:

$$\frac{PMN}{STU} = \frac{ABC}{DEF}.$$

10. Therefore, by repeated division and the property of equal ratios, the sum of all prisms in pyramid $ABCG$ is to the sum of all prisms in pyramid $DEFH$ as base ABC is to base DEF .

Therefore. Equal numbers of corresponding prisms cut from the two pyramids are in the ratio of the triangular bases.



PROPOSITION 4.

IF THERE ARE TWO PYRAMIDS WITH THE SAME HEIGHT, HAVING TRIANGULAR BASES, AND EACH OF THEM IS DIVIDED INTO TWO PYRAMIDS EQUAL TO ONE ANOTHER, AND SIMILAR TO THE WHOLE, AND INTO TWO EQUAL PRISMS THEN AS THE BASE OF ONE PYRAMID IS TO THE BASE OF THE OTHER PYRAMID, SO THE (SUM OF) ALL THE PRISMS IN ONE PYRAMID WILL BE TO (THE SUM OF ALL) THE EQUAL NUMBER OF PRISMS IN THE OTHER PYRAMID.

LEMMA TO PROPOSITION 4

Comparison of the corresponding prisms

Statement. Triangle LOC is to triangle RVF as the prism on base LOC , opposite PMN , is to the prism on base RVF , opposite STU .

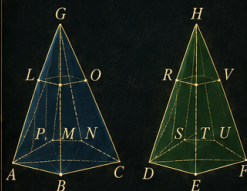
Proof.

1. Consider the same figure as in Proposition XII.4.
2. Draw perpendiculars from G and H to the planes ABC and DEF . These are equal, since the pyramids have equal heights.
3. Since planes ABC and PMN cut the parallel lines GC and the perpendicular from G , they cut them in the same ratio.
4. GC is cut in half by PMN , so the perpendicular from G is also cut in half by PMN .
5. Similarly, the perpendicular from H is cut in half by STU .
6. Since the perpendiculars from G and H to ABC and DEF are equal, the perpendiculars from the planes PMN and STU to ABC and DEF are also equal.
7. Therefore the prisms on bases LOC and RVF , with opposite planes PMN and STU , are of equal height.
8. The parallelepiped solids described on these prisms are of equal height and their ratio is equal to the ratio of their bases LOC and RVF .
9. Since the prisms are halves of these parallelepipeds, the prisms themselves are in the ratio of their bases.

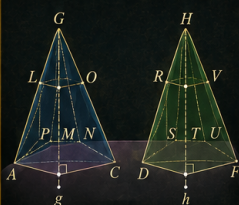
Therefore.

$$\frac{\text{prism on } LOC}{\text{prism on } RVF} = \frac{\triangle LOC}{\triangle RVF}.$$

- 1 Consider the same figure as in Proposition 4.

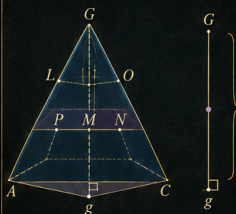


- 2 Draw perpendiculars from G and H to the planes ABC and DEF . These are equal as the pyramids have equal heights.

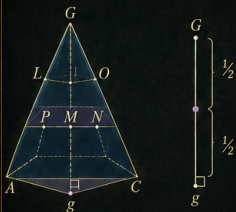


$$gg = hh \quad (\text{equal heights})$$

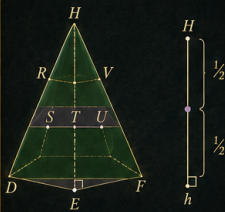
- 3 Since planes ABC and PMN cut the parallel lines GC and the perpendicular from G , they cut them in the same ratio.



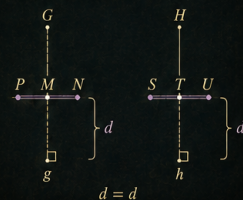
- 4 GC is cut in half by PMN , so the perpendicular from G is also cut in half by PMN .



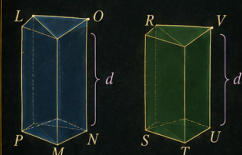
- 5 Similarly, the perpendicular from H is cut in half by STU .



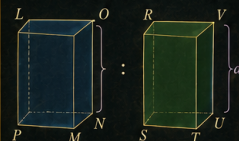
- 6 Since the perpendiculars from G and H to ABC and DEF are equal, the perpendiculars from the planes PMN and STU to ABC and DEF are also equal.



- 7 Therefore, the prisms on bases LOC and RVF (with opposite planes PMN and STU) are of equal height.

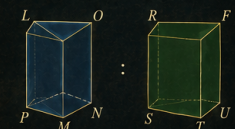


- 8 The parallelepiped solids described on these prisms are of equal height and their ratio is equal to the ratio of their bases (LOC and RVF).



$$\frac{\text{Volume}(\text{parallelepiped on } LOC)}{\text{Volume}(\text{parallelepiped on } RVF)} = \frac{\text{Area}(LOC)}{\text{Area}(RVF)}$$

- 9 Since the prisms are halves of these parallelepipeds, the prisms themselves are in the ratio of their bases (LOC / RVF).



$$\frac{\text{Volume}(\text{prism on } LOC)}{\text{Volume}(\text{prism on } RVF)} = \frac{\text{Area}(LOC)}{\text{Area}(RVF)}$$

$\therefore \text{Prism on } LOC : \text{Prism on } RVF = \text{Area}(LOC) : \text{Area}(RVF)$

LEMMA (FOR PROPOSITION 4)

TRIANGLE LOC IS TO TRIANGLE RVF AS THE PRISM ON BASE LOC (OPPOSITE PMN) IS TO THE PRISM ON BASE RVF (OPPOSITE STU).

PROPOSITION 5

Statement. Pyramids which are of the same height, and have triangular bases, are to one another as their bases.

Proof.

1. Let there be two pyramids of equal height, $ABCG$ and $DEFH$, with bases ABC and DEF .
2. Assume the contrary: as base ABC is to base DEF , so pyramid $ABCG$ is to some solid W , where W is either less than or greater than pyramid $DEFH$. First, let W be lesser.
3. Divide pyramid $DEFH$ into two equal prisms and two smaller pyramids, as in Proposition XII.3.
4. The sum of the prisms is greater than half the whole pyramid.
5. Repeatedly divide the smaller pyramids, leaving a set of smaller pyramids whose total volume is less than the excess by which pyramid $DEFH$ exceeds W .
6. The remaining prisms in pyramid $DEFH$ are therefore greater than W .
7. Divide pyramid $ABCG$ in the same manner and the same number of times as $DEFH$.
8. By Proposition XII.4,

$$\frac{\text{sum of prisms in } ABCG}{\text{sum of prisms in } DEFH} = \frac{ABC}{DEF}.$$

9. By the assumption,

$$\frac{ABC}{DEF} = \frac{ABCG}{W}.$$

10. Therefore

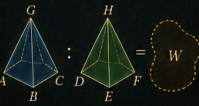
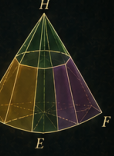
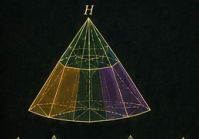
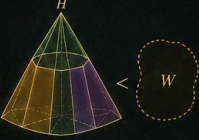
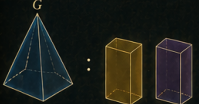
$$\frac{ABCG}{W} = \frac{\text{prisms in } ABCG}{\text{prisms in } DEFH}.$$

11. Alternately,

$$\frac{ABCG}{\text{prisms in } ABCG} = \frac{W}{\text{prisms in } DEFH}.$$

12. But $ABCG$ is greater than its prisms, so W must be greater than the prisms in $DEFH$.
13. However, W is less than the prisms in $DEFH$, from step 6. Contradiction.
14. Therefore the assumption that W is lesser is false.
15. Similarly, it can be shown that the relation does not hold for a greater solid W .
16. Therefore, as base ABC is to base DEF , so pyramid $ABCG$ is to pyramid $DEFH$.

Therefore. Equal-height triangular pyramids are proportional to their bases.

