

Euclid's Elements

Book XI

Solid Geometry

Contents

Book XI

Chapter 1

Definitions

1.1 Definition 1

A solid is that which has length, breadth, and depth.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

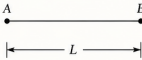
1

DEFINITION 1 – SOLID

ACTION: Take three dimensions: length, breadth, and depth.

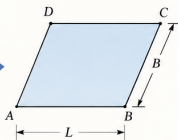
RESULT: The combined extent is a solid.

1. LENGTH



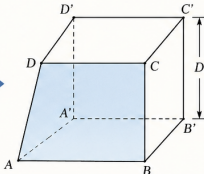
One dimension:
a line with length L .

2. BREADTH



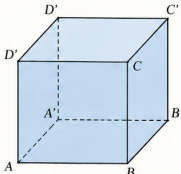
Two dimensions:
a plane figure (parallelogram)
with length L and breadth B .

3. DEPTH



Give the plane figure a depth D
in a direction perpendicular
to its plane.

RESULT
A SOLID



The combined extent in length (L),
breadth (B), and depth (D)
is a SOLID:
a parallelepiped (rectangular solid).

THREE DIMENSIONS $\rightarrow$ LENGTH (L), BREADTH (B), DEPTH (D)
THE COMBINED EXTENT IS A SOLID.

1.2 Definition 2

The extremity of a solid is a surface.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

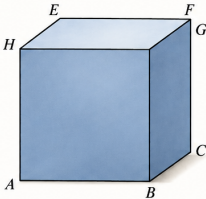
2

DEFINITION 2 – EXTREMITY OF A SOLID

ACTION: Take the boundary of a solid.

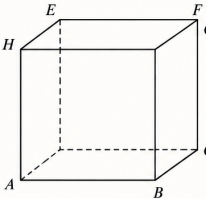
RESULT: That boundary is a surface.

1. A SOLID



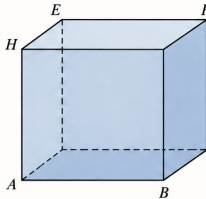
Consider a solid (a cube).

2. TAKE THE BOUNDARY



Take the boundary (the limiting outline) of the solid.

3. RESULT: A SURFACE



That boundary is a surface (the extremity of the solid).

RESULT: The boundary (extremity) of a solid is a SURFACE.

1.3 Definition 3

A straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and lie in the plane.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 3 – STRAIGHT-LINE PERPENDICULAR TO A PLANE

GIVEN: A straight-line l and a plane α .

ACTION: The line l meets the plane α at a point P .

CONDITION: For every straight-line in the plane α that passes through P , the line l forms a right angle with it.

RESULT: The line l is at right-angles (perpendicular) to the plane α .

1. GIVEN

Given a straight-line l and a plane α .

2. MEETS AT A POINT

The line l meets the plane α at the point P .

3. RIGHT ANGLES WITH EVERY LINE THROUGH P

For every straight-line in the plane α passing through P (such as $a, b, c, d, \dots$), the line l forms a right angle with it.

4. RESULT

Therefore, the line l is at right-angles (perpendicular) to the plane α .

RESULT: The straight-line l is perpendicular to the plane α .

1.4 Definition 4

A plane is perpendicular to a plane when the straight lines drawn in one of the planes at right angles to their common section are at right angles to the other plane.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 4 – PLANE PERPENDICULAR TO A PLANE

GIVEN: Two intersecting planes α and β .

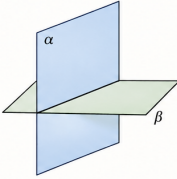
ACTION 1: Identify their common section (intersection line) ℓ .

ACTION 2: In plane α , draw any line m perpendicular to the common section ℓ .

CONDITION: That drawn line m must be perpendicular to the other plane β .

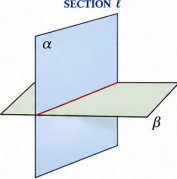
RESULT: The plane α is at right-angles (perpendicular) to the plane β .

1. GIVEN:
TWO INTERSECTING PLANES



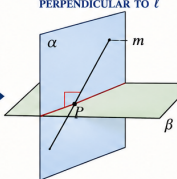
Two planes α and β intersect in a straight-line ℓ (their common section).

2. ACTION 1:
IDENTIFY THEIR COMMON SECTION ℓ



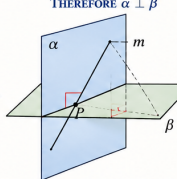
The line of intersection of α and β is ℓ .

3. ACTION 2:
IN PLANE α , DRAW ANY LINE m PERPENDICULAR TO ℓ



In plane α , draw any line m perpendicular to the line ℓ , meeting it at point P .

4. CONDITION AND RESULT:
 m IS PERPENDICULAR TO β ;
THEREFORE $\alpha \perp \beta$



The line m in plane α , perpendicular to ℓ , is perpendicular to plane β . Hence, α is at right-angles to β ($\alpha \perp \beta$).

RESULT: Plane α is at right-angles (perpendicular) to plane β .

1.5 Definition 5

The inclination of a straight line to a plane is the acute angle contained by that straight line and the straight line drawn from the point where it meets the plane to the foot of the perpendicular drawn from any point on it to the plane.

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DEFINITION 5 – INCLINATION OF A LINE TO A PLANE

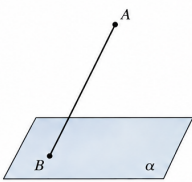
GIVEN: A line AB standing on the plane α (not perpendicular to the plane).

ACTION 1: From the upper end A of the line, drop a perpendicular AP to the plane α , meeting it at P .

ACTION 2: From P , draw a straight line PB back to the lower end B of the line (where it meets the plane α).

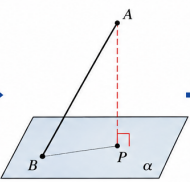
RESULT: The angle between the original line AB and the line APB is the inclination of the line AB to the plane α .

1. GIVEN



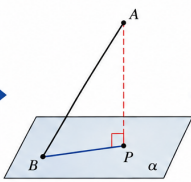
A line AB stands on the plane α . It is not perpendicular to the plane.

2. ACTION 1



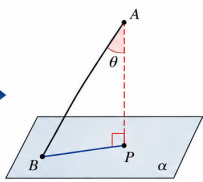
From the upper end A , drop a perpendicular AP to the plane α , meeting it at P .

3. ACTION 2



From P , draw a straight line PB back to the lower end B of the line (where it meets the plane α).

4. RESULT



The angle θ between the original line AB and the line APB is the inclination of the line AB to the plane α .

RESULT: The angle between the line AB and the line APB is the inclination of the line AB to the plane α .

1.6 Definition 6

The inclination of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to their common section, one in each plane, at the same point of the section.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

6 DEFINITION 6 – INCLINATION OF A PLANE TO A PLANE

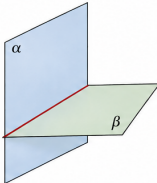
GIVEN: Two intersecting planes α and β .

ACTION 1: Locate their common section (intersection line) ℓ .

ACTION 2: At any one point P on that section ℓ , draw a line PA in plane α perpendicular to ℓ , and a line PB in plane β perpendicular to ℓ .

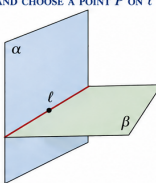
RESULT: The acute angle between these two drawn lines is the inclination of one plane to the other.

