

EUCLID'S ELEMENTS

Book XIV

Comparison of the Regular Solids

Propositions 1–6

The Dodecahedron and Icosahedron,
their Surfaces, Volumes, and Relations to the Cube

Illustrated edition

Contents

Book XIV

Proposition 1

Statement. The perpendicular drawn from the centre of any circle to the side of the pentagon inscribed in that circle is half the sum of the sides of the hexagon and the decagon inscribed in the same circle.

Proof.

1. Draw circle ABF with centre Δ .
2. Let BF be a side of the pentagon.
3. Draw the perpendicular ΔE to BF .
4. Produce ΔE , and draw EZ and ΔA .
5. Join $\Delta \Gamma$ and ΓZ .
6. Set $HE = EZ$.
7. From H draw HF to Γ .
8. The circumference is five times the arc BZF .
9. The arc AFZ is half the circumference, while ZF is half of BZF .
10. Therefore the arc AFZ is five times the arc ZF .
11. Thus AF is four times ZF .
12. By the proportional relation of arcs and angles,

$$AF : ZF = \angle AFZ : \angle ZAF.$$

13. Therefore

$$\angle A\Delta F = 4\angle Z\Delta F.$$

14. But

$$\angle A\Delta F = 2\angle EZF.$$

15. Therefore

$$\angle EZF = 2\angle H\Delta F.$$

16. Also

$$\angle EZF = \angle EHF.$$

17. Therefore

$$\angle EHF = 2\angle H\Delta F.$$

18. Hence

$$\Delta H = HF.$$

19. But

$$HF = ZF.$$

20. Therefore

$$\Delta H = ZF.$$

21. Also

$$HE = EZ.$$

22. Therefore

$$\Delta E = EZ + ZF.$$

23. Adding the common ΔE ,

$$\Delta Z + ZF = 2\Delta E.$$

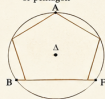
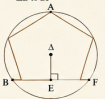
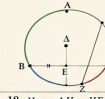
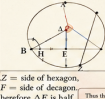
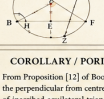
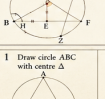
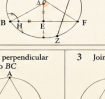
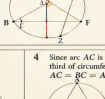
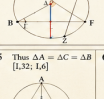
24. Now ΔZ is the side of the hexagon, and ZF is the side of the decagon.

25. Therefore ΔE is half the sum of the sides of the hexagon and the decagon.

Corollary / Porism

From Proposition XIII.12, the perpendicular from the centre to a side of an inscribed equilateral triangle is half the radius.

Therefore, the perpendicular from the centre of the circle to the side of the inscribed pentagon is half the sum of the corresponding hexagon and decagon sides.

PROPOSITION 1. The perpendicular drawn from the centre of any circle to the side of the pentagon inscribed in that circle is half the sum of the sides of the hexagon and the decagon inscribed in the same circle.					
1 Draw circle ABF with centre Δ 	2 Let BF be side of pentagon 	3 Draw perpendicular ΔE to BF 	4 Produce ΔE and draw EZ and ΔA 	5 Join $\Delta\Gamma$ and ΓZ 	6 Set $HE = EZ$ 
7 From H draw HF to Γ 	8 Note circumference is five times arc BZF 	9 Note arc AFZ is half circumference, ZF is half of BZF 	10 Therefore arc AFZ is five times arc ZF 	11 Thus AF is four times ZF 	12 Use $AF:ZF = AFZ:ZAF$ [V.1,33] 
13 Therefore $\angle\Delta MF = 4\angle Z\Delta F$ 	14 But $\angle\Delta MF = 2\angle EZF$ [I.1,20] 	15 Therefore $\angle EZF = 2\angle H\Delta F$ 	16 Also $\angle EZF = \angle EHF$ [I.4] 	17 Therefore $\angle EHF = 2\angle H\Delta F$ 	18 Hence $\Delta H = HF$ [I.32; I.6] 
19 But $HF = ZF$ [I.4] 	20 Therefore $\Delta H = ZF$ 	21 Also $HE = EZ$ 	22 Therefore $\Delta E = EZ + ZF$ 	23 Add common ΔE : $\Delta Z + ZF = 2\Delta E$ 	24 $\Delta Z =$ side of hexagon, $ZF =$ side of decagon. Therefore ΔE is half the sum. Thus the perpendicular from the centre to a side of the pentagon equals half the sum of the sides of the hexagon and the decagon inscribed in the same circle.
COROLLARY / PORISM From Proposition [12] of Book XIII, the perpendicular from centre to side of inscribed equilateral triangle is half the radius. Illustrate it in a panel of cinematic sequence.					
1 Draw circle ABC with centre Δ 	2 Draw perpendicular ΔE to BC 	3 Join ΔA 	4 Since arc AC is one third of circumference, $AC = BC = AB$ 	5 Thus $\Delta A = \Delta C = \Delta B$ [I.32; I.6] 	6 Therefore ΔE is half the radius, ΔA . Hence the perpendicular from the centre to a side of the inscribed equilateral triangle is half the radius.

Proposition 2

Statement. One and the same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

This result was previously set out by Aristaeus and Apollonius.

Lemma

Statement. If an equilateral and equiangular pentagon is inscribed in a circle, the square on the straight line subtending two of its sides together with the square on the side of the pentagon is five times the square on the radius.

Proof.