1 Let there be two pyramids of equal height, $ABCG$ and $DEFH$, with bases ABC and DEF. 	2 Assume the contrary: as base ABC is to base DEF, so pyramid $ABCG$ is to some solid W, where W is either less than or greater than pyramid $DEFH$. First, let W be lesser.  W is lesser.	3 Divide pyramid $DEFH$ into two equal prisms and two smaller pyramids (as per Prop. 12.3). 	4 The sum of the prisms is greater than half the whole pyramid.  $(\text{Prisms}) > \frac{1}{2} (\text{Pyr } DEFH)$
5 Repeatedly divide the smaller pyramids, leaving a set of smaller pyramids whose total volume is less than the excess by which pyramid $DEFH$ exceeds W.  $(\text{sum of small pyramids}) < (DEFH - W)$	6 The remaining prisms in pyramid $DEFH$ are therefore greater than W.  $(\text{Prisms in } DEFH) > W$	7 Divide pyramid $ABCG$ in the same manner and number of times as $DEFH$.  $\text{Pyr } ABCG = \text{sum of prisms in } ABCG$	8 By Proposition 12.4, $\frac{(\text{sum of prisms in } ABCG)}{(\text{sum of prisms in } DEFH)} = \frac{ABC}{DEF}$ 
10 Therefore, $\frac{ABCG}{W} = \frac{(\text{prisms in } ABCG)}{(\text{prisms in } DEFH)}$ 	11 Alternating, $\frac{ABCG}{(\text{prisms in } ABCG)} = \frac{W}{(\text{prisms in } DEFH)}$ 	12 But $ABCG$ is greater than its prisms, so W must be greater than the prisms in $DEFH$.  $W > (\text{prisms in } DEFH)$	13 However, W is less than the prisms in $DEFH$ (from step 6). This is a contradiction.  $W < (\text{prisms in } DEFH)$ Contradiction!
14 Therefore, the assumption that W is lesser is false.  Assumption false.	15 Similarly, it can be shown that the relation does not hold for a greater solid W. (Case where W is greater)  Also impossible.	16 Therefore, as base ABC is to base DEF, so pyramid $ABCG$ is to pyramid $DEFH$. $\frac{ABC}{DEF} = \frac{ABCG}{DEFH}$ 	

PROPOSITION 5.
 PYRAMIDS WHICH ARE OF THE SAME HEIGHT, AND HAVE TRIANGULAR BASES,
 ARE TO ONE ANOTHER AS THEIR BASES.

PROPOSITION 6

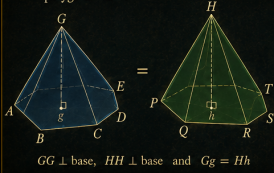
Statement. Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.

Proof.

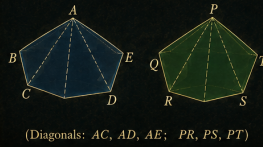
1. Let there be two pyramids of equal height with polygonal bases.
2. Divide the polygonal bases into triangles by drawing diagonals from a vertex.
3. This divides the pyramids into corresponding pyramids with triangular bases.
4. By Proposition XII.5, each triangular pyramid in one is to its corresponding triangular pyramid in the other as the triangular base is to the corresponding triangular base.
5. Since the sums of the triangular bases are the polygonal bases, and the sums of the corresponding pyramids are the whole pyramids, the ratio of the whole pyramids equals the ratio of the polygonal bases.

Therefore. Equal-height pyramids with polygonal bases are proportional to their bases.

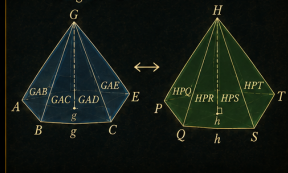
- 1 Let there be two pyramids of equal height with polygonal bases.



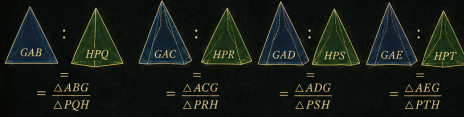
- 2 Divide the polygonal bases into triangles by drawing diagonals from a vertex.



- 3 This divides the pyramids into corresponding pyramids with triangular bases.

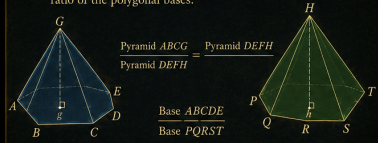


- 4 By Proposition 12.5, each triangular pyramid in one is to its corresponding triangular pyramid in the other as the triangular base is to the corresponding triangular base.



In general, (triangular pyramids in one) : (corresponding triangular pyramids in other)
= (triangular base) : (corresponding triangular base)

- 5 Since the sums of the triangular bases are the polygonal bases, and the sums of the corresponding pyramids are the whole pyramids, the ratio of the whole pyramids equals the ratio of the polygonal bases.



PROPOSITION 6.

PYRAMIDS WHICH ARE OF THE SAME HEIGHT, AND HAVE POLYGONAL BASES, ARE TO ONE ANOTHER AS THEIR BASES.

COROLLARY TO PROPOSITION 6

Similar polygonal pyramids

Statement. Similar pyramids having polygonal bases are in the cubed ratio of their corresponding sides.

Proof.

1. Divide the similar polygonal bases into similar triangles, equal in number and corresponding.
2. This divides the pyramids into corresponding triangular pyramids.
3. By Proposition XII.8, each triangular pyramid in one is to its corresponding pyramid in the other in the cubed ratio of corresponding sides.
4. Since the sums of the pyramids form the whole polygonal pyramids, the whole pyramids are also in the cubed ratio of their corresponding sides.

Therefore. Similar polygonal pyramids are in the cubed ratio of corresponding sides.

Cinematic Sequence: Corollary to Proposition 6

Similar Polygonal Pyramids Are in the Cubed Ratio of Their Corresponding Sides

1. THE TWO PYRAMIDS

Similar pyramids having polygonal bases.

We begin with two similar pyramids. Their bases are similar polygons, and their heights correspond in the same ratio as their sides.

2. DIVIDING THE BASES INTO TRIANGLES

Divide the similar polygonal bases into similar triangles, equal in number and corresponding.

From a single vertex, draw diagonals across each base. This decomposes each polygon into the same number of corresponding triangles, similar in both pyramids.

3. THE PYRAMIDS SPLIT INTO TRIANGULAR PYRAMIDS

This divides the pyramids into corresponding triangular pyramids.

Each triangular base, together with the apex, forms a triangular pyramid. Thus the whole polygonal pyramid is divided into corresponding triangular pyramids.

4. APPLYING PROPOSITION XII.8

$V_1 : V_2 = a^3 : b^3$

By Proposition XII.8, each triangular pyramid is to its corresponding triangular pyramid in the cubed ratio of corresponding sides.

By Proposition XII.8, similar triangular pyramids are in the cubed ratio of their corresponding sides. This holds for each corresponding pair.

5. SUMMING THE PARTS

Since the sums of the pyramids form the whole polygonal pyramids, the whole pyramids are also in the cubed ratio of their corresponding sides.

The sum of the corresponding triangular pyramids makes up the whole polygonal pyramid. Therefore, the whole similar polygonal pyramids are also in the cubed ratio of their sides.

6. CONCLUSION

Small Pyramid : Large Pyramid = $a^3 : b^3$
where $a : b$ is the ratio of any pair of corresponding sides.

Similar polygonal pyramids are in the cubed ratio of corresponding sides.

Thus, similar pyramids having polygonal bases are in the cubed ratio of their corresponding sides. ■

EUCLID'S ELEMENTS

Book XII

Corollary to Proposition 6

Similar polygonal pyramids are in the cubed ratio of their corresponding sides.

$V_1 : V_2 = a^3 : b^3$
 $a : b$
(corresponding sides)

Finis.

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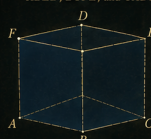
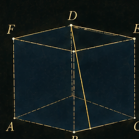
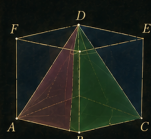
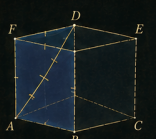
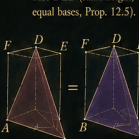
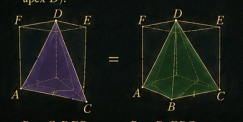
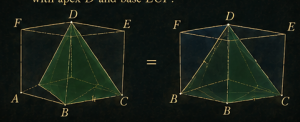
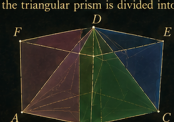
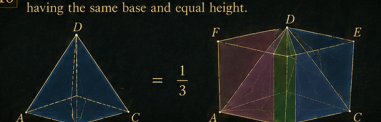
PROPOSITION 7

Statement. Any prism having a triangular base is divided into three pyramids which have triangular bases and are equal to one another.

Proof.

1. Let prism $ABCDEF$ have triangular bases ABC and DEF , with lateral faces $ABED$, $BCFE$, $CADF$.
2. Draw diagonal BD .
3. Consider pyramids with vertices D and bases ABD , DBC , DBE .
4. Since $ABED$ is a parallelogram, diagonal DB bisects it. Therefore triangle ABD is equal to triangle DEB .
5. The pyramid with apex C and base ABD is equal to the pyramid with apex C and base DEB , having the same height and equal bases by Proposition XII.5.
6. The pyramid $C-DEB$ is the same solid as pyramid $D-EBC$, with base EBC and apex D .
7. Since $FCBE$ is a parallelogram, triangle CEF is equal to triangle CBE .
8. Therefore the pyramid with apex D and base BCE is equal to the pyramid with apex D and base ECF .
9. Since pyramids $C-ABD$, $D-EBC$, and $D-ECF$ are all equal, the triangular prism is divided into three equal pyramids.
10. Corollary: any pyramid is the third part of the prism having the same base and equal height.

Therefore. A triangular prism is composed of three equal triangular pyramids.