1. GIVEN
TWO INTERSECTING PLANES



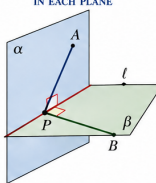
Two planes α and β intersect in the straight line ℓ (their common section).

2. ACTION 1
LOCATE THE COMMON SECTION ℓ AND CHOOSE A POINT P ON ℓ



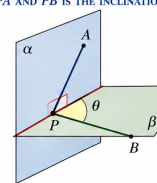
Let P be any one point on the common section ℓ .

3. ACTION 2
DRAW PERPENDICULARS AT P IN EACH PLANE



In plane α , draw $PA \perp \ell$ at P .
In plane β , draw $PB \perp \ell$ at P .

4. RESULT
THE ACUTE ANGLE BETWEEN PA AND PB IS THE INCLINATION



The acute angle θ between the lines PA and PB is the inclination of plane α to plane β .

RESULT: The acute angle between the lines drawn in each plane perpendicular to their common section at a point on it is the inclination of one plane to the other.

1.7 Definition 7

A plane is said to be similarly inclined to a plane as another plane is to another when the angles of inclination are equal.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 7 – SIMILARLY INCLINED PLANES

GIVEN:

Two pairs of planes (first pair, second pair).

ACTION:

Measure the inclination angles (as in Definition 6) for both pairs.

CONDITION:

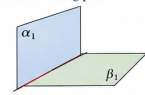
The angles are equal.

RESULT:

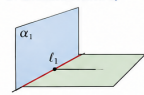
The pairs are similarly inclined.

1. FIRST PAIR OF PLANES

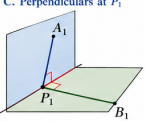
A. Intersecting planes



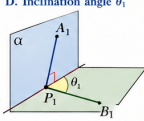
B. Common section l_1



C. Perpendiculars at P_1

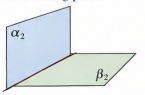


D. Inclination angle theta_1

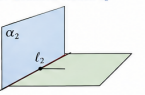


2. SECOND PAIR OF PLANES

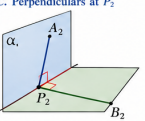
A. Intersecting planes



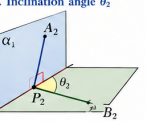
B. Common section l_2



C. Perpendiculars at P_2



D. Inclination angle theta_2



3. CONDITION

The inclination angles are equal:

theta_1 = theta_2

4. RESULT

Therefore, the first pair (alpha_1, beta_1) and the second pair (alpha_2, beta_2) are **SIMILARLY INCLINED**.

RESULT: Two pairs of planes are similarly inclined when the inclination angles between each pair are equal.

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1.8 Definition 8

Parallel planes are those which do not meet.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 8 – PARALLEL PLANES

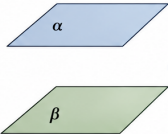
GIVEN: Two planes α and β .

ACTION: Extend the two planes indefinitely.

CONDITION: They never meet (that is, they have no common point).

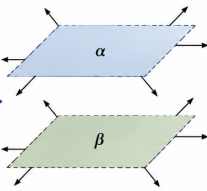
RESULT: The planes α and β are parallel planes.

1. GIVEN



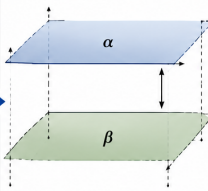
Two planes α and β .

2. EXTEND IN ALL DIRECTIONS



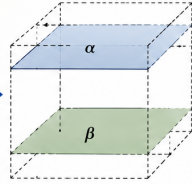
Extend both planes indefinitely in every direction.

3. NEVER MEET



No matter how far they are produced, they never meet; they have no common point.

4. RESULT



Therefore, the planes α and β are parallel planes ($\alpha \parallel \beta$).

RESULT: Two planes are parallel when, being produced indefinitely, they never meet.

1.9 Definition 9

Similar solid figures are those contained by an equal number of similar planes.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 9 – SIMILAR SOLID FIGURES

GIVEN:

Two solid figures.

ACTION:

Count the faces (planes) and compare them.

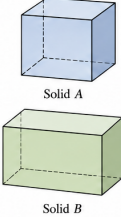
CONDITION:

The faces are equal in number, each is similar to its counterpart, and they are arranged in the same way.

RESULT:

The solids are similar.

1. GIVEN

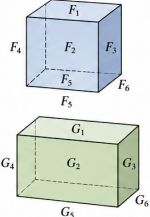


Solid A

Solid B

Given two solid figures, Solid A and Solid B.

2. ACTION

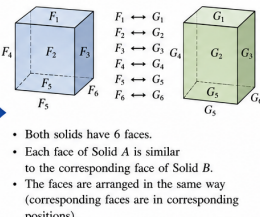


F_1
 F_2
 F_3
 F_4
 F_5
 F_6

G_1
 G_2
 G_3
 G_4
 G_5
 G_6

Count the faces (planes) and compare them.

3. CONDITION

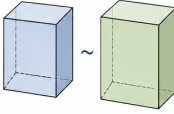


$F_1 \leftrightarrow G_1$
 $F_2 \leftrightarrow G_2$
 $F_3 \leftrightarrow G_3$
 $F_4 \leftrightarrow G_4$
 $F_5 \leftrightarrow G_5$
 $F_6 \leftrightarrow G_6$

- Both solids have 6 faces.
- Each face of Solid A is similar to the corresponding face of Solid B.
- The faces are arranged in the same way (corresponding faces are in corresponding positions).

Equal number of faces, each similar to its counterpart, and similarly arranged.

4. RESULT



$Solid A \sim Solid B$

Therefore, the solids are **similar**.

RESULT: Two solid figures are similar when they have the same number of faces, each face is similar to its corresponding face, and the faces are arranged in the same way.

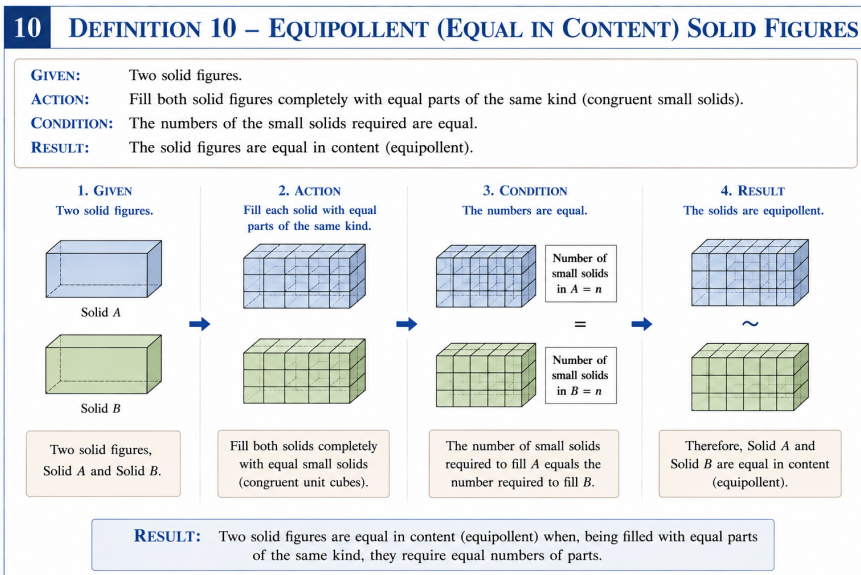
10

1.10 Definition 10

Equal and similar solid figures are those contained by similar planes equal in number and magnitude.