1. Draw circle ABF with centre Δ .
2. Let AF be a side of the pentagon.
3. Draw the perpendicular ΔZ to AF , produced to B and E .
4. Join AB .
5. Join AE , which is the side of the decagon.
6. Since

$$BE = 2E\Delta,$$

we have

$$BE^2 = 4E\Delta^2.$$

7. But

$$BA^2 + AE^2 = BE^2.$$

8. Therefore

$$BA^2 + AE^2 = 4\Delta E^2.$$

9. Hence

$$BA^2 + AE^2 + EA^2 = 5\Delta E^2.$$

10. But, by Proposition XIII.10,

$$AF^2 = \Delta E^2 + EA^2.$$

11. Therefore

$$AB^2 + AF^2 = 5\Delta E^2.$$

Proof of Proposition 2

1. Let AB be the diameter of the sphere.
2. Inscribe a dodecahedron and an icosahedron in the sphere.
3. Let $\Gamma\Delta EZH$ be a pentagon of the dodecahedron.
4. Let $K\Lambda\Theta$ be a triangle of the icosahedron.
5. It is required to show that the radii of the circles circumscribing these two figures are equal.
6. Join ΔH , the side of the cube, by Proposition XIII.17.
7. Set MN so that

$$AB^2 = 5MN^2.$$

8. Since the square on the diameter is five times the square on the radius of the circle from which the icosahedron is described, MN equals that radius.
9. Cut MN in extreme and mean ratio at Ξ , with $M\Xi$ the greater segment.
10. Then $M\Xi$ is the side of the decagon.
11. Since

$$AB^2 = 5MN^2$$

and

$$BA^2 = 3\Delta H^2$$

by Proposition XIII.15,

$$3\Delta H^2 = 5MN^2.$$

12. Therefore

$$3\Delta H^2 : 3H^2 = 5MN^2 : 5M\Xi^2.$$

13. By Propositions XIII.16 and XIII.10,

$$5M\Xi^2 + 5MN^2 = 5K\Lambda^2.$$

14. Therefore

$$5K\Lambda^2 = 3H^2 + 3\Delta H^2.$$

15. By Proposition XIII.12,

$$5K\Lambda^2$$

is equal to fifteen squares on the radius of the circle about triangle $\Theta K\Lambda$.

16. By the lemma,

$$3\Delta H^2 + 3H^2$$

is equal to fifteen squares on the radius of the circle about pentagon $\Gamma\Delta EZH$.

17. Therefore fifteen squares on the one radius are equal to fifteen squares on the other.

18. Hence the radii are equal.

Therefore, one and the same circle circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

PROPOSITION 2.

One and the same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

Previously set out by Aristotle and Apollonius.

LEMMA. If an equilateral and equiangular pentagon is inscribed in a circle, the square on the straight line subtending two of its sides together with the square on the side of the pentagon is five times the square on the radius.

1 Draw circle ABF with centre Δ

2 Let AF be side of pentagon

3 Draw perpendicular ΔZ to AF , produced to B and F

4 Join AB

5 Join AE (side of decagon)

6 Since $BE = ZE$, $BE^2 = AE^2$

7 But $BA^2 + AE^2 = BE^2$

8 Therefore $BA^2 + AE^2 = 4EA^2$

9 Hence $BA^2 + AE^2 + EA^2 = 5EA^2$

10 But $AF^2 = EA^2 + EA^2$ [XIII.10]

11 Therefore $AB^2 + AF^2 = 5EA^2$

Blue square = BA^2

Purple square = AE^2

Yellow square = EA^2

Green square = AF^2

Red square = EA^2 (radius squared)

PROOF OF PROPOSITION 2.

1 Let AB be diameter of sphere

2 Inscribe dodecahedron and icosahedron in sphere

3 Let $\Gamma\Delta Z\eta\theta$ be pentagon of dodecahedron

4 Let $KA\Theta$ be triangle of icosahedron

5 Need to show radii of circumscribed circles are equal

6 Join ΔH (side of cube) [XIII.17]

7 Set MN such that $AB^2 = 5MN^2$

8 Since diameter squared = 5 times radius squared of icosahedron's circle [XIII.16 post], MN equals that radius

9 Cut MN in extreme and mean ratio at Ξ , with ME greater segment

10 ME is side of decagon

11 Since $AB^2 = 5MN^2$ not $BA^2 = 3\Delta H^2$ [XIII.15], $3\Delta H^2 = 5MN^2$

12 Use $3\Delta H^2 = 3H^2 = 5MN^2 = 5ME^2$

13 $5ME^2 + 5MN^2 = 5KA^2$ [XIII.16.10]

14 Therefore $5KA^2 = 3H^2 + 3\Delta H^2$

15 $5KA^2 = 15$ squares on radius of circle about triangle ΘKA [XIII.12]

16 $3\Delta H^2 + 3H^2 = 15$ squares on radius of circle about $\Gamma\Delta Z\eta\theta$ (since $\Delta H^2 + H^2 = 5$ times the square on radius)

17 Therefore fifteen squares on one radius = fifteen squares on the other

18 Hence the radii of the two circumscribed circles are equal.

8

Proposition 3

Statement. If an equilateral and equiangular pentagon is given, and a circle is circumscribed about it, and a perpendicular is drawn from the centre to any one of the sides, then thirty times the rectangle contained by one side and the perpendicular is equal to the surface of the dodecahedron.

Proof.