1 Let prism $ABCDEF$ have triangular bases ABC and DEF (with lateral faces $ABED$, $BCFE$, and $CADF$).  Bases: ABC and DEF Lateral faces: $ABED$, $BCFE$, $CADF$	2 Draw diagonal BD. 	3 Consider pyramids with vertices D and bases ABD, DBC, and DBE.  Pyramids: $D-ABD$, $D-DBC$, $D-DBE$	4 Since $ABED$ is a parallelogram, diagonal DB bisects it. Therefore, $\triangle ABD = \triangle DEB$.  $ABED$ is a parallelogram $\Rightarrow \triangle ABD = \triangle DEB$	5 The pyramid with apex C and base ABD is equal to the pyramid with apex C and base DEB (same height, equal bases, Prop. 12.5).  $\text{Pyr. } C-ABD = \text{Pyr. } C-DEB$
6 The pyramid $C-DEB$ is the same solid as pyramid $D-EBC$ (base EBC, apex D).  $\text{Pyr. } C-DEB = \text{Pyr. } D-EBC$	7 Since $FCBE$ is a parallelogram, triangle CEF is equal to triangle CBE.  $FCBE$ is a parallelogram $\Rightarrow \triangle CEF = \triangle CBE$	8 Therefore, the pyramid with apex D and base BCE is equal to the pyramid with apex D and base ECF.  $\text{Pyr. } D-BCE = \text{Pyr. } D-ECF$		
9 Since pyramids $C-ABD$, $D-EBC$, and $D-ECF$ are all equal, the triangular prism is divided into three equal pyramids.  $\text{Pyr. } C-ABD = \text{Pyr. } D-EBC = \text{Pyr. } D-ECF$ The prism $ABCDEF$ is divided into three equal pyramids.		10 COROLLARY: Any pyramid is the third part of the prism having the same base and equal height.  $= \frac{1}{3}$		

PROPOSITION 7.
ANY PRISM HAVING A TRIANGULAR BASE IS DIVIDED INTO THREE PYRAMIDS WHICH HAVE TRIANGULAR BASES, (WHICH ARE) EQUAL TO ONE ANOTHER.

PROPOSITION 8

Statement. Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Proof.

1. Let $ABCG$ and $DEFH$ be similar and similarly laid out pyramids. Let BC and EF be corresponding sides.
2. Complete the parallelepiped solids $BGML$ and $EHQP$ on the pyramids.
3. By the definition of similar pyramids,

$$\angle ABC = \angle DEF, \quad \angle GBC = \angle HEF, \quad \angle ABG = \angle DEH,$$

and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{BG}{EH}.$$

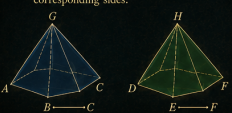
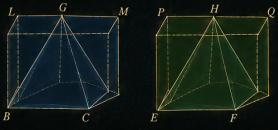
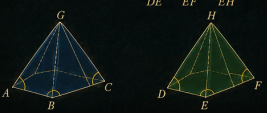
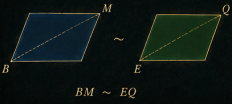
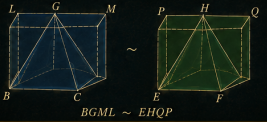
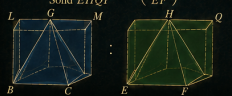
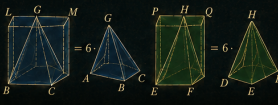
4. Therefore parallelogram BM is similar to parallelogram EQ .
5. Similarly, parallelograms BN and BK are similar to ER and EO respectively.
6. The three parallelograms forming the solids are similar, so parallelepiped solids $BGML$ and $EHQP$ are similar.
7. By Proposition XI.33, similar parallelepipeds are in the cubed ratio of corresponding sides. Thus

$$\frac{BGML}{EHQP} = \left(\frac{BC}{EF} \right)^3.$$

8. The solid $BGML$ is six times pyramid $ABCG$, and $EHQP$ is six times pyramid $DEFH$, by the corollary to Proposition XII.7.
9. Therefore

$$\frac{ABCG}{DEFH} = \frac{BGML}{EHQP} = \left(\frac{BC}{EF} \right)^3.$$

Therefore. Similar triangular pyramids are in the cubed ratio of corresponding sides.

1 Let $ABCG$ and $DEFH$ be similar and similarly laid out pyramids. Let BC and EF be corresponding sides. 	2 Complete the parallelepiped solids $BGML$ and $EHQP$ on the pyramids. 	3 By the definition of similar pyramids, $\angle ABC = \angle DEF$, $\angle GBC = \angle HEF$, $\angle ABG = \angle DEH$, and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{BG}{EH}$ 
4 Therefore, parallelogram BM is similar to parallelogram EQ.  $BM \sim EQ$	5 Similarly, parallelograms BN and BK are similar to ER and EO respectively.  $BN \sim ER$ and $BK \sim EO$	6 The three parallelograms forming the solids are similar, so the parallelepiped solids $BGML$ and $EHQP$ are similar.  $BGML \sim EHQP$
7 By Proposition 11.33, similar parallelepipeds are in the cubed ratio of corresponding sides. $\frac{\text{Solid } BGML}{\text{Solid } EHQP} = \left(\frac{BC}{EF}\right)^3$ 	8 The solid $BGML$ is six times the pyramid $ABCG$, and $EHQP$ is six times the pyramid $DEFH$ (by Prop. 12.7 Corollary). 	9 Therefore, $\frac{\text{Pyramid } ABCG}{\text{Pyramid } DEFH} = \frac{\text{Solid } BGML}{\text{Solid } EHQP} = \left(\frac{BC}{EF}\right)^3$ 

PROPOSITION 8.

SIMILAR PYRAMIDS WHICH ALSO HAVE TRIANGULAR BASES ARE IN THE CUBED RATIO OF THEIR CORRESPONDING SIDES.

PROPOSITION 9

Statement. The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.

Proof.

1. Let $ABCG$ and $DEFH$ be equal pyramids with bases ABC and DEF and heights corresponding to their apexes G and H .
2. Complete the parallelepiped solids $BGML$ and $EHQP$.
3. Since the pyramids are equal, the solids $BGML$ and $EHQP$ are equal, as each is six times its respective pyramid.
4. For equal parallelepiped solids, their bases are reciprocally proportional to their heights by Proposition XI.34. Therefore

$$\frac{BM}{EQ} = \frac{\text{height of } EHQP}{\text{height of } BGML}.$$

5. But $BM = 2(ABC)$ and $EQ = 2(DEF)$. Therefore

$$\frac{ABC}{DEF} = \frac{\text{height of } EHQP}{\text{height of } BGML}.$$

6. The height of the solids is the same as the height of their respective pyramids. Hence

$$\frac{ABC}{DEF} = \frac{\text{height of } DEFH}{\text{height of } ABCG}.$$

7. For the converse, assume

$$\frac{ABC}{DEF} = \frac{\text{height of } DEFH}{\text{height of } ABCG}.$$

8. This implies

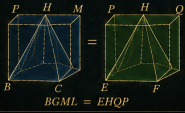
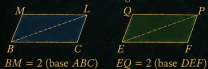
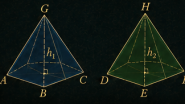
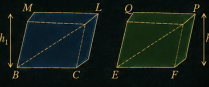
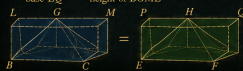
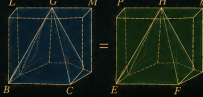
$$\frac{BM}{EQ} = \frac{\text{height of } EHQP}{\text{height of } BGML}.$$

9. For parallelepiped solids, if the bases are reciprocally proportional to the heights, the solids are equal by Proposition XI.34.

10. Therefore $BGML = EHQP$.

11. Hence their sixth parts, pyramids $ABCG$ and $DEFH$, are equal.

Therefore. Equal triangular pyramids have reciprocally proportional bases and heights, and conversely.

1 Let $ABCG$ and $DEFH$ be equal pyramids with bases ABC and DEF and heights corresponding to their apexes G and H. 	2 Complete the parallelepiped solids $BGML$ and $EHQP$. 	3 Since the pyramids are equal, the solids $BGML$ and $EHQP$ are equal (as each is six times its respective pyramid). 	4 For equal parallelepiped solids, their bases are reciprocally proportional to their heights (Prop. 11.34). Therefore, $\frac{\text{base } BM}{\text{base } EQ} = \frac{\text{height of } EHQP}{\text{height of } BGML}$ 
5 But base $BM = 2(\text{base } ABC)$ and base $EQ = 2(\text{base } DEF)$. Therefore, $\frac{2(\text{base } ABC)}{2(\text{base } DEF)} = \frac{\text{height of } EHQP}{\text{height of } BGML}$ $\frac{\text{base } ABC}{\text{base } DEF} = \frac{\text{height of } EHQP}{\text{height of } BGML}$ 	6 The height of the solids is the same as the height of their respective pyramids. So, $\frac{\text{base } ABC}{\text{base } DEF} = \frac{\text{height of } DEFH}{\text{height of } ABCG}$  Thus, the bases are reciprocally proportional to the heights.	7 For the converse, assume $\frac{\text{base } ABC}{\text{base } DEF} = \frac{\text{height of } DEFH}{\text{height of } ABCG}$ 	8 This implies $\frac{\text{base } BM}{\text{base } EQ} = \frac{\text{height of } EHQP}{\text{height of } BGML}$ 
9 For parallelepiped solids, if the bases are reciprocally proportional to the heights, the solids are equal (Prop. 11.34). $\frac{\text{base } BM}{\text{base } EQ} = \frac{\text{height of } EHQP}{\text{height of } BGML} \Rightarrow BGML = EHQP$ 	10 Therefore, $BGML = EHQP$ 	11 Hence, their sixth parts, pyramids $ABCG$ and $DEFH$, are equal.  (Each parallelepiped solid is six times its pyramid—by the Corollary to Prop. 12.7.)	