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objects and incidences of the definition $\implies$ defined solid-geometric relation



1.11 Definition 11

A solid angle is the inclination constituted by more than two straight lines which meet one another and are not in the same plane, or by more than two plane angles which are not in the same plane and are constructed at one point.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 11 – SOLID ANGLE

A solid angle is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines.

Otherwise: A solid angle is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

1 ACTION 1

Take three rays that meet at one point O .

Three rays OA , OB , OC are taken, meeting at the point O .

2 ACTION 2

The three pairwise plane angles they determine do not all lie in one plane.

The plane angles $\angle AOB$, $\angle BOC$ and $\angle COA$ are formed; but they do not all lie in the same plane, since OC is not in the plane containing OA and OB .

3 ACTION 3

The region they enclose is the solid angle at O .

The three rays bound a three-dimensional opening (an infinite pyramidal region). This region is the solid angle at O .

DEFINITION (SUMMARY): A solid angle is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines.

Equivalently: It is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

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1.12 Definition 12

A pyramid is a solid figure contained by planes which are constructed from one plane to one point.

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objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 12 – PYRAMID

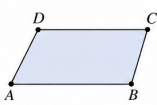
ACTION 1: Take one plane figure as the base.

ACTION 2: Take a single point outside the plane of that base.

ACTION 3: Join that point to all the vertices of the base.

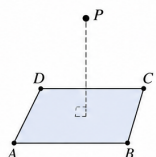
RESULT: The solid figure contained by the base and the triangular faces formed by those joins is a pyramid.

1 ACTION 1
Take one plane figure as the base.



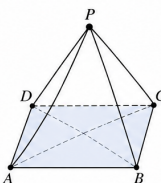
Let $ABCD$ be a plane figure (the base).

2 ACTION 2
Take a single point outside the plane of that base.



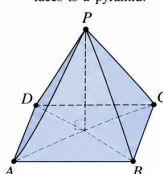
Let P be a point not in the plane of $ABCD$.

3 ACTION 3
Join that point to all the vertices of the base.



Join P to A, B, C and D .

4 RESULT
The solid figure contained by the base and the triangular faces is a pyramid.



The figure bounded by $ABCD$ and the triangles PAD, PAB, PBC, PCD is a pyramid.

DEFINITION (SUMMARY): A pyramid is the solid figure contained by a plane figure (the base) and the triangular faces formed by joining a point outside the plane of the base to all the vertices of the base.

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1.13 Definition 13

A prism is a solid figure contained by planes, two of which, namely those which are opposite, are equal, similar, and parallel, while the rest are parallelograms.

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13 DEFINITION 13 – PRISM

ACTION 1: Take two plane figures that are equal, similar, and parallel.

ACTION 2: Place them in parallel planes so that corresponding vertices match.

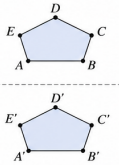
ACTION 3: Join each pair of corresponding vertices with straight lines.

CONDITION: The side faces formed by these joins are parallelograms.

RESULT: The solid figure contained by the two parallel equal and similar bases and the parallelogram side faces is a prism.

1 ACTION 1

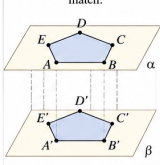
Take two plane figures that are equal, similar, and parallel.



Let $ABCDE$ and $A'B'C'D'E'$ be equal and similar plane figures.

2 ACTION 2

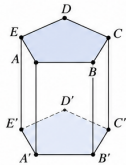
Place them in parallel planes so that corresponding vertices match.



The figures lie in parallel planes α and β with matching vertices: $A \leftrightarrow A', B \leftrightarrow B', \dots$

3 ACTION 3

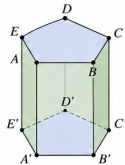
Join each pair of corresponding vertices with straight lines.



Draw AA', BB', CC', DD', EE' . These are the lateral edges.

4 CONDITION

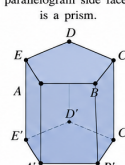
The side faces formed by these joins are parallelograms.



Each lateral face ($AA'E'E, BB'C'C$, etc.) is a parallelogram.

5 RESULT

The solid figure contained by the two parallel equal and similar bases and the parallelogram side faces is a prism.



This solid is the prism $ABCDE-A'B'C'D'E'$.

DEFINITION (SUMMARY): A prism is the solid figure contained by two plane figures that are equal, similar, and parallel, with the straight lines joining corresponding vertices, provided the side faces so formed are parallelograms.



1.14 Definition 14

A sphere is the figure comprehended when, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved.

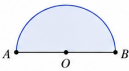
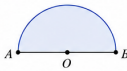
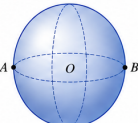
Geometrical characterization

The diameter AB is the fixed axis: A and B remain motionless throughout the revolution. In a side-view cinematic sequence the semicircle is above AB at 0° , projects essentially to the unchanged segment AB at 90° , appears below the same AB at 180° , again projects to AB at 270° , and returns at 360° .

 $AB \text{ fixed} \implies \text{semicircle revolves about } AB \implies \text{sphere}$

14 DEFINITION 14 – SPHERE

- ACTION 1:** Take a semicircle.
ACTION 2: Fix its diameter.
ACTION 3: Rotate the semicircle completely about that fixed diameter until it returns to its starting position.
RESULT: The solid figure generated by this revolution is a sphere.

1 ACTION 1 Take a semicircle.  Semicircle ADB with diameter AB.	2 ACTION 2 Fix its diameter.  The diameter AB (line AOB) is fixed. It is the axis of rotation.	3 ACTION 3 Rotate the semicircle completely about that fixed diameter until it returns to its starting position. 0°  Starting position. 90°  Edge-on: only the diameter AB is visible. 180°  Semicircle on the opposite side of AB. 270°  Edge-on: only the diameter AB is visible. 360°  Returns to starting position. The semicircle is revolved through 360° about the fixed diameter AB. Points A and B remain fixed while the rest of the semicircle leaves the plane, passes through the edge-on positions (90° and 270°), appears on the opposite side at 180°, and returns to its starting position.	4 RESULT The solid figure generated by this revolution is a sphere.  The sphere with center O and diameter AB.
---	--	--	---

DEFINITION (SUMMARY): A sphere is the solid figure generated by the revolution of a semicircle about its diameter.



1.15 Definition 15

The axis of a sphere is the straight line which remains fixed and about which the semicircle is turned.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

15

DEFINITION 15 – AXIS OF A SPHERE

ACTION:

Identify the fixed straight line about which the semicircle rotates in Definition 14.

RESULT:

That fixed straight line is the *axis* of the sphere.

1 IDENTIFY

In Definition 14, the semicircle is rotated completely about its fixed diameter *AB*.

Semicircle *ADB* with diameter *AB*.

2 REVOLUTION ABOUT *AB*

Rotate the semicircle completely about the fixed diameter *AB* until it returns to its starting position.

0°

90°

180°

270°

360°

Starting position.

Edge-on: only the diameter *AB* is visible.

Semicircle on the opposite side of *AB*.

Edge-on: only the diameter *AB* is visible.