1. Let ABE be an equilateral and equiangular pentagon with circumscribed circle $A\Delta$.
2. Let Z be the centre of the circle.
3. Draw the perpendicular ZH to $\Gamma\Delta$.
4. The claim is

$$30(\Gamma\Delta \cdot ZH) = 12 \text{ pentagons } AB\Gamma\Delta E.$$

5. Join ΓZ and $Z\Delta$.

6. Since

$$\Gamma\Delta \cdot ZH = 2[\triangle\Gamma\Delta Z],$$

7. it follows that

$$5(\Gamma\Delta \cdot ZH) = 10[\triangle\Gamma\Delta Z].$$

8. Taking each six times,

$$30(\Gamma\Delta \cdot ZH)$$

is equal to the surface of the dodecahedron.

Analogous Result for the Icosahedron

1. Let ABF be an equilateral triangle with a circumscribed circle.

2. Let Δ be the centre, and draw ΔE perpendicular to BF .

3. The claim is

$$30(BF \cdot \Delta E) = \text{surface of the icosahedron}.$$

4. Since

$$\Delta E \cdot BF = 2[\triangle \Delta BF],$$

5. taking each three times gives

$$6[\triangle \Delta BF] = 3(\Delta E \cdot BF).$$

6. But

$$6[\triangle \Delta BF] = 2[\triangle ABF].$$

7. Taking each ten times,

$$30(\Delta E \cdot BF) = 20[\triangle ABF].$$

Corollary

For solids inscribed in the same sphere,

$$\frac{\text{surface of dodecahedron}}{\text{surface of icosahedron}} = \frac{\text{side of pentagon} \times \text{its perpendicular}}{\text{side of triangle} \times \text{its perpendicular}}.$$

Therefore, the stated rectangles provide the corresponding surface measures of the dodecahedron and icosahedron.

PROPOSITION 3.
 If an equilateral and equiangular pentagon is given, and a circle is circumscribed about it, and a perpendicular is drawn from the centre to any one of the sides, then thirty times the rectangle contained by one side and the perpendicular is equal to the surface of the dodecahedron.

1 Let $ABFE$ be an equilateral and equiangular pentagon with circumscribed circle AA .

2 Z is the centre of the circle.

3 Draw perpendicular ZH to EA .

4 Claim: $30 \cdot \Gamma\Delta \times ZH = 12$ pentagons $AB\Gamma\Delta E$.

5 Join ΓZ and ZA .

6 Since $\Gamma A \times ZH = 2 \cdot$ triangle $\Gamma\Delta Z$.

7 Then $5 \cdot \Gamma\Delta \times ZH = 10$ triangles $\Gamma\Delta Z$.

8 Take each of these triangles six times.

9 Therefore $30 \cdot \Gamma\Delta \times ZH =$ surface of dodecahedron.

One rectangle $\Gamma A \times ZH = \frac{1}{30}$ of the surface of the dodecahedron.

SIMILAR FOR THE ICOSAHDREDON

1 Let ABF be an equilateral triangle with circumscribed circle.

2 Δ is the centre. Draw perpendicular ΔE to BF .

3 Claim: $30 \cdot BF \times \Delta E =$ surface of icosahedron.

4 $\Delta E \times BF = 2 \cdot$ triangle ΔBF .

5 Each taken three times: $6 \cdot$ triangle $\Delta BF = 3 \cdot \Delta E \times BF$.

6 But $6 \cdot$ triangle $\Delta BF = 2 \cdot$ triangle ABF .

7 Take each of the triangle ABF ten times.

8 Therefore $30 \cdot \Delta E \times BF =$ surface of icosahedron.

9 Hence $30 \cdot \Delta E \times BF =$ surface of icosahedron.

One rectangle $BF \times \Delta E = \frac{1}{30}$ of the surface of the icosahedron.

COROLLARY.
 As the surface of the dodecahedron is to the surface of the icosahedron, so is the rectangle (side of pentagon $\times$ perpendicular) to the rectangle (side of triangle $\times$ perpendicular), for solids inscribed in the same sphere.

Surface of dodecahedron = $\frac{\Gamma A \times ZH \text{ (rectangle from pentagon)}}{BF \times \Delta E \text{ (rectangle from triangle)}}$

From pentagon: $\Gamma\Delta$ ZH
 Rectangle = $\Gamma\Delta \times ZH$

From triangle: $B\Gamma$ ΔE
 Rectangle = $BF \times \Delta E$

Proposition 4

Statement. The side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

First Demonstration

Proof.

1. Draw circle ABF circumscribing both pentagon and triangle.
2. Inscribe $\Gamma\Delta$ as the side of the icosahedron, and AF as the side of the dodecahedron.
3. Let E be the centre of the circle.
4. Draw perpendiculars EZ and EH to $\Delta\Gamma$ and ΓA , respectively.
5. Produce EH to B .
6. Join BF .
7. Let Θ be the side of the cube.
8. It is required to prove

$$\Theta : \Gamma\Delta = \text{surface of dodecahedron} : \text{surface of icosahedron}.$$

9. When $BE + BF$ is cut in extreme and mean ratio, its greater segment is BE , by Proposition XIII.9.
10. By Proposition 1,

$$EH = \frac{1}{2}(BE + BF).$$

11. By the porism to Proposition 1,

$$EZ = \frac{1}{2}BE.$$

12. Therefore, when EH is cut in extreme and mean ratio, its greater segment is EZ .

13. When Θ is cut in extreme and mean ratio, its greater segment is $\Gamma\Delta$, by the porism to Proposition XIII.17.