PROPOSITION 9.

THE BASES OF EQUAL PYRAMIDS WHICH ALSO HAVE TRIANGULAR BASES ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS.
 AND THOSE PYRAMIDS WHICH HAVE TRIANGULAR BASES WHOSE BASES ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS ARE EQUAL.

PROPOSITION 10

Statement. Every cone is the third part of the cylinder which has the same base as it and an equal height.

Proof.

1. Let a cone and a cylinder have the same base, circle $ABCD$, and equal height.
2. Assume the cylinder is not three times the cone. Suppose it is more than three times the cone.
3. Inscribe a square $ABCD$ in the base circle. The square is more than half the circle.
4. Erect a prism of equal height on this square. This prism is more than half the cylinder.
5. Bisect the arcs AB, BC, CD, DA and inscribe triangles. Each triangle is more than half its circular segment.
6. Erect prisms on these triangles. Each such prism is more than half the corresponding segment of the cylinder.
7. Repeating this, leave segments of the cylinder whose total volume is less than the excess by which the cylinder exceeds three times the cone.
8. Thus the remaining prism, on the inscribed polygon, is greater than three times the cone.
9. But by the corollary to Proposition XII.7, this prism is exactly three times the pyramid on the same polygonal base and with apex at the cone's apex.
10. Therefore this pyramid is greater than the cone, which is impossible because it is contained within the cone.
11. Therefore the cylinder is not more than three times the cone.
12. Suppose the cylinder is less than three times the cone. Inversely, the cone is greater than a third of the cylinder.
13. Inscribe square $ABCD$ in the base. It is more than half the circle.

14. Erect a pyramid of the same apex as the cone on the square. This pyramid is more than half the cone.
15. Repeat the process of inscribing polygons and erecting pyramids to leave segments of the cone whose total volume is less than the excess by which the cone exceeds a third of the cylinder.
16. The remaining pyramid on the inscribed polygon is therefore greater than a third of the cylinder.
17. But this pyramid is a third of the prism on the same polygonal base and of the cylinder's height.
18. Therefore this prism is greater than the cylinder, which is impossible because it is contained within the cylinder.
19. Therefore the cylinder is not less than three times the cone.
20. Thus the cylinder is exactly three times the cone.

Therefore. Every cone is one third of the cylinder having the same base and equal height.

PROPOSITION 10 — PART I

EVERY CONE IS THE THIRD PART OF THE CYLINDER WHICH HAS THE SAME BASE AS IT, AND AN EQUAL HEIGHT.

I. Suppose the cylinder is more than three times the cone.

1 Let a cone and a cylinder have the same base (circle $ABCD$) and equal height.

2 Assume the cylinder is not three times the cone. Suppose it is more than three times the cone.

3 Inscribe a square $ABCD$ in the base circle. The square is more than half the circle.

4 Erect a prism of equal height on this square. This prism is more than half the cylinder.

5 Bisect the arcs AB , BC , CD , DA and inscribe triangles. Each triangle is more than half its circular segment.

6 Erect prisms on these triangles. Each such prism is more than half the corresponding segment of the cylinder.

7 Repeating this, leave segments of the cylinder whose total volume is less than the excess by which the cylinder exceeds three times the cone.

8 Thus, the remaining prism (on the inscribed polygon) is greater than three times the cone.

9 But by Proposition 12.7 Corollary, this prism is exactly three times the pyramid on the same polygonal base and with apex at the cone's apex.

10 Therefore, this pyramid is greater than the cone, which is impossible as it is contained within the cone.

PROPOSITION 10 — PART II

EVERY CONE IS THE THIRD PART OF THE CYLINDER WHICH HAS THE SAME BASE AS IT, AND AN EQUAL HEIGHT.

II. Suppose the cylinder is less than three times the cone.

11 Therefore, the cylinder is not more than three times the cone.

12 Suppose the cylinder is less than three times the cone. Inversely, the cone is greater than a third of the cylinder.

13 Inscribe square $ABCD$ in the base. It is more than half the circle.

14 Erect a pyramid of the same apex as the cone on the square. This pyramid is more than half the cone.

15 Repeat the process of inscribing polygons and erecting pyramids to leave segments of the cone whose total volume is less than the excess by which the cone exceeds a third of the cylinder.

16 The remaining pyramid on the inscribed polygon is therefore greater than a third of the cylinder.

17 But this pyramid is a third of the prism on the same polygonal base and of the cylinder's height.

18 Therefore, this prism is greater than the cylinder, which is impossible as it is contained within the cylinder.

19 Therefore, the cylinder is not less than three times the cone.

20 Thus, the cylinder is exactly three times the cone.

PROPOSITION 11

Statement. Cones and cylinders having the same height are to one another as their bases.

Proof.

1. Let AL and EN be cones of equal height on bases $ABCD$ and $EFGH$.
2. Assume the contrary: as base $ABCD$ is to base $EFGH$, so cone AL is to some solid O , either less than or greater than cone EN . Suppose O is lesser.
3. Let X be the difference between EN and O , so cone $EN = O + X$.
4. Inscribe a square $EFGH$ in the base $EFGH$. The square is more than half the circle.
5. Erect a pyramid of the same height on the square. This pyramid is more than half the cone.
6. Repeatedly inscribe polygons, erecting pyramids on them, until the remaining segments of cone EN are less than X .
7. The remaining pyramid in cone EN is therefore greater than O .
8. Inscribe a similar polygon in circle $ABCD$ and erect a pyramid of the same height.
9. By Proposition XII.1,

$$\frac{\text{polygon in } ABCD}{\text{polygon in } EFGH} = \frac{AC^2}{EG^2} = \frac{\text{circle } ABCD}{\text{circle } EFGH}.$$

10. By Proposition XII.6, the pyramids on the polygons are in the same ratio as the polygons.
11. By the assumption,

$$\frac{\text{cone } AL}{O} = \frac{\text{circle } ABCD}{\text{circle } EFGH}.$$

12. Therefore

$$\frac{\text{cone } AL}{O} = \frac{\text{pyramid in } AL}{\text{pyramid in } EN}.$$

13. Alternately,

$$\frac{\text{cone } AL}{\text{pyramid in } AL} = \frac{O}{\text{pyramid in } EN}.$$

14. Since cone AL is greater than its pyramid, O is greater than the pyramid in EN , a contradiction.

15. Therefore O cannot be less than cone EN .

16. By a similar argument, using the converse of the lemma, O cannot be greater than cone EN .

17. Therefore, as circle $ABCD$ is to circle $EFGH$, so cone AL is to cone EN .

18. Since cylinders on the same bases and heights are three times the cones, the cylinders are in the same ratio as their bases.

Therefore. Equal-height cones and cylinders are proportional to their bases.

PROPOSITION 11 — PART I

CONES AND CYLINDERS HAVING THE SAME HEIGHT ARE TO ONE ANOTHER AS THEIR BASES.

- 1 Let AL and EN be cones of equal height on bases $ABCD$ and $EFGH$.

- 2 Assume the contrary: as base $ABCD$ is to base $EFGH$, so cone AL is to some solid O , either less than or greater than cone EN . Suppose O is less.

- 3 Let X be the difference between EN and O , so cone $EN = O + X$.

- 4 Inscribe a square $EFGH$ in the base $EFGH$. The square is more than half the circle.

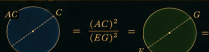

Area square $EFGH$
 $> \frac{1}{2}$ area circle
- 5 Erect a pyramid of the same height on the square. This pyramid is more than half the cone.


Volume pyramid on $EFGH$
 $> \frac{1}{2}$ volume cone EN
- 6 Repeatedly inscribe polygons, erecting pyramids on them, until the remaining segments of cone EN are less than X .


Remaining segments
 $< X$
- 7 The remaining pyramid in cone EN is therefore greater than O .

- 8 Inscribe a similar polygon in circle $ABCD$ and erect a pyramid of the same height.

- 9 By Proposition 12.1,

$$\frac{\text{polygon } ABCD}{\text{polygon } EFGH} = \frac{(AC)^2}{(EG)^2} = \frac{(AC)^2}{(EG)^2} = \frac{\text{circle } ABCD}{\text{circle } EFGH}$$


PROPOSITION 11 — PART II

CONES AND CYLINDERS HAVING THE SAME HEIGHT ARE TO ONE ANOTHER AS THEIR BASES.