Returns to starting position.

The straight line *AB* remains fixed throughout.

It is the line about which the semicircle is revolved.

3 RESULT

The fixed straight line *AB* about which the revolution occurs is the *axis* of the sphere.

Axis of the sphere is the line *AB*.

DEFINITION (SUMMARY):

The *axis* of a sphere is the fixed straight line (the diameter) about which the semicircle is revolved in forming the sphere.

1.16 Definition 16

The centre of a sphere is the same as that of the semicircle.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

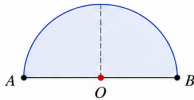
16 DEFINITION 16 – CENTER OF A SPHERE

ACTION: Take the center of the semicircle from which the sphere was generated.

RESULT: That same point is the center of the sphere.

1 TAKE THE CENTER

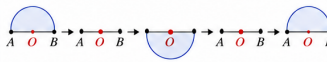
Take the center of the semicircle from which the sphere was generated.



Semicircle ADB with diameter AB .
The center of the semicircle is O .

2 REVOLUTION ABOUT AB

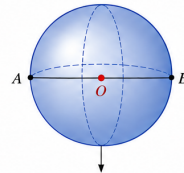
Rotate the semicircle completely about the fixed diameter AB until it returns to its starting position.



The semicircle is revolved through 360° about the fixed diameter AB .
The point O stays fixed throughout.

3 CENTER OF THE SPHERE

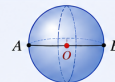
That same point is the **center** of the sphere.



Point O , the center of the semicircle, is the **center** of the sphere.

DEFINITION (SUMMARY):

The **center** of a sphere is the center of the semicircle from which the sphere is generated.



1.17 **Definition 17**

A diameter of a sphere is any straight line drawn through the centre and terminated in both directions by the surface of the sphere.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

17

DEFINITION 17 – DIAMETER OF A SPHERE

ACTION 1:

Draw a straight line through the center of the sphere.

ACTION 2:

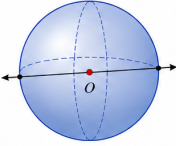
Extend the straight line in both directions until its two ends meet the surface of the sphere.

RESULT:

The straight line so drawn through the center and terminated in both directions by the sphere is a **diameter** of the sphere.

1 ACTION 1

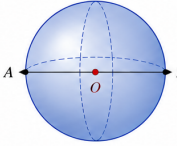
Draw a straight line through the center of the sphere.



A straight line is drawn through the center O of the sphere.

2 ACTION 2

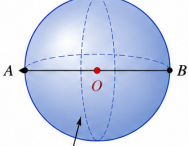
Extend the straight line in both directions until its two ends meet the surface of the sphere.



The straight line is extended in both directions until it meets the surface of the sphere at A and B .

3 RESULT

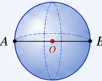
The straight line so drawn through the center and terminated in both directions by the sphere is a **diameter** of the sphere.



The straight line AB is a **diameter** of the sphere.

DEFINITION (SUMMARY):

A **diameter** of a sphere is a straight line drawn through the center and terminating in both directions at the surface of the sphere.



O : Center of the sphere

A, B : Points on the surface of the sphere

AB : Diameter of the sphere

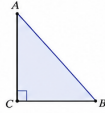
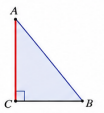
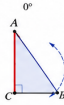
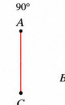
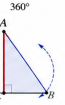
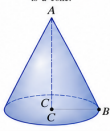
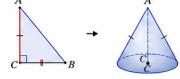
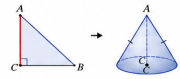
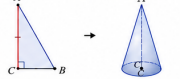
1.18 Definition 18

A cone is the figure comprehended when, one of the sides about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved; and similarly for the corresponding obtuse- and acute-angled cases.

Geometrical characterization

The fixed side AC is the axis. Points A and C never move. The moving vertex B travels in the circle lying in the plane through C perpendicular to AC . In the edge-on positions B projects onto C , so the triangle projects onto AC .

$$AC \text{ fixed} \implies B \text{ revolves about } AC \implies \text{cone}$$

18 DEFINITION 18 – CONE			
ACTION 1: Take a right-angled triangle. ACTION 2: Fix one of the sides containing the right angle. ACTION 3: Rotate the triangle completely about that fixed side until it returns to its starting position. RESULT: The solid figure generated by this revolution is a cone.	CLASSIFICATION OF CONES If the fixed side is equal to the other side containing the right angle, the cone is right-angled. If the fixed side is less than the other side, the cone is obtuse-angled. If the fixed side is greater than the other side, the cone is acute-angled.		
1 ACTION 1 Take a right-angled triangle.  Right-angled triangle ABC right angle at C. The sides containing the right angle are AC and CB.	2 ACTION 2 Fix one of the sides containing the right angle.  Fix the side AC. This will be the axis of rotation.	3 ACTION 3 Rotate the triangle completely about the fixed side AC until it returns to its starting position. 0°  Starting position. 90°  Edge-on: only the axis AC is visible. 180°  Triangle on the opposite side of the axis. 270°  Edge-on: only the axis AC is visible. 360°  Returns to starting position. The triangle is revolved through 360° about the fixed side AC. Points A and C remain fixed, and point B describes a circle in the plane through C perpendicular to AC, returning to its starting position.	4 RESULT The solid figure generated by this revolution is a cone.  Cone with vertex A, axis AC, and base a circle in the plane through C perpendicular to AC.
CLASSIFICATION OF CONES			
(a) RIGHT-ANGLED CONE Fixed side AC is equal to the other side CB. Hence $AC = CB$.  The axis is perpendicular to the base. (Right cone)	(b) OBTUSE-ANGLED CONE Fixed side AC is less than the other side CB. Hence $AC < CB$.  The axis is shorter than the other side. (Obtuse cone)	(c) ACUTE-ANGLED CONE Fixed side AC is greater than the other side CB. Hence $AC > CB$.  The axis is longer than the other side. (Acute cone)	

1.19 Definition 19

The axis of a cone is the straight line which remains fixed and about which the triangle is turned.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

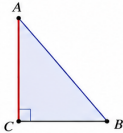
19

DEFINITION 19 – AXIS OF A CONE

ACTION: Identify the fixed straight line about which the right-angled triangle rotates in **Definition 18**.

RESULT: That fixed straight line is the **axis of the cone**.

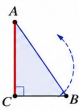
1 IDENTIFY
In **Definition 18**, a right-angled triangle is rotated about the fixed side AC .



Right-angled triangle ABC with right angle at C . The side AC is the fixed side. This is the **axis** of rotation.

2 ROTATION ABOUT AC
Rotate the triangle completely about the fixed side AC until it returns to its starting position.

0°



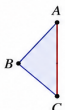
Starting position.

90°



Edge-on: only the axis AC is visible.

180°



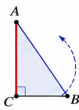
Triangle on the opposite side of the axis.

270°



Edge-on: only the axis AC is visible.

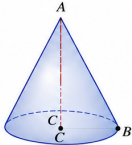
360°



Returns to starting position.

During the rotation, the points A and C remain fixed. The point B describes a circle in the plane through C perpendicular to AC .

3 RESULT
The fixed straight line AC about which the triangle rotates is the **axis** of the cone.



The line AC (shown dashed) is the **axis** of the cone. It passes through the vertex A and the center C of the base.