14. Hence

$$\Theta : \Gamma\Delta = EH : EZ.$$

15. Therefore

$$ZE \cdot \Theta = \Gamma\Delta \cdot EH.$$

16. Since

$$\Theta : \Gamma\Delta = ZE \cdot \Theta : \Gamma\Delta \cdot ZE,$$

17. and

$$\Gamma\Delta \cdot EH = ZE \cdot \Theta,$$

18. it follows that

$$\Theta : \Gamma\Delta = \Gamma\Delta \cdot EH : \Gamma\Delta \cdot ZE.$$

19. By Proposition 3, this is

surface of dodecahedron : surface of icosahedron.

Therefore, the side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

PROPOSITION 4.
The side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

1 Draw circle ABF circumscribing both pentagon and triangle [p.10]

2 Inscribe $\Gamma\Delta$ as side of icosahedron, AE as side of dodecahedron.

3 E = centre of circle.

4 Draw perpendiculars EZ to ΔF , EH to $\Gamma\Delta$.

5 Produce EH to B .

6 Join BF .

7 Let Θ = side of cube.

8 Claim: $\Theta : \Gamma\Delta$ = surface of dodecahedron : surface of icosahedron.

9 When $BE + BF$ is cut in extreme and mean ratio, greater segment = BE [XIII.9].

10 $EH = \frac{1}{2} (BE + BF)$ [Prop. 1].

11 $EZ = \frac{1}{2} BE$ [Porism to Prop. 1].

12 Therefore when EH is cut in extreme and mean ratio, greater segment = EZ .

13 When Θ is cut in extreme and mean ratio, greater segment = $\Gamma\Delta$ [XIII.17 por.].

14 Therefore $\Theta : \Gamma\Delta = EH : EZ$.

$$\frac{\Theta}{\Gamma\Delta} = \frac{EH}{EZ}$$

15 Hence $ZE \times \Theta = \Gamma\Delta \times EH$.

$$ZE \times \Theta = \Gamma\Delta \times EH$$

16 Since $\Theta : \Gamma\Delta = ZE \times \Theta : \Gamma\Delta \times ZE$.

$$\frac{\Theta}{\Gamma\Delta} = \frac{ZE \times \Theta}{\Gamma\Delta \times ZE}$$

17 And $\Gamma\Delta \times EH = ZE \times \Theta$.

$$\Gamma\Delta \times EH = ZE \times \Theta$$

18 Therefore $\Theta : \Gamma\Delta = \Gamma\Delta \times EH : \Gamma\Delta \times ZE$.

$$\frac{\Theta}{\Gamma\Delta} = \frac{\Gamma\Delta \times EH}{\Gamma\Delta \times ZE}$$

19 Thus $\Theta : \Gamma\Delta =$ surface of dodecahedron : surface of icosahedron [p.14].

Therefore
Side of cube : side of icosahedron = surface of dodecahedron : surface of icosahedron.

Proposition 4 — Alternative Demonstration with Lemma

Lemma

Proof.

1. Draw circle ABF .
2. Inscribe AB and AF as sides of an equilateral pentagon.
3. Join BF , and let Δ be the centre.
4. Join $A\Delta$, and produce it to E .

5. Set

$$\Delta Z = \frac{1}{2}\Delta A.$$

6. Set

$$HF = 3\Gamma\Theta.$$

7. Claim:

$$AZ \cdot BO = \text{one pentagon}.$$

8. Draw $B\Delta$.

9. Since

$$\Delta A = 2\Delta Z,$$

we have

$$AZ = \frac{3}{2}\Delta A.$$

10. Since

$$HF = 3\Gamma\Theta,$$

and

$$HO = 2\Theta\Gamma,$$

we have

$$HF = \frac{3}{2}\Theta H.$$

11. Therefore

$$ZA : \Delta A = \Gamma H : HO.$$

12. Hence

$$AZ \cdot HO = \Delta A \cdot \Gamma H.$$

13. But

$$\Gamma H = BH.$$

14. Therefore

$$A\Delta \cdot BH = ZA \cdot HO.$$

15. But

$$A\Delta \cdot BH = 2[\triangle AB\Delta].$$

16. Therefore

$$5(AZ \cdot HO) = 10[\triangle AB\Delta].$$

17. The ten triangles $AB\Delta$ make two pentagons.

18. Since

$$HO = 2\Theta\Gamma,$$

we have

$$AZ \cdot HO = 2(AZ \cdot \Theta\Gamma).$$

19. Hence

$$10(AZ \cdot \Theta\Gamma) = 5(AZ \cdot HO) = 2 \text{ pentagons.}$$

20. Therefore

$$5(AZ \cdot \Theta\Gamma) = 1 \text{ pentagon.}$$

21. But

$$5(AZ \cdot \Theta\Gamma) = AZ \cdot \Theta B,$$

since $\Theta B = 5\Theta\Gamma$, with common height AZ .

22. Therefore

$$AZ \cdot B\Theta = \text{one pentagon.}$$

Therefore, the rectangle $AZ \cdot B\Theta$ is equal to one pentagon.

ALTERNATIVE DEMONSTRATION OF PROPOSITION 4 WITH LEMMA

LEMMA. $AZ \times BO$ (side of cube) = one pentagon, where AZ is drawn from the centre to the midpoint of a side of the inscribed equilateral pentagon.