- 10 By Proposition 12.6, the pyramids on the polygons are in the same ratio as the polygons.

$$\frac{\text{cone } AL}{\text{circle } EFGH} = \frac{\text{circle } ABCD}{\text{polygon } EFGH}$$

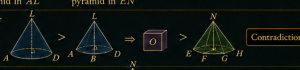
- 11 By the assumption,

$$\frac{\text{cone } AL}{O} = \frac{\text{circle } ABCD}{\text{circle } EFGH}$$

- 12 Therefore,

$$\frac{\text{cone } AL}{O} = \frac{\text{pyramid in } AL}{\text{pyramid in } EN}$$

- 13 Alternating,

$$\frac{\text{cone } AL}{\text{pyramid in } AL} = \frac{O}{\text{pyramid in } EN}$$
- 14 Since cone AL is greater than its pyramid, O is greater than the pyramid in EN , which is a contradiction.


Contradiction
- 15 Therefore, O cannot be less than cone EN .

- 16 By a similar argument (using the converse of the lemma), it can be shown that O cannot be greater than cone EN .

- 17 Therefore, as circle $ABCD$ is to circle $EFGH$, so cone AL is to cone EN .

$$\frac{\text{circle } ABCD}{\text{circle } EFGH} = \frac{\text{cone } AL}{\text{cone } EN}$$

- 18 Since the cylinders on the same bases and heights are three times the cones, the cylinders are in the same ratio as their bases.

$$\frac{\text{cylinder on } ABCD}{\text{cylinder on } EFGH} = \frac{\text{cone } AL}{\text{cone } EN} = \frac{\text{circle } ABCD}{\text{circle } EFGH}$$


Therefore,

PROPOSITION 11.

CONES AND CYLINDERS HAVING THE SAME HEIGHT ARE TO ONE ANOTHER AS THEIR BASES.

PROPOSITION 12

Statement. Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Proof.

1. Let cones, and likewise cylinders, $ABCDL$ and $EFGHN$ be similar. Let BD and FH be diameters of their bases, and KL and MN their axes.
2. Assume cone $ABCDL$ has to some solid O , less than cone $EFGHN$, the cubed ratio that BD has to FH .
3. Inscribe square $EFGH$ in the base circle. The square is more than half the circle.
4. Erect a pyramid with apex N on the square. This pyramid is more than half the cone.
5. Repeatedly inscribe polygons, erecting pyramids, until the remaining segments of cone $EFGHN$ are less than the excess by which cone $EFGHN$ exceeds O .
6. Therefore the remaining pyramid, on polygon $EPFQGRHS$, is greater than O .
7. Inscribe a similar polygon $ATBUCVDW$ in circle $ABCD$ and erect a pyramid with apex L .
8. Establish that the pyramids on these polygons are similar to each other, by constructing smaller similar triangles.
9. Therefore, by Proposition XII.8, the ratio of these whole polygonal pyramids is the cubed ratio of corresponding sides, which is the cubed ratio of BD to FH .
10. By the assumption,

$$\frac{\text{cone } ABCDL}{O} = \left(\frac{BD}{FH} \right)^3.$$

11. Therefore

$$\frac{\text{cone } ABCDL}{O} = \frac{\text{pyramid in } AL}{\text{pyramid in } EN}.$$

12. Alternately,

$$\frac{\text{cone } AL}{\text{pyramid in } AL} = \frac{O}{\text{pyramid in } EN}.$$

13. Since the cone is greater than the pyramid within it, O must be greater than the pyramid in EN , contradicting step 6.

14. Therefore cone $ABCDL$ does not have the cubed ratio of BD to FH to a solid less than cone $EFGHN$.

15. By a similar argument, it does not have this ratio to a solid greater than cone $EFGHN$.

16. Hence

$$\frac{\text{cone } ABCDL}{\text{cone } EFGHN} = \left(\frac{BD}{FH} \right)^3.$$

17. Since cylinders are three times their corresponding cones, the cylinders are also in the cubed ratio of the diameters of their bases.

Therefore. Similar cones and similar cylinders are in the cubed ratio of corresponding base diameters.

PROPOSITION 12 — PART I		PROPOSITION 12 — PART II	
SIMILAR CONES AND CYLINDERS ARE TO ONE ANOTHER IN THE CUBED RATIO OF THE DIAMETERS OF THEIR BASES.		SIMILAR CONES AND CYLINDERS ARE TO ONE ANOTHER IN THE CUBED RATIO OF THE DIAMETERS OF THEIR BASES.	
1	Let cone (and cylinder) $ABCDL$ and $EFGHN$ be similar. Let BD and FH be diameters of their bases, and KL and MN be their axes.	10	By the assumption, cone $ABCDL/O = (BD/FH)^3$.
2	Assume cone $ABCDL$ has to some solid O , less than cone $EFGHN$, the cubed ratio that BD has to FH .	11	Therefore, cone $ABCDL$ pyramid in AL pyramid in EN =
3	Inscribe square $EFGH$ in base circle. The square is more than half the circle.	12	Alternating, cone AL pyramid in AL pyramid in EN =
4	Erect a pyramid with apex N on the square. This pyramid is more than half the cone.	13	Since the cone is greater than the pyramid within it, O must be greater than the pyramid in EN , contradicting step 6.
5	Repeatedly inscribe polygons, erecting pyramids, until the remaining segments of cone $EFGHN$ are less than the excess by which cone $EFGHN$ exceeds O .	14	Therefore, cone $ABCDL$ does not have the cubed ratio of BD to FH to a solid less than cone $EFGHN$.
6	Therefore, the remaining pyramid (on polygon $EFGH$) is greater than O .	15	By a similar argument, it does not have this ratio to a solid greater than cone $EFGHN$.
7	Inscribe a similar polygon $ABRCVUW$ in circle $ABCD$ and erect a pyramid with apex L .	16	Hence, cone $ABCDL$ cone $EFGHN$ = $(BD/FH)^3$.
8	Establish that the pyramids on these polygons are similar to each other. (By constructing smaller similar triangles).	17	Since cylinders are three times their corresponding cones, the cylinders are also in the cubed ratio of the diameters of their bases.
9	Therefore, by Prop. 12.8, the ratio of these whole polyhedral pyramids is the cubed ratio of their corresponding sides, which is the cubed ratio of BD to FH .		

PROPOSITION 12.
SIMILAR CONES AND CYLINDERS ARE TO ONE ANOTHER IN THE CUBED RATIO OF THE DIAMETERS OF THEIR BASES.

PROPOSITION 13

Statement. If a cylinder is cut by a plane which is parallel to the opposite planes of the cylinder, then as the cylinder is to the cylinder, so the axis will be to the axis.

Proof.

1. Let cylinder AD be cut by plane GH , parallel to its bases AB and CD , meeting the axis EF at K .
2. Produce the axis EF in both directions to L and M .
3. Set out any number of lengths EN and NL equal to EK , and any number of lengths FO and OM equal to FK .
4. Conceive a cylinder on axis LM with bases PQ and VW .
5. Through points N and O , draw planes parallel to the bases, creating circles RS and TU around centers N and O .
6. Since axes LN, NE, EK are equal, the cylinders QR, RB, BG are equal, as they have equal bases.
7. Thus, as many multiples as axis KL is of axis EK , so many multiples is cylinder QG of cylinder GB .
8. Similarly, as many multiples as axis MK is of axis KF , so many multiples is cylinder WG of cylinder GD .
9. If $KL > KM$ then $QG > WG$; if equal, equal; if less, less.
10. Therefore, by Definition V.5,

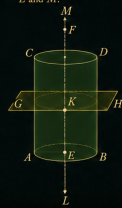
$$\frac{EK}{KF} = \frac{BG}{GD}.$$

Therefore. The two parts of the cylinder are in the ratio of the corresponding parts of the axis.

1 Let cylinder AD be cut by plane GH parallel to its bases AB and CD , meeting the axis EF at point K .



2 Produce the axis EF in both directions to L and M .



3 Set out any number of lengths EN and NL equal to EK , and any number of lengths FO and OM equal to FK .



$FO = FK$
 $OM = FK$

$EN = EK$
 $NL = EK$

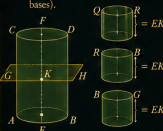
4 Conceive a cylinder on axis LM with bases PQ and VW .



5 Through points N and O , draw planes parallel to the bases, creating circles RS and TU around centers N and O .

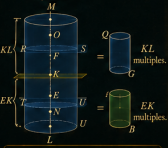


6 Since axes LN , NE , and EK are equal, the cylinders QR , RB , and BG are equal (as they have equal bases).



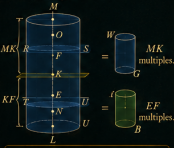
Thus, $QR = RB = BG$.