DEFINITION (SUMMARY): The **axis** of a cone is the fixed straight line (the side containing the right angle) about which the right-angled triangle is revolved in forming the cone.



A : Vertex of the cone
 C : Center of the base
 AC : Axis of the cone

1.20 Definition 20

The base of a cone is the circle described by the moving straight line.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

20 DEFINITION 20 – BASE OF A CONE

ACTION 1: Look at the side of the right-angled triangle that is not fixed.

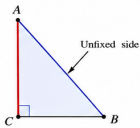
ACTION 2: Rotate the triangle about the fixed side, as in Definition 18.

ACTION 3: The unfixed side sweeps out a circle.

RESULT: That circle is the **base of the cone**.

1 ACTION 1

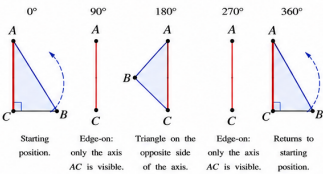
Look at the side of the right-angled triangle that is not fixed.



In triangle ABC right-angled at C , the side AB is not fixed. (The fixed side is AC .)

2 ACTION 2

Rotate the triangle about the fixed side AC , as in Definition 18.

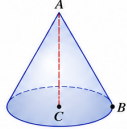


Starting position. Edge-on: only the axis AC is visible. Triangle on the opposite side of the axis. Edge-on: only the axis AC is visible. Returns to starting position.

During the rotation, the side AB sweeps through space. The points A and C remain fixed. The point B describes a circle in the plane through C , perpendicular to AC .

3 ACTION 3

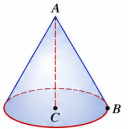
The unfixed side AB sweeps out a **circle**.



The side AB describes a circle in the plane through C perpendicular to AC .

4 RESULT

That circle is the **base** of the cone.



The circle with center C in the plane through C perpendicular to AC is the **base** of the cone.

DEFINITION (SUMMARY): The **base** of a cone is the circle described by the unfixed side of the right-angled triangle when the triangle is revolved about the fixed side.



A : Vertex of the cone

C : Center of the base

AB : Unfixed side (generates the base)

Base : Circle in the plane through C perpendicular to AC

1.21 Definition 21

A cylinder is the figure comprehended when, one side of a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

21

DEFINITION 21 – CYLINDER

ACTION 1

Take a right-angled parallelogram (a rectangle).

ACTION 2

Fix one of its sides.

ACTION 3

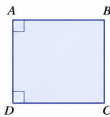
Rotate the parallelogram completely about that fixed side until it returns to its starting position.

RESULT

The solid figure generated by this revolution is a **cylinder**.

1 ACTION 1

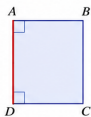
Take a right-angled parallelogram (rectangle).



Rectangle $ABCD$ with right angles at A and D .

2 ACTION 2

Fix one of its sides.

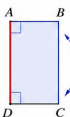


Fix the side AD . This will be the axis of rotation.

3 ACTION 3

Rotate the parallelogram completely about the fixed side AD until it returns to its starting position.

0°



Starting position.

90°



Edge-on: only the axis AD is visible.

180°



Opposite side of the axis is visible.

270°



Edge-on: only the axis AD is visible.

360°

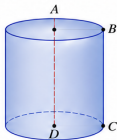


Returns to starting position.

During the rotation, the side AD remains fixed as the axis. The side BC describes a circle in the plane through D perpendicular to AD , and likewise the side AB describes a congruent circle.

4 RESULT

The solid figure generated by this revolution is a **cylinder**.



Cylinder with axis AD , bases the circles through D perpendicular to AD , and radius $DB = DC$.

DEFINITION (SUMMARY):

A **cylinder** is the solid figure generated when a right-angled parallelogram (rectangle) is revolved about one of its sides.



AD : Axis of the cylinder

A, D : Extremities of the axis

B, C : Points on the base circles (perpendicular to AD)

$DB = DC$: Radius of the base circles

22

1.22 Definition 22

The axis of a cylinder is the straight line which remains fixed and about which the parallelogram is turned.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

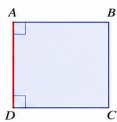
22 DEFINITION 22 – AXIS OF A CYLINDER

ACTION: Identify the fixed straight line about which the right-angled parallelogram rotates in **Definition 21**.

RESULT: That fixed straight line is the **axis of the cylinder**.

1 IDENTIFY

In **Definition 21**, a right-angled parallelogram (rectangle) is rotated about the fixed side AD .

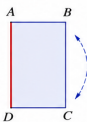


The side AD is the fixed side.
This is the **axis of rotation**.

2 ROTATION ABOUT AD

Rotate the rectangle completely about the fixed side AD until it returns to its starting position.

0°



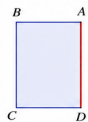
Starting position.

90°



Edge-on: only the axis AD is visible.

180°



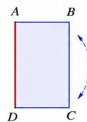
Opposite side of the axis is visible.

270°



Edge-on: only the axis AD is visible.

360°

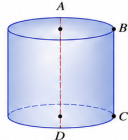


Returns to starting position.

During the rotation, the points A and D remain fixed.
The side BC describes a circle in the plane through D perpendicular to AD , and the side AB describes a congruent circle.
Thus a cylinder is generated.

3 RESULT

The fixed straight line AD about which the rectangle rotates is the **axis** of the cylinder.



The line AD (shown dashed) is the **axis** of the cylinder.
It passes through the centers of the two bases and is perpendicular to them.

DEFINITION (SUMMARY): The **axis of a cylinder** is the fixed straight line (the side containing the right angles) about which the right-angled parallelogram is revolved in forming the cylinder.



AD : Axis of the cylinder

A, D : Extremities of the axis

B, C : Points on the base circles (centers of the sides AB and BC)

$AB = BC$: Radii of the base circles

23

1.23 Definition 23

The bases of a cylinder are the circles described by the two opposite moving sides of the parallelogram.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

23

DEFINITION 23 – BASES OF A CYLINDER

ACTION 1: Look at the two opposite sides of the rectangle that are not fixed.

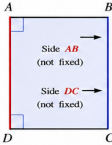
ACTION 2: Rotate the rectangle about the fixed side AD , as in Definition 21.

ACTION 3: As those two opposite sides rotate, each describes a circle.

RESULT: Those two circles are the **bases** of the cylinder.

1 ACTION 1

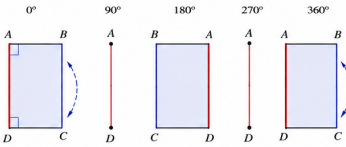
Look at the two opposite sides of the rectangle that are not fixed.



The fixed side is AD . The two opposite sides that are not fixed are AB and DC .

2 ACTION 2

Rotate the rectangle about the fixed side AD , as in Definition 21.

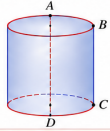


Starting position. Edge-on: only the axis AD is visible. Opposite side of the axis is visible. Edge-on: only the axis AD is visible. Returns to starting position.

During the rotation, the points A and D remain fixed on the axis AD . The side AB sweeps around and the side DC sweeps around as well.

3 ACTION 3

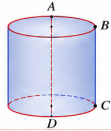
As the two opposite sides rotate, each describes a circle.