1 Draw circle ABF .

2 Inscribe AB, AF as sides of equilateral pentagon A .

3 Join BF , Δ = centre.

4 Join ΔA to A , produce to E .

5 Set $\Delta Z = \frac{1}{2} \Delta A$.

6 Set $HF = 3\Gamma\Theta$.

7 Claim: $AZ \times BO =$ the pentagon.

8 Draw BA to Δ .

9 Since $\Delta A = 2\Delta Z$, $AZ = \frac{1}{2} \Delta A$.

10 Since $HF = 3\Gamma\Theta$, $HO = 2\Theta\Gamma$, so $HF = \frac{3}{2} \Theta H$.

11 Therefore $ZA : \Delta A = \Gamma H : HO$.

12 Hence $AZ \times HO = \Delta A \times \Gamma H$.

Thus we have proved the lemma.

$AZ \times BO =$ one pentagon.

13 But $\Gamma H = BH$.

14 Therefore $\Delta A \times BH = ZA \times HO$.

15 $\Delta A \times BH = 2 \cdot \triangle ABA$.

16 Therefore $5 \cdot AZ \times HO = 10 \triangle ABA$.

17 10 triangles $ABA = 2$ pentagons.

18 Since $HO = 2\Theta\Gamma$, $AZ \times HO = 2 \cdot AZ \times \Theta\Gamma$.

19 So $AZ \times \Theta\Gamma = 5 \cdot AZ \times HO = 2$ pentagons.

20 Therefore $AZ = \frac{1}{5} \Theta\Gamma$.

21 But $5 \cdot AZ \times \Theta\Gamma = AZ \times OB$ (since $OB = 5\Theta\Gamma$, AZ common height).

Side of cube : side of icosahedron = surface of dodecahedron : surface of icosahedron.

Proposition 4 — Proof via Lemma

Proof.

1. Draw circle ABF circumscribing both figures.
2. Inscribe BA and AF as sides of the pentagon.
3. Join BF , and let E be the centre.
4. Join AE , and produce it through A to Z .
5. Set

$$AE = 2EH.$$

6. Set

$$KF = 3F\Theta.$$

7. From H draw HM perpendicular to AZ , and produce it to Δ .
8. ΔM is the side of the equilateral triangle.
9. Join the remaining sides so that $\triangle\triangle\triangle M$ is equilateral.
10. By the lemma,

$$AH \cdot OB = \text{pentagon}.$$

11. Also

$$AH \cdot H\Delta = \triangle\triangle\triangle M.$$

12. Therefore

$$AH \cdot OB : \Delta H \cdot HA = \text{pentagon} : \text{triangle}.$$

13. But

$$BO \cdot AH : \Delta H \cdot HA = BO : \Delta H.$$

14. Therefore

$$12 \text{ pentagons} : 20 \text{ triangles} = 12OB : 20\Delta H.$$

15. This is the ratio

surface dodecahedron : surface icosahedron.

16. Since $BO = 5O\Gamma$ and $BF = 6O\Gamma$,

$$12OB = 10BF.$$

17. Since

$$\Delta M = 2\Delta H,$$

we have

$$20\Delta H = 10\Delta M.$$

18. Therefore

$$10BF : 10\Delta M = \text{surface dodecahedron} : \text{surface icosahedron}.$$

19. But BF is the side of the cube, by Proposition XIII.17, and ΔM is the side of the icosahedron.

Therefore, the side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

PROPOSITION 4. PROOF VIA LEMMA
The side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

1 Draw circle ABF circumscribing both [p.10].

2 Inscribe BA, AF as sides of equilateral pentagon.

3 Join BF , $E = \text{centre}$.

4 Join AE to A , produce to Z .

5 Set $AE = ZEH$.

6 Set $KF = 3FO$.

7 From H draw HM perpendicular to AZ , produce to Δ .

8 $\Delta M = \text{side of equilateral triangle}$ [p.6].

9 Join $\Delta\Delta$, $AM : \Delta\Delta M$ is equilateral triangle.

10 $AH \times OB = \text{pentagon}$ [p.20].

11 $AH \times H\Delta = \text{triangle } \Delta\Delta M$.

12 Therefore $AH \times OB : \Delta H \times HA = \text{pentagon} : \text{triangle}$.

13 But $BO \times AH : \Delta H \times HA = BO : \Delta H$.

14 Therefore 12 pentagons : 20 triangles = $12OB : 20\Delta H$.
 $12 \times \frac{OB}{5} : 20 \times \frac{\Delta H}{2} = 12 \times \frac{OB}{5} : 20 \times \frac{\Delta H}{2}$

15 This equals surface dodecahedron : surface icosahedron.

16 $12OB = 10BF$ (since $BO = 5OF$, $BF = 6OF$, so $6BO = 5BF$).

17 $20\Delta H = 10\Delta M$ (since $\Delta M = 2\Delta H$).

18 Therefore $10BF : 10\Delta M = \text{surface dodecahedron} : \text{surface icosahedron}$.
 $10 \times \frac{BF}{5} : 10 \times \frac{\Delta M}{2} = 10 \times \frac{BF}{5} : 10 \times \frac{\Delta M}{2}$

19 But $BF = \text{side of cube}$ [XIII.17], $\Delta M = \text{side of icosahedron}$.

Hence the side of the cube is to the side of the icosahedron as the surface of the dodecahedron is to the surface of the icosahedron.