7 Thus, as many multiples as axis KL is of axis EK , so many multiples is cylinder QG of cylinder GB .



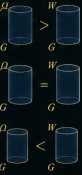
$\frac{KL}{EK} \text{ multiples} \Rightarrow \frac{QG}{GB} = \frac{KL}{EK}$

8 Similarly, as many multiples as axis MK is of axis KF , so many multiples is cylinder WG of cylinder GD .



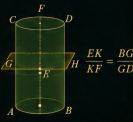
$\frac{MK}{KF} \text{ multiples} \Rightarrow \frac{WG}{GD} = \frac{MK}{KF}$

9 If $KL > KM$ then $QG > WG$; if equal, equal; if less, less.



$KL > KM \Rightarrow QG > WG$
 $KL = KM \Rightarrow QG = WG$
 $KL < KM \Rightarrow QG < WG$

10 Therefore, by the definition of proportion (Def. 5.8), axis EK is to axis KF as cylinder BG is to cylinder GD .



$\frac{EK}{KF} = \frac{BG}{GD}$

PROPOSITION 13.

IF A CYLINDER IS CUT BY A PLANE WHICH IS PARALLEL TO THE OPPOSITE PLANES (OF THE CYLINDER) THEN AS THE CYLINDER (IS) TO THE CYLINDER, SO THE AXIS WILL BE TO THE AXIS.

PROPOSITION 14

Statement. Cones and cylinders which are on equal bases are to one another as their heights.

Proof.

1. Let EB and FD be cylinders on equal bases AB and CD . Let their heights be GH and KL .
2. Produce axis KL to N , and make $LN = GH$.
3. Conceive cylinder CM on axis LN .
4. Since cylinders EB and CM have the same height, their ratio is equal to the ratio of their bases by Proposition XII.11.
5. Since their bases are equal, cylinders EB and CM are equal.
6. Cylinder FM has been cut by plane CD parallel to its bases, so by Proposition XII.13,

$$\frac{CM}{FD} = \frac{LN}{KL}.$$

7. Since $CM = EB$ and $LN = GH$,

$$\frac{EB}{FD} = \frac{GH}{KL}.$$

8. As the cylinder ratio is the same as the cone ratio by Proposition XII.10, the corresponding cones also have the same ratio as their heights.

Therefore. Cones and cylinders on equal bases are proportional to their heights.

1 Let EB and FD be cylinders on equal bases AB and CD . Let their heights be GH and KL .

Bases AB and CD are equal.
 GH, KL are the heights.

2 Produce the axis KL to N , and make LN equal to GH .

$LN = GH$

3 Conceive cylinder CM on axis LN .

4 Since cylinders EB and CM have the same height, their ratio is equal to the ratio of their bases (Prop. 12.11).

$\frac{EB}{CM} = \frac{\text{area}(AB)}{\text{area}(CD)}$

5 Since their bases are equal, cylinders EB and CM are equal.

$AB = CD \Rightarrow EB = CM.$

6 Cylinder FM has been cut by plane CD parallel to its bases, so by Proposition 12.13,

$$\frac{CM}{FD} = \frac{LN}{KL}.$$

LN
 KL

7 Since $CM = EB$ and $LN = GH$,

$$\frac{EB}{FD} = \frac{GH}{KL}.$$

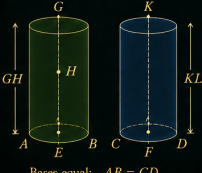
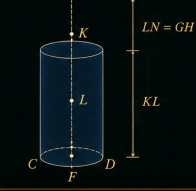
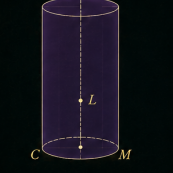
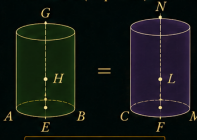
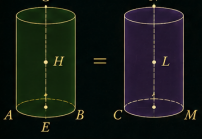
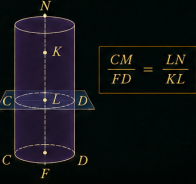
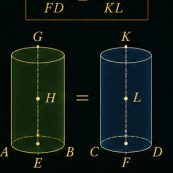
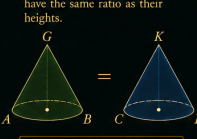
$GH = KL$

8 As the cylinder ratio is the same as the cone ratio (Prop. 12.10), the corresponding cones also have the same ratio as their heights.

$$\frac{\text{cone on } AB}{\text{cone on } CD} = \frac{GH}{KL}.$$

PROPOSITION 14.
CONES AND CYLINDERS WHICH ARE ON EQUAL BASES ARE TO ONE ANOTHER AS THEIR HEIGHTS.

ALTERNATIVE ILLUSTRATION

1 Let EB and FD be cylinders on equal bases AB and CD. Let their heights be GH and KL.  Bases equal: $AB = CD$	2 Produce axis KL to N, and make LN equal to GH.  $LN = GH$	3 Conceive cylinder CM on axis LN. 	4 Since cylinders EB and CM have the same height, their ratio is equal to the ratio of their bases (Prop. 12.11).  $\frac{EB}{CM} = \frac{\text{base } AB}{\text{base } CM}$
5 Since their bases are equal, cylinders EB and CM are equal.  $EB = CM$	6 Cylinder FM has been cut by plane CD parallel to its bases, so by Proposition 12.13,  $\frac{CM}{FD} = \frac{LN}{KL}$	7 Since $CM = EB$ and $LN = GH$, $\frac{EB}{FD} = \frac{GH}{KL}$ 	8 As the cylinder ratio is the same as the cone ratio (Prop. 12.10), the corresponding cones also have the same ratio as their heights.  $\frac{\text{Cone on } AB}{\text{Cone on } CD} = \frac{GH}{KL}$

PROPOSITION 14.

CONES AND CYLINDERS WHICH ARE ON EQUAL BASES ARE TO ONE ANOTHER AS THEIR HEIGHTS.

PROPOSITION 15

Statement. The bases of equal cones and cylinders are reciprocally proportional to their heights. And those cones and cylinders whose bases are reciprocally proportional to their heights are equal.

Proof.

1. Let cones and cylinders on bases $ABCD$ and $EFGH$ be equal. Let their heights be KL and MN .
2. Complete the cylinders AO and EP .
3. Case 1: heights are equal. Since the cylinders are equal and have the same height, their bases are equal by Proposition XII.11. This satisfies the reciprocal proportion.
4. Case 2: heights are unequal. Suppose $MN > KL$.
5. Cut off QN from MN equal to KL .
6. Cut cylinder EP with a plane through Q parallel to its bases, forming cylinder ES with height NQ and base $EFGH$.
7. Since $AO = EP$,

$$\frac{AO}{ES} = \frac{EP}{ES}.$$

8. By Proposition XII.11,

$$\frac{AO}{ES} = \frac{ABCD}{EFGH}.$$

9. By Proposition XII.13,

$$\frac{EP}{ES} = \frac{MN}{QN}.$$

10. Therefore

$$\frac{ABCD}{EFGH} = \frac{MN}{QN}.$$

11. Since $QN = KL$,

$$\frac{ABCD}{EFGH} = \frac{MN}{KL}.$$

Thus the bases and heights are reciprocally proportional.

12. For the converse, assume

$$\frac{ABCD}{EFGH} = \frac{MN}{KL}.$$

13. Construct as before with $QN = KL$.

14. Then

$$\frac{ABCD}{EFGH} = \frac{MN}{QN}.$$

15. Therefore

$$\frac{AO}{ES} = \frac{EP}{ES},$$

by Propositions XII.11 and XII.13.

16. Hence $AO = EP$. The cylinders are equal. The same proof applies to cones.

Therefore. Equal cones and cylinders have bases reciprocally proportional to their heights, and conversely.

I. THE BASES OF EQUAL CONES AND CYLINDERS ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS.

1 Let cones and cylinders on bases $ABCD$ and $EFGH$ be equal. Let their heights be KL and MN .

2 Complete the cylinders AO and EP .

3 CASE 1: Heights are equal. If $KL = MN$, then the cylinders AO and EP are equal and have the same height; therefore, their bases are equal (Prop. 12.11).

4 CASE 2: Heights are unequal. Suppose $MN > KL$.

5 Cut off QN from MN equal to KL .

6 Cut cylinder EP with a plane through Q parallel to its base, forming cylinder ES of height NQ and base $EFGH$.

7 Since $AO = EP$, $\frac{AO}{ES} = \frac{EP}{ES}$.

8 By Prop. 12.11, $\frac{AO}{ES} = \frac{ABCD}{EFGH}$.

9 By Prop. 12.13, $\frac{EP}{ES} = \frac{MN}{QN}$.

10 Therefore, $\frac{ABCD}{EFGH} = \frac{MN}{QN}$.

11 Since $QN = KL$, $\frac{ABCD}{EFGH} = \frac{MN}{KL}$. Thus, the bases and heights are reciprocally proportional.

II. THOSE CONES AND CYLINDERS WHOSE BASES ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS ARE EQUAL.

12 For the converse, assume $\frac{ABCD}{EFGH} = \frac{MN}{KL}$.

13 Construct as before with $QN = KL$.

14 Then $\frac{ABCD}{EFGH} = \frac{MN}{QN}$.

15 Therefore, by Props. 12.11 and 12.13, $\frac{AO}{ES} = \frac{EP}{ES}$.

16 Hence, $AO = EP$. The cylinders are equal.

The same proof applies to cones.