The side AB describes the top circle in the plane through A perpendicular to AD . The side DC describes the bottom circle in the plane through D perpendicular to AD .

4 RESULT

Those two circles are the **bases** of the cylinder.



The circle with center A in the plane perpendicular to AD is the **upper base**. The circle with center D in the plane perpendicular to AD is the **lower base**. These are the **bases** of the cylinder.

DEFINITION (SUMMARY): The **bases** of a cylinder are the two circles described by the two opposite sides of the right-angled parallelogram that are not fixed when the parallelogram is revolved about the fixed side.



AD : Axis of the cylinder (fixed side)
 A, D : Extremities of the axis
 AB : Generates the upper base (center A)
 DC : Generates the lower base (center D)
 $AB = DC$: Radii of the bases

24

1.24 Definition 24

Similar cones and cylinders are those in which the axes and the diameters of the bases are proportional.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

24

DEFINITION 24 – SIMILAR CONES AND CYLINDERS

GIVEN:

TWO CONES, OR TWO CYLINDERS.

ACTION:

Form the ratio of the axis to the diameter of the base for each solid.

CONDITION:

The two ratios are equal.

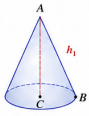
RESULT:

The cones, or the cylinders, are **similar**.

I. TWO CONES

1 MEASURE THE AXIS

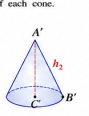
Measure the axis (height) of each cone.



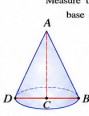
Axis = AC = h_1

2 MEASURE THE DIAMETER OF THE BASE

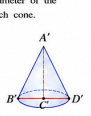
Measure the diameter of the base of each cone.



Axis = A'C' = h_2



Diameter = BD = d_1



Diameter = B'D' = d_2

3 FORM THE RATIO

Form the ratio axis : diameter of base for each cone.

First cone

axis : diameter
= AC : BD
= h_1 : d_1

Second cone

axis : diameter
= A'C' : B'D'
= h_2 : d_2

4 CONDITION AND RESULT

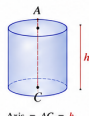
If the two ratios are equal, then the cones are **similar**.

If
 $h_1 : d_1 = h_2 : d_2$
then the cones are **similar**.

II. TWO CYLINDERS

1 MEASURE THE AXIS

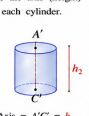
Measure the axis (height) of each cylinder.



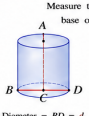
Axis = AC = h_1

2 MEASURE THE DIAMETER OF THE BASE

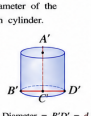
Measure the diameter of the base of each cylinder.



Axis = A'C' = h_2



Diameter = BD = d_1



Diameter = B'D' = d_2

3 FORM THE RATIO

Form the ratio axis : diameter of base for each cylinder.

First cylinder

axis : diameter
= AC : BD
= h_1 : d_1

Second cylinder

axis : diameter
= A'C' : B'D'
= h_2 : d_2

4 CONDITION AND RESULT

If the two ratios are equal, then the cylinders are **similar**.

If
 $h_1 : d_1 = h_2 : d_2$
then the cylinders are **similar**.

DEFINITION (SUMMARY):

Two cones, or two cylinders, are similar when the ratio of the axis to the diameter of the base of one is equal to the ratio of the axis to the diameter of the base of the other.



h1



h2

Similar

$h_1 : d_1 = h_2 : d_2$



h1



h2

25

1.25 Definition 25

A cube is a solid figure contained by six equal squares.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

25

DEFINITION 25 – CUBE

ACTION 1:

Take six equal squares.

ACTION 2:

Arrange them so that opposite squares are parallel.

ACTION 3:

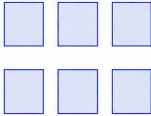
Arrange adjacent squares so that they meet at right angles.

RESULT:

The solid enclosed by the six equal square faces is a **cube**.

1 ACTION 1

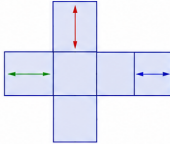
Take six equal squares.



Six equal squares.

2 ACTION 2

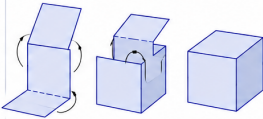
Arrange them so that opposite squares are parallel.



Opposite squares are parallel.
(Red pair, Green pair, Blue pair)

3 ACTION 3

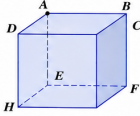
Arrange adjacent squares so that they meet at right angles.



Adjacent squares meet at right angles to form a closed solid.

4 RESULT

The solid enclosed by the six equal square faces is a **cube**.

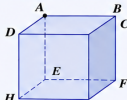


This cube has six equal square faces:

ABCD, BCGF, CDHE, DAEH, EFGH, ABFE.

DEFINITION (SUMMARY):

A **cube** is the solid figure contained by six equal squares.



ABCD : bottom face

EFGH : top face

ADHE : left face

BCGF : right face

ABFE : front face

CDHG : back face

26

1.26 Definition 26

An octahedron is a solid figure contained by eight equal and equilateral triangles.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

26 DEFINITION 26 – OCTAHEDRON

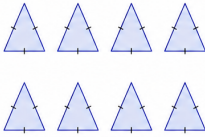
ACTION 1: Take eight equal equilateral triangles.

ACTION 2: Join them so that four triangles meet at each vertex.

RESULT: The solid enclosed by the eight equal equilateral triangular faces is an **octahedron**.

1 ACTION 1

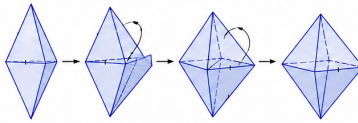
Take eight equal equilateral triangles.



Eight equal equilateral triangles.

2 ACTION 2

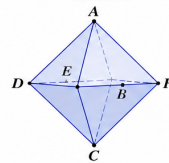
Join them so that four triangles meet at each vertex.



As the triangles are joined, four triangles meet at every vertex.

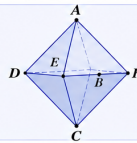
3 RESULT

The solid enclosed by the eight equal equilateral triangular faces is an **octahedron**.



This solid is an **octahedron**.

DEFINITION (SUMMARY): An **octahedron** is the solid figure contained by eight equal equilateral triangular faces, such that four triangles meet at each vertex.



A, C : Vertex at top and bottom
D, F : Opposite side vertices
B, E : Opposite side vertices

Faces (triangles):

ADE, AEF, AFB, ADB,
CDE, CEF, CFB, CDB

1.27 Definition 27

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

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DEFINITION 27 – ICOSAHEDRON

ACTION 1:

Take twenty equal equilateral triangles.

ACTION 2:

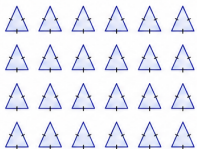
Arrange and join them so that five triangles meet at each vertex.

RESULT:

The solid enclosed by the twenty equal equilateral triangular faces is an **icosahedron**.

1 ACTION 1

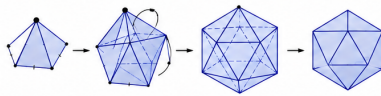
Take twenty equal equilateral triangles.



Twenty equal equilateral triangles.

2 ACTION 2

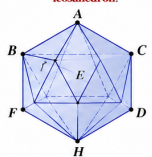
Arrange and join them so that five triangles meet at each vertex.