Proposition 5

Statement. If any straight line is cut in extreme and mean ratio, then the side of the cube is to the side of the icosahedron inscribed in the same sphere as the straight line whose square equals the squares on the whole and on the greater segment is to the straight line whose square equals the squares on the whole and on the lesser segment.

Proof.

1. Draw circle AOB circumscribing both figures.
2. Let Γ be the centre.
3. Draw FB through Γ , and cut it at Δ in extreme and mean ratio, $\Gamma\Delta$ being the greater segment.
4. Then $\Gamma\Delta$ is the side of the decagon.
5. Let E be the side of the icosahedron, Z the side of the dodecahedron, and H the side of the cube.
6. Thus E is the side of an equilateral triangle and Z the side of a pentagon in the common circle.
7. By the porism to Proposition XIII.17, Z is the greater segment when H is cut in extreme and mean ratio.
8. Since E is the side of an equilateral triangle,

$$E^2 = 3BF^2.$$

9. Also,

$$FB^2 + BA^2 = 3\Gamma A^2.$$

10. Therefore

$$E^2 : FB^2 = (FB^2 + BA^2) : \Gamma A^2.$$

11. Alternately,

$$E^2 : (FB^2 + BA^2) = FB^2 : \Gamma A^2.$$

12. From the previous relation,

$$BF^2 : \Gamma A^2 = H^2 : Z^2.$$

13. Therefore

$$E^2 : (FB^2 + BA^2) = H^2 : Z^2.$$

14. Alternately and inversely,

$$H^2 : E^2 = Z^2 : (FB^2 + BA^2).$$

15. By Proposition XIII.10,

$$BF^2 + \Gamma A^2 = Z^2.$$

16. Therefore

$$H^2 : E^2 = (BF^2 + \Gamma A^2) : (FB^2 + BA^2).$$

17. Since H is the side of the cube and E the side of the icosahedron, the required ratio follows.

Therefore, the ratio of the cube side to the icosahedron side is represented by the corresponding straight lines formed from the whole and its greater and lesser segments in extreme and mean ratio.

PROPOSITION 5.

If any straight line is cut in extreme and mean ratio, then the side of the cube is to the side of the icosahedron inscribed in the same sphere as the straight line whose square equals the squares on the whole and on the greater segment is to the straight line whose square equals the squares on the whole and on the lesser segment.

1 Draw circle AOB circumscribing both [p.10].

2 Γ = centre.

3 Draw FB through Γ , cut at Δ in extreme and mean ratio, $\Gamma\Delta$ greater segment.

4 $\Gamma\Delta$ = side of decagon [p.13 n.1].

5 Δ = side of icosahedron H = side of cube.

6 E = side of equilateral triangle, Z = side of pentagon.

7 Z = greater segment of H cut in extreme and mean ratio [XIII,17 por.].

8 Since E = side of equilateral triangle, $E^2 = 3BF^2$ [XIII,12].

9 $FB^2 + BA^2 = 3\Gamma A^2$ [XIII,4].

10 Therefore $E^2 : FB^2 = FB^2 + BA^2 : \Gamma A^2$.

$$\frac{E^2}{BF^2} = \frac{FB^2 + BA^2}{\Gamma A^2}$$

11 Alternately: $E^2 : (FB^2 + BA^2) = FB^2 : \Gamma A^2$.

$$\frac{E^2}{FB^2 + BA^2} = \frac{FB^2}{\Gamma A^2}$$

12 $BF^2 : \Gamma A^2 = H^2 : Z^2$ [p.32].

$$\frac{BF^2}{\Gamma A^2} = \frac{H^2}{Z^2}$$

13 Therefore $E^2 : (FB^2 + BA^2) = H^2 : Z^2$.

$$\frac{E^2}{FB^2 + BA^2} = \frac{H^2}{Z^2}$$

14 Alternately and inversely: $H^2 : E^2 = Z^2 : (FB^2 + BA^2)$.

$$\frac{H^2}{E^2} = \frac{Z^2}{FB^2 + BA^2}$$

15 But $BF^2 + \Gamma A^2 = Z^2$ [XIII,10 = square on side of pentagon = squares on sides of hexagon and decagon].

$$BF^2 + \Gamma A^2 = Z^2$$

16 Therefore $H^2 : E^2 = (BF^2 + \Gamma A^2) : (FB^2 + BA^2)$.

$$\frac{H^2}{E^2} = \frac{BF^2 + \Gamma A^2}{FB^2 + BA^2}$$

17 H = side of cube, E = side of icosahedron.

Hence the side of the cube is to the side of the icosahedron as the straight line whose square equals the squares on the whole and on the greater segment is to the straight line whose square equals the squares on the whole and on the lesser segment.

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Proposition 6

Statement. The volume of the dodecahedron is to the volume of the icosahedron as the side of the cube is to the side of the icosahedron.

Proof.

1. Equal circles circumscribe the pentagon of the dodecahedron and the triangle of the icosahedron.
2. In a sphere, equal circles are equally distant from the centre, by Theodosius, *Sphaerica* I.6.
3. Therefore the perpendiculars from the centre of the sphere to the planes of those circles are equal.
4. Those perpendiculars fall upon the centres of the circles.
5. Hence pyramids with pentagonal bases and triangular bases have the same height.
6. Pyramids of the same height are to one another as their bases, by Proposition XII.6.
7. Therefore

$$\text{pentagon} : \text{triangle} = \text{pyramid with pentagonal base} : \text{pyramid with triangular base}$$

8. Hence

$$12 \text{ pentagons} : 20 \text{ triangles} = 12 \text{ pentagonal pyramids} : 20 \text{ triangular pyramids}$$

9. But twelve pentagons make the surface of the dodecahedron, and twenty triangles make the surface of the icosahedron.
10. Therefore

$$\text{surface dodecahedron} : \text{surface icosahedron} = \text{volume dodecahedron} : \text{volume icosahedron}$$

11. By Proposition 4,

side of cube : side of icosahedron = surface dodecahedron : surface icosahedron

12. Therefore

side of cube : side of icosahedron = volume dodecahedron : volume icosahedron

Therefore, the volume of the dodecahedron is to the volume of the icosahedron as the side of the cube is to the side of the icosahedron.