PROPOSITION 15.

THE BASES OF EQUAL CONES AND CYLINDERS ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS.
AND, THOSE CONES AND CYLINDERS WHOSE BASES ARE RECIPROCALLY PROPORTIONAL TO THEIR HEIGHTS ARE EQUAL.

PROPOSITION 16

Statement. There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.

Proof.

1. Let $ABCD$ and $EFGH$ be two circles about the same center K .
2. Draw diameter BKD .
3. Through point G , draw GA perpendicular to BD , extending it to C . Therefore AC is tangent to the smaller circle.
4. Let arc AD be bisected repeatedly until a chord LD is left that is less than arc AD .
5. Through L , draw LM perpendicular to BD , meeting the circle at N .
6. Join LD and DN . Then $LD = DN$.
7. Since $LN \parallel AC$, both being perpendicular to BD , and AC touches the inner circle, LN does not touch the inner circle. Therefore LD and DN do not touch it.
8. By repeatedly inserting chords equal to LD into circle $ABCD$, an equilateral and even-sided polygon is inscribed. None of its sides can touch the inner circle.

Therefore. An equilateral even-sided polygon has been inscribed in the greater circle without touching the lesser concentric circle.

PROPOSITION 16.

THERE BEING TWO CIRCLES ABOUT THE SAME CENTER, TO INSCRIBE AN EQUILATERAL AND EVEN-SIDED POLYGON IN THE GREATER CIRCLE, NOT TOUCHING THE LESSER CIRCLE.

- 1 Let $ABCD$ and $EFGH$ be two circles about the same center K .



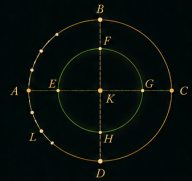
- 2 Draw diameter BKD .



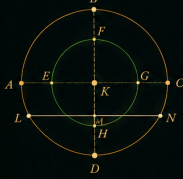
- 3 Through point G , draw GA perpendicular to BD , extending it to C . Therefore, AC is tangent to the smaller circle.



- 4 Let arc AD be bisected repeatedly until a chord LD is left that is less than the arc AD .



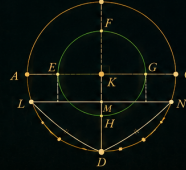
- 5 Through L , draw LM perpendicular to BD , meeting the circle at N .



- 6 Join LD and DN . $LD = DN$.



- 7 Since LN is parallel to AC (both perpendicular to BD), and AC touches the inner circle, LN does not touch the inner circle. Therefore, LD and DN do not touch it.



- 8 By repeatedly inserting chords equal to LD into circle $ABCD$, an equilateral and even-sided polygon is inscribed. None of its sides can touch the inner circle, as they are all either chords like LN or LD .



PROPOSITION 17

Statement. There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Proof.

1. Let the spheres have the same center A . Let $BCDE$ be a great circle of the larger sphere, and FGH a great circle of the smaller.
2. In the greater circle, inscribe an equilateral and even-sided polygon that does not touch the lesser circle, as in Proposition XII.16. Let BK, KL, LM, ME be its sides in quadrant BE .
3. Join KA and extend it to N .
4. Draw AO perpendicular to the plane of circle $BCDE$, meeting the surface of the larger sphere at O .
5. Construct great circles through AO and the diameters BD and KN , making semicircles BOD and KON .
6. Divide the arcs BO and KO similarly to BE , marking points P, Q, R on BO and S, T, U on KO , such that chords BP, PQ, QR, RO are equal to BK, KL, LM, ME , and KS, ST, TU, UO are similarly equal.
7. Join SP, TQ, UR .
8. Draw perpendiculars PV and SW from P and S to the plane $BCDE$, falling on BD and KN at V and W .
9. Join WV .
10. Establish that $PV = SW$ and $BV = KW$, hence $WV \parallel KB$. Also WV is parallel and equal to SP . Therefore quadrilateral $KBPS$ is in one plane.
11. Similarly, quadrilaterals $SPQT, TQRU$ are in one plane, and triangle URO is in one plane.
12. If we connect points P, S, Q, T, R, U to A , we get a polyhedral solid composed of pyramids with bases $KBPS, SPQT, TQRU, URO$ and apex A .

13. By repeating this construction for the other sides and quadrants, the polyhedron is completed.
14. Draw AX perpendicular to the plane of quadrilateral $KBPS$, meeting it at X .
15. Join XB and XK .
16. Since $AB = AK$, the square on AB equals the square on AK .
17. As AX is perpendicular to the plane, triangles AXB and AXK are right-angled, so

$$AB^2 = AX^2 + XB^2, \quad AK^2 = AX^2 + XK^2.$$

18. Therefore $XB = XK$. A circle with center X passes through B and K , and it can be shown it also passes through P and S .
19. In this circle, $KB = BP = KS$, and PS is shorter. Thus the square on KB is greater than twice the square on BX .
20. Drop perpendicular KY from K to BV .
21. Establish that the square on KB is less than twice the square on KY .
22. Therefore $KY^2 > BX^2$. Since $AB = AK$,

$$AX^2 + XB^2 = AY^2 + YK^2.$$

Hence $AX^2 > AY^2$, and therefore $AX > AY$.

23. AX is the perpendicular distance from the center to one of the polyhedron's faces. AG is a radius of the smaller sphere.
24. Since $AX > AY$ and $AY > AG$, as AG is shorter than any line from A to the chord LN , AX is much greater than AG .
25. Therefore the face $KBPS$ of the polyhedron is farther from the center than the surface of the inner sphere. Thus the polyhedron does not touch the inner sphere.

Therefore. A polyhedral solid has been inscribed in the greater sphere without touching the lesser concentric sphere.

PROPOSITION 17.

THERE BEING TWO SPHERES ABOUT THE SAME CENTER, TO INSCRIBE A POLYHEDRAL SOLID IN THE GREATER SPHERE, NOT TOUCHING THE LESSER SPHERE ON ITS SURFACE.

- 1 Let the spheres have the same center A . Let $BCDE$ be a great circle of the larger sphere, and FGH a great circle of the smaller.



- 2 In the greater circle, inscribe an equilateral and even-sided polygon that does not touch the lesser circle (as in Prop. 12.16). Let BK, KL, LM, ME be its sides in quadrant BE .



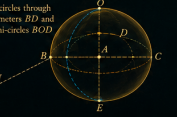
- 3 Join KA and extend it to N .



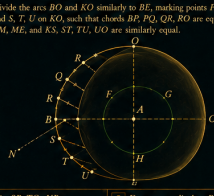
- 4 Draw AO perpendicular to the plane of circle $BCDE$, meeting the surface of the larger sphere at O .



- 5 Construct great circles through AO and the diameters BD and KN , making semi-circles BOD and KON .



- 6 Divide the arcs BO and KO similarly to BE , marking points P, Q, R on BO and S, T, U on KO , such that chords BP, PQ, QR, RO are equal to BK, KL, LM, ME , and KS, ST, TU, UO are similarly equal.



- 7 Join SP, TQ, UR .



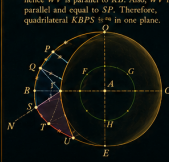
- 8 Draw perpendiculars PV and SW from P and S to the plane $BCDE$, falling on BD and KO at V and W .



- 9 Join WV .



- 10 Establish that $PV = SW$ and $BV = KW$, hence WV is parallel to KB . Also, WV is parallel and equal to SP . Therefore, quadrilateral $KBPS$ is in one plane.



- 12 If we connect points P, S, Q, T, R, U to A , we get a polyhedral solid composed of pyramids with bases KBP, SPQ, TQR, URO and apex A .



- 13 By repeating this construction for the other sides and quadrants, the polyhedron is completed.



- 14 Draw AX perpendicular to the plane of quadrilateral $KBPS$, meeting it at X .



- 15 Join XB and XK .

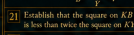


- 16 Since $AB = AK$, the square on AB equals the square on AK .

- 17 As AX is perpendicular to the plane, triangles AXB and AXK are right-angled, so $AB^2 = AX^2 + XB^2$ and $AK^2 = AX^2 + XK^2$.

- 19 In this circle, $KB = BP = KS$, and PS is shorter. Thus, the square on KB is greater than twice the square on BV .

- 20 Drop perpendicular KY from K to BV .



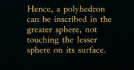
- 21 Establish that the square on KB is less than twice the square on KY .

- 22 Therefore, $KY^2 > BV^2$. Since $AB = AK$, $AX^2 + XB^2 = AY^2 + YK^2$. Hence, $AX^2 > AY^2$ and $AX > AY$.

- 23 AX is the perpendicular distance from the center to one of the polyhedron's faces, AG is a radius of the smaller sphere.

- 24 Since $AX > AY$ and $AY > AG$ (as AG is shorter than any line from A to the chord KN), AX is much greater than AG .

- 25 Therefore, the face $KBPS$ of the polyhedron is further from the center than the surface of the inner sphere. Thus, the polyhedron does not touch the inner sphere.



Hence, a polyhedron can be inscribed in the greater sphere, not touching the lesser sphere on its surface.