As the triangles are arranged and joined, five triangles meet at every vertex.

3 RESULT

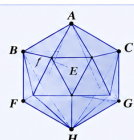
The solid enclosed by the twenty equal equilateral triangular faces is an **icosahedron**.



This solid is an **icosahedron**.

DEFINITION (SUMMARY):

An **icosahedron** is the solid figure contained by twenty equal equilateral triangular faces, such that five triangles meet at each vertex.



A, H : Opposite vertices (top and bottom)

B, D, F, G : Vertices around the middle

C, E : Opposite vertices around the middle

Faces (triangles): 20 in all

$ABC, ACF, ADE, ABE, ADF,$
 $BCF, BCE, BDE, BFG, BDG,$
 $CDF, CEG, CFG, CDE, EFG,$
 EFH, EGH, DGH, DFH, AFH

28

1.28 Definition 28

A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

Symbolic / spatial summary

objects and incidences of the definition $\implies$ defined solid-geometric relation

28

DEFINITION 28 – DODECAHEDRON

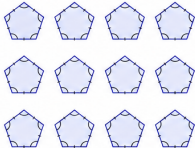
ACTION 1: Take twelve equal, equilateral, and equiangular pentagons.

ACTION 2: Arrange and join them so that three pentagons meet at each vertex.

RESULT: The solid enclosed by the twelve equal pentagonal faces is a **dodecahedron**.

1 ACTION 1

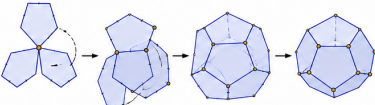
Take twelve equal, equilateral, and equiangular pentagons.



Twelve equal, equilateral, and equiangular pentagons.

2 ACTION 2

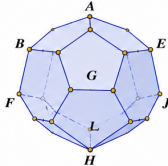
Arrange and join them so that three pentagons meet at each vertex.



As the pentagons are arranged and joined, three pentagons meet at every vertex.

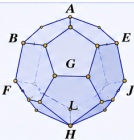
3 RESULT

The solid enclosed by the twelve equal pentagonal faces is a **dodecahedron**.



This solid is a **dodecahedron**.

DEFINITION (SUMMARY): A **dodecahedron** is the solid figure contained by twelve equal, equilateral, and equiangular pentagonal faces, such that three pentagons meet at each vertex.



A, H : Opposite vertices (top and bottom)

B, E, F, J : Vertices around the "equator"

G, L : Opposite vertices around the middle

Faces (pentagons): 12 in all

ABFGE, AEFJG, AGHEB, BEDHF, BFLHA, FJGLH, EJGFL, AJGHB, ALHBF, DLHJB, DJBFE, DLAGH.

29

Chapter 2

Propositions

2.1 Proposition 1

Statement

No part of a straight-line can lie in a reference plane while another part lies in a more elevated plane.

Given / Construction

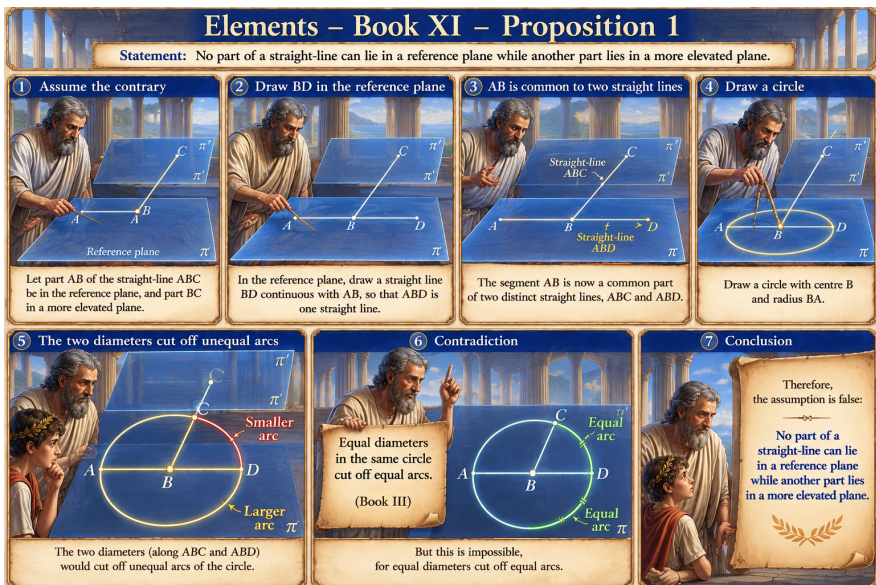
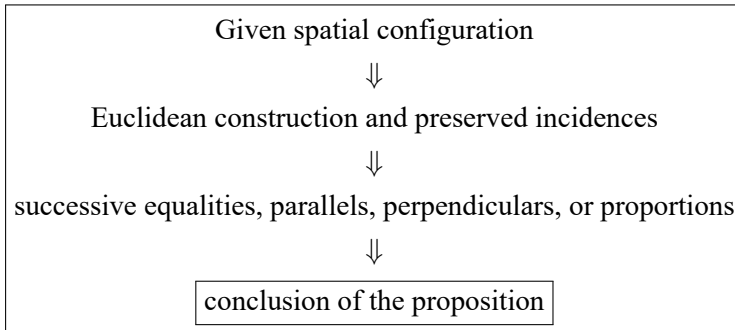
Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Assume the contrary: let part AB of straight-line ABC be in the reference plane, and part BC in an elevated plane.
2. In the reference plane, draw a straight line BD continuous with AB , so that ABD is one straight line.
3. Observe that AB is now a common segment of two distinct straight lines ABC and ABD .
4. Draw a circle with centre B and radius AB .
5. The two diameters, one along ABC and the other along ABD , would cut off unequal arcs of the circle.

6. This contradicts the fact that equal diameters cut off equal arcs.
7. Therefore the assumption is false; a straight line cannot have parts in different planes.

Complete Cinematic Logic



2.2 Proposition 2

Statement

If two straight-lines cut one another, they lie in one plane; and every triangle formed from them lies in one plane.

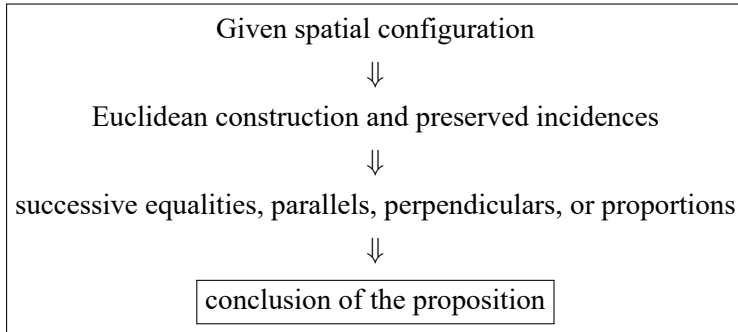
Given / Construction

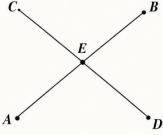
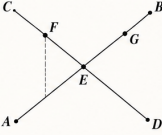
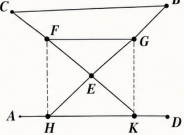
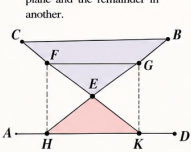
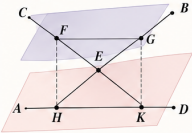
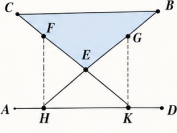
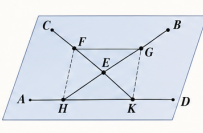
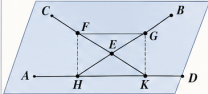
Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let AB and CD cut at E .
2. Take arbitrary points F on EC and G on EB .
3. Join CB and FG ; draw FH and GK across.
4. Consider triangle ECB . Suppose part of it is in the reference plane and the remainder in another.
5. Then parts of EC and/or EB would also be split between planes, contradicting Proposition 1.
6. Hence triangle ECB is wholly in one plane.
7. Since EC and EB lie in that plane, and AB and CD are extensions of them, the two original lines are in that same plane.
8. Therefore the two lines and every triangle formed from them are coplanar.