PROPOSITION 6.

The volume of the dodecahedron is to the volume of the icosahedron as the side of the cube is to the side of the icosahedron.

1 Equal circles circumscribe the pentagon of the dodecahedron and the triangle of the icosahedron [p.10].

2 In spheres, equal circles equally distant from the centre [Theodosius, Sphaerica I.6].

3 Therefore the perpendiculars from the centre of the sphere to the planes of the circles are equal.

4 The perpendiculars fall upon the centres of the circles.

5 Pyramids with pentagon bases and triangular bases have the same height.

6 Pyramids of the same height are to each other as their bases [XII.6].

7 Therefore pentagon : triangle = pyramid (pentagonal base) : pyramid (triangular base).

8 Hence 12 pentagons : 20 triangles = 12 pyramids (pentagonal bases) : 20 pyramids (triangular bases).

9 12 pentagons = surface of dodecahedron, 20 triangles = surface of icosahedron.

10 Therefore surface of dodecahedron : surface of icosahedron = volume of dodecahedron : volume of icosahedron.

11 But side of cube : side of icosahedron = surface of dodecahedron : surface of icosahedron [p.16].

12 Therefore side of cube : side of icosahedron = volume of dodecahedron : volume of icosahedron.

Hence the volume of the dodecahedron is to the volume of the icosahedron as the side of the cube is to the side of the icosahedron.

Supplementary Lemma

Statement. If two straight lines are cut in extreme and mean ratio, they stand in the assumed proportion.

Proof.

1. Let AB be cut at Γ in extreme and mean ratio, $A\Gamma$ being the greater segment.
2. Let ΔE be cut at Z in extreme and mean ratio, ΔZ being the greater segment.
3. It is required to prove

$$AB : A\Gamma = \Delta E : \Delta Z.$$

4. Since

$$AB \cdot B\Gamma = A\Gamma^2$$

and

$$\Delta E \cdot EZ = \Delta Z^2,$$

5. we have

$$AB \cdot B\Gamma : A\Gamma^2 = \Delta E \cdot EZ : \Delta Z^2.$$

6. Therefore

$$4AB \cdot B\Gamma : A\Gamma^2 = 4\Delta E \cdot EZ : \Delta Z^2.$$

7. By componendo,

$$(4AB \cdot B\Gamma + A\Gamma^2) : A\Gamma^2 = (4\Delta E \cdot EZ + \Delta Z^2) : \Delta Z^2.$$

8. By Proposition II.8,

$$(AB + B\Gamma)^2 : A\Gamma^2 = (\Delta E + EZ)^2 : \Delta Z^2.$$

9. Taking square roots,

$$(AB + B\Gamma) : A\Gamma = (\Delta E + EZ) : \Delta Z.$$

10. But

$$AB + B\Gamma = 2AB$$

and

$$\Delta E + EZ = 2\Delta E.$$

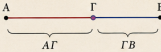
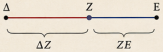
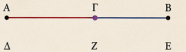
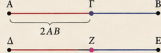
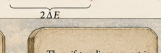
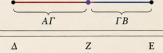
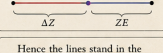
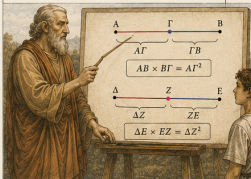
Therefore

$$2AB : A\Gamma = 2\Delta E : \Delta Z.$$

11. Taking halves,

$$AB : A\Gamma = \Delta E : \Delta Z.$$

Therefore, two straight lines cut in extreme and mean ratio stand in the assumed proportion.