COROLLARY TO PROPOSITION 17

Cubed ratio for similar inscribed polyhedra

Statement. If a similar polyhedral solid is inscribed in another sphere, the polyhedral solid in the first sphere is to that in the other in the cubed ratio of the diameters of their respective spheres.

Proof.

1. If the polyhedra are divided into pyramids, these pyramids are similar.
2. Similar pyramids are in the cubed ratio of corresponding sides.
3. The corresponding sides are the radii of the spheres.
4. Therefore the ratio of the pyramids, and thus the whole polyhedra, is the cubed ratio of the radii, which is also the cubed ratio of the diameters.

Therefore. Similar polyhedra inscribed in spheres are in the cubed ratio of the diameters of those spheres.

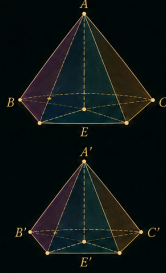
COROLLARY.

IF A SIMILAR POLYHEDRAL SOLID IS INSCRIBED IN ANOTHER SPHERE,
THE POLYHEDRAL SOLID IN THE FIRST SPHERE IS TO THAT IN THE OTHER
IN THE CUBED RATIO OF THE DIAMETERS OF THEIR RESPECTIVE SPHERES.

- 1 If the polyhedra are divided into pyramids, these pyramids are similar.

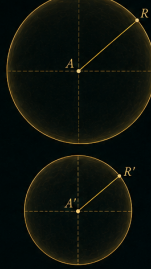


- 2 Similar pyramids are in the cubed ratio of corresponding sides.



$$\frac{(\text{Pyramid } ABCD)}{(\text{Pyramid } A'B'C'D')} = \left(\frac{AB}{A'B'}\right)^3$$

- 3 The corresponding sides are the radii of the spheres.



$$\begin{aligned} AB = AC = AD = AE = \dots = R \\ A'B' = A'C' = A'D' = A'E' = \dots = R' \end{aligned}$$

- 4 Therefore, the ratio of the pyramids, and thus of the whole polyhedra, is the cubed ratio of the radii, which is the cubed ratio of the diameters.



$$\begin{aligned} \text{Polyhedron in sphere } (R) &= \left(\frac{R}{R'}\right)^3 = \left(\frac{2R}{2R'}\right)^3 \\ \text{Polyhedron in sphere } (R') & \end{aligned}$$

Hence, if a similar polyhedral solid is inscribed in another sphere,
the polyhedral solid in the first sphere is to that in the other in the cubed ratio of the diameters of their respective spheres.

PROPOSITION 18

Statement. Spheres are to one another in the cubed ratio of their respective diameters.

Proof.

1. Let ABC and DEF be two spheres with diameters BC and EF .
2. Assume sphere ABC has to a sphere GHK , less than DEF , the cubed ratio that BC has to EF .
3. Conceive sphere DEF as concentric with GHK .
4. Inscribe a polyhedral solid in sphere DEF that does not touch the inner sphere GHK , by Proposition XII.17.
5. Inscribe a similar polyhedral solid in sphere ABC .
6. By the corollary to Proposition XII.17,

$$\frac{\text{solid in } ABC}{\text{solid in } DEF} = \left(\frac{BC}{EF} \right)^3.$$

7. By assumption,

$$\frac{\text{sphere } ABC}{\text{sphere } GHK} = \left(\frac{BC}{EF} \right)^3.$$

8. Therefore

$$\frac{\text{sphere } ABC}{\text{sphere } GHK} = \frac{\text{solid in } ABC}{\text{solid in } DEF}.$$

9. Alternately,

$$\frac{\text{sphere } ABC}{\text{solid in } ABC} = \frac{\text{sphere } GHK}{\text{solid in } DEF}.$$

10. Since sphere ABC is greater than the solid within it, sphere GHK must be greater than the solid in DEF .
11. But sphere GHK is less than sphere DEF , and the solid in DEF lies outside the surface of GHK in the relevant comparison. Hence sphere GHK is less than that solid. Contradiction.

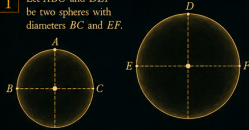
12. Therefore sphere ABC does not have the cubed ratio of BC to EF to any sphere less than DEF .
13. Similarly, it can be shown that it does not have this ratio to any sphere greater than DEF .
14. Therefore

$$\frac{\text{sphere } ABC}{\text{sphere } DEF} = \left(\frac{BC}{EF} \right)^3 .$$

Therefore. Spheres are to one another in the cubed ratio of their respective diameters.

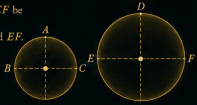
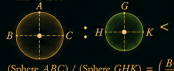
PROPOSITION 18.

SPHERES ARE TO ONE ANOTHER IN THE CUBED RATIO OF THEIR RESPECTIVE DIAMETERS.

1 Let ABC and DEF be two spheres with diameters BC and EF. 	2 Assume sphere ABC has to a sphere GHK, less than DEF, the cubed ratio that BC has to EF.  $=$  $\text{Sphere } ABC = \left(\frac{BC}{EF}\right)^3$ with $\text{Sphere } GHK < \text{Sphere } DEF$	3 Conceive sphere DEF as concentric with GHK. 		
4 Inscribe a polyhedral solid in sphere DEF that does not touch the inner sphere GHK (Prop. 12.17). 	5 Inscribe a similar polyhedral solid in sphere ABC. 	6 By the corollary to Prop. 12.17, $\frac{\text{Solid in } ABC}{\text{Solid in } DEF} = \left(\frac{BC}{EF}\right)^3$	7 By assumption, $\frac{\text{Sphere } ABC}{\text{Sphere } GHK} = \left(\frac{BC}{EF}\right)^3$	8 Therefore, $\frac{\text{Sphere } ABC}{\text{Sphere } GHK} = \frac{\text{Solid in } ABC}{\text{Solid in } DEF}$
9 Alternately, $\frac{\text{Sphere } ABC}{\text{Solid in } ABC} = \frac{\text{Sphere } GHK}{\text{Solid in } DEF}$	10 Since sphere ABC is greater than the solid within it, sphere GHK must be greater than the solid in DEF.  $<$ 	11 But sphere GHK is less than sphere DEF and contains the solid in DEF, so sphere GHK is less than that solid. Contradiction.  $<$ 		
12 Therefore, sphere ABC does not have the cubed ratio of BC to EF to any sphere less than DEF.  $= \left(\frac{BC}{EF}\right)^3$  Impossible.	13 Similarly, it can be shown it does not have this ratio to any sphere greater than DEF.  $= \left(\frac{BC}{EF}\right)^3$  Also impossible.	14 Therefore, sphere ABC is to sphere DEF as the cube of BC is to the cube of EF.  $= \frac{\text{Sphere } ABC}{\text{Sphere } DEF} = \left(\frac{BC}{EF}\right)^3$ 		

ALTERNATIVE ILLUSTRATION

PROPOSITION 18.
SPHERES ARE TO ONE ANOTHER IN THE CUBED RATIO OF THEIR RESPECTIVE DIAMETERS.

1 Let ABC and DEF be two spheres with diameters BC and EF. 	2 Assume sphere ABC has to a sphere GHK, less than DEF, the cubed ratio that BC has to EF.  $(\text{Sphere } ABC) / (\text{Sphere } GHK) = \left(\frac{BC}{EF}\right)^3$	3 Conceive sphere DEF as concentric with GHK. 
4 Inscribe a polyhedral solid in sphere DEF that does not touch the inner sphere GHK (Prop. 12.17). 	5 Inscribe a similar polyhedral solid in sphere ABC. 	6 By the corollary to Prop. 12.17, $(\text{Solid in } ABC) = \left(\frac{BC}{EF}\right)^3 (\text{Solid in } DEF)$ 
7 By assumption, $\frac{(\text{Sphere } ABC)}{(\text{Sphere } GHK)} = \left(\frac{BC}{EF}\right)^3$ 	8 Therefore, $\frac{(\text{Sphere } ABC)}{(\text{Sphere } GHK)} = \frac{(\text{Solid in } ABC)}{(\text{Solid in } DEF)}$	9 Alternately, $\frac{(\text{Sphere } ABC)}{(\text{Sphere } GHK)} = \frac{(\text{Solid in } ABC)}{(\text{Solid in } DEF)}$ 
10 Since sphere ABC is greater than the solid within it, sphere GHK must be greater than the solid in DEF. 	11 But sphere GHK is less than sphere DEF and contains the solid in DEF, so sphere GHK is less than that solid. 	12 Therefore, sphere ABC does not have the cubed ratio of BC to EF to any sphere less than DEF. not $= \left(\frac{BC}{EF}\right)^3$ 
13 Similarly, it can be shown it does not have this ratio to any sphere greater than DEF. not $= \left(\frac{BC}{EF}\right)^3$ 	14 Therefore, sphere ABC is to sphere DEF as the cube of BC is to the cube of EF. $\frac{ABC}{DEF} = \left(\frac{BC}{EF}\right)^3$ 	

END OF BOOK XII