Complete Cinematic Logic



PROPOSITION 2 If two straight-lines cut one another, they lie in one plane; and every triangle (formed from them) lies in one plane.			
1 Let AB and CD cut at E . 	2 Take arbitrary points F on EC and G on EB . 	3 Join CB and FG ; draw FH and GK across. 	4 Consider triangle ECB . Suppose part of it (say FHC or GBK) is in the reference plane and the remainder in another. 
5 Then parts of EC and/or EB would also be split between planes – which contradicts Prop. 1. 	6 Hence triangle ECB is wholly in one plane. 	7 Since EC and EB lie in that plane, and AB and CD are extensions of them, the two original lines are also in that same plane. 	8 Conclusion: the two lines and every triangle formed from them are co-planar.  Therefore, if two straight-lines cut one another, they lie in one plane; and every triangle formed from them lies in one plane.
PROPOSITION 2 (SUMMARY): If two straight-lines cut one another, they lie in one plane; and every triangle (formed from them) lies in one plane.			

2.3 Proposition 3

Statement

If two planes cut one another, their common section is a straight-line.

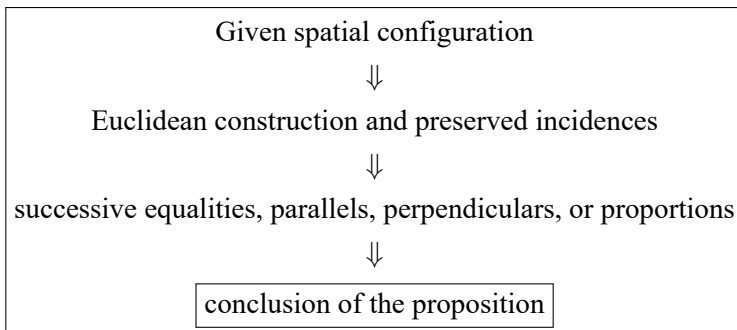
Given / Construction

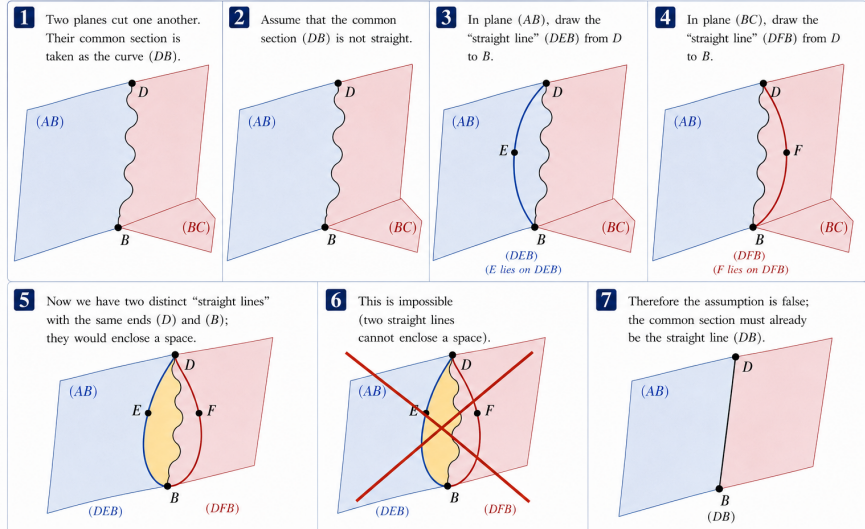
Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let planes AB and BC intersect, with common section from the persistent point D to the persistent point B .
2. Suppose the common section DB is not straight.
3. In plane AB , draw the line DEB from the same D to the same B , with E on it; in the reductio illustration this is the line claimed to be straight.
4. In plane BC , draw the line DFB from the same D to the same B , with F on it; again it is the line claimed to be straight.
5. Thus two distinct purported straight lines have exactly the same end-points D and B and would enclose a space.
6. Two straight lines cannot enclose a space.
7. Therefore the common section must be the straight line DB .

Complete Cinematic Logic



PROPOSITION 3**If two planes cut one another, their common section is a straight line.**

2.4 Proposition 4

Statement

If a straight-line is set up at right-angles to two intersecting lines at their point of intersection, then it is also perpendicular to the plane through those two lines.

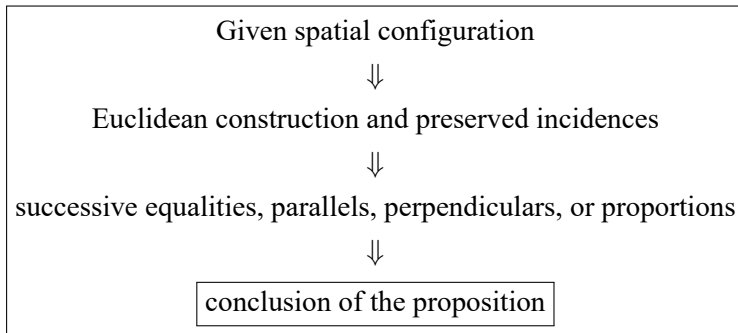
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

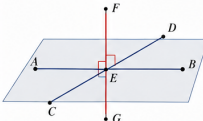
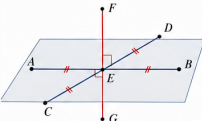
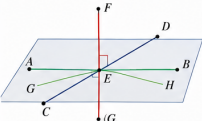
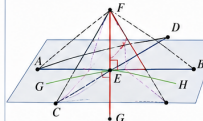
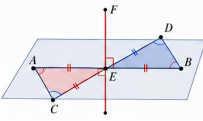
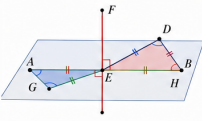
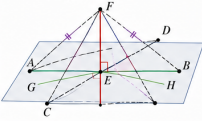
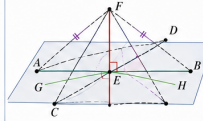
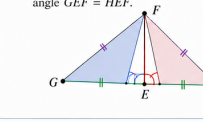
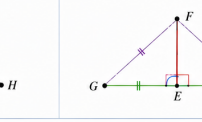
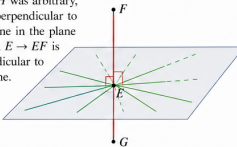
1. Let EF stand at right-angles to AB and CD at their intersection E .
2. Cut off $AE = EB = CE = ED$.
3. Draw arbitrary GEH through E in the plane of AB and CD .
4. Join $AD, CB, FA, FG, FD, FC, FH, FB$.
5. Prove triangles AED and CEB