SUPPLEMENTARY LEMMA. If two straight lines are cut in extreme and mean ratio, they stand in the assumed proportion.	1 Let the straight line AB be cut in extreme and mean ratio at Γ, $A\Gamma$ being the greater segment.  Extreme and mean ratio $\Rightarrow$ $AB \times B\Gamma = A\Gamma^2$	2 Let the straight line ΔE be cut in extreme and mean ratio at Z, ΔZ being the greater segment.  Extreme and mean ratio $\Rightarrow$ $\Delta E \times EZ = \Delta Z^2$	3 Claim: $AB : A\Gamma = \Delta E : \Delta Z$.  
4 Since $AB \times B\Gamma = A\Gamma^2$ and $\Delta E \times EZ = \Delta Z^2$. $AB \times B\Gamma = A\Gamma^2$ $\Delta E \times EZ = \Delta Z^2$	5 Therefore $AB \times B\Gamma : A\Gamma^2 = \Delta E \times EZ : \Delta Z^2$. $\frac{AB \times B\Gamma}{A\Gamma^2} = \frac{\Delta E \times EZ}{\Delta Z^2}$	6 Hence, multiplying both terms by 4, $\begin{aligned} 4 \cdot AB \times B\Gamma : A\Gamma^2 &= \\ 4 \cdot \Delta E \times EZ : \Delta Z^2 &= \\ \frac{4AB \times B\Gamma}{A\Gamma^2} &= \frac{4\Delta E \times EZ}{\Delta Z^2} \end{aligned}$	7 Componendo [ratio addition]: $\begin{aligned} (4 \cdot AB \times B\Gamma + A\Gamma^2) : A\Gamma^2 &= \\ (4 \cdot \Delta E \times EZ + \Delta Z^2) : \Delta Z^2 &= \\ \frac{4AB \times B\Gamma + A\Gamma^2}{A\Gamma^2} &= \frac{4\Delta E \times EZ + \Delta Z^2}{\Delta Z^2} \end{aligned}$
8 But $AB \times B\Gamma + A\Gamma^2 = (AB + B\Gamma)^2$ [II,8], and $\Delta E \times EZ + \Delta Z^2 = (\Delta E + EZ)^2$. Therefore $(AB + B\Gamma)^2 : A\Gamma^2 = (\Delta E + EZ)^2 : \Delta Z^2$ $\frac{(AB + B\Gamma)^2}{A\Gamma^2} = \frac{(\Delta E + EZ)^2}{\Delta Z^2}$	9 Taking square roots (for the magnitudes are positive), we have in length: $(AB + B\Gamma) : A\Gamma = (\Delta E + EZ) : \Delta Z,$ that is, $AB + B\Gamma : A\Gamma = \Delta E + EZ : \Delta Z$	10 But $AB + B\Gamma = 2AB$ and $\Delta E + EZ = 2\Delta E$. Therefore $2AB : A\Gamma = 2\Delta E : \Delta Z.$  	11 Taking halves, $AB : A\Gamma = \Delta E : \Delta Z$.  
 $\begin{aligned} AB \times B\Gamma &= A\Gamma^2 \\ \Delta E \times EZ &= \Delta Z^2 \\ \Rightarrow AB \times B\Gamma : A\Gamma^2 &= \Delta E \times EZ : \Delta Z^2 \\ \Rightarrow (AB + B\Gamma) : A\Gamma &= (\Delta E + EZ) : \Delta Z \\ \Rightarrow 2AB : A\Gamma &= 2\Delta E : \Delta Z \\ \Rightarrow AB : A\Gamma &= \Delta E : \Delta Z \end{aligned}$ Thus, if two lines are cut in extreme and mean ratio, they stand in the assumed proportion. Q.E.D.			

Conclusion of Book XIV

It has been shown:

1. For any straight line cut in extreme and mean ratio,

$$\frac{\text{side of cube}}{\text{side of icosahedron}} = \frac{\text{line whose square equals the squares on the whole}}{\text{line whose square equals the squares on the whole}}$$

2. For the dodecahedron and icosahedron inscribed in the same sphere,

$$\frac{\text{surface of dodecahedron}}{\text{surface of icosahedron}} = \frac{\text{side of cube}}{\text{side of icosahedron}}.$$

3. Also,

$$\frac{\text{volume of dodecahedron}}{\text{volume of icosahedron}} = \frac{\text{surface of dodecahedron}}{\text{surface of icosahedron}},$$

because the pentagonal and triangular faces are circumscribed by equal circles in the same sphere and the corresponding pyramids have equal heights.

Final Conclusion

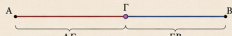
If the dodecahedron and icosahedron are inscribed in the same sphere, and if any straight line is cut in extreme and mean ratio, then

$\frac{\text{Dodecahedron}}{\text{Icosahedron}} = \frac{\text{line whose square equals the squares on the whole and greater}}{\text{line whose square equals the squares on the whole and less}}$

CONCLUSION OF BOOK XIV

Let the dodecahedron and icosahedron be inscribed in the same sphere.

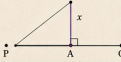
- 1** Any straight line cut in extreme and mean ratio:
 side of cube : side of icosahedron =
 line whose square equals the squares on the whole
 and on the greater part :
 line whose square equals the squares on the whole
 and on the lesser part. [p. 24]



AT, ΓB in extreme and mean ratio

Line whose square equals
squares on whole and
greater part.

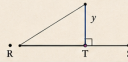
PQ (the whole)
 Let x be such that $x^2 = PQ^2 + PA^2$



Side of cube
 Side of icosahedron

Line whose square equals
squares on whole and
lesser part.

RS (the whole)
 Let y be such that $y^2 = RS^2 + RT^2$



$\frac{x}{y}$

- 2** The surface of the dodecahedron
 to the surface of the icosahedron
 inscribed in the same sphere
 is equal to the side of the cube
 to the side of the icosahedron.



Surface of
dodecahedron



Surface of
icosahedron

=



Side of
cube



Side of
icosahedron

- 3** The dodecahedron is to the icosahedron
 as the surface of the dodecahedron
 is to the surface of the icosahedron,
 because the pentagon and the triangle
 are circumscribed by the same circle.



Dodecahedron



Icosahedron



Surface of
dodecahedron



Surface of
icosahedron

FINAL CONCLUSION

If the dodecahedron and the icosahedron
 are inscribed in the same sphere,
 then for any straight line cut in extreme and mean ratio,



Dodecahedron



Icosahedron

= $\frac{x}{y}$

Line whose square equals
the squares on the whole
and on the greater part

Line whose square equals
the squares on the whole
and on the lesser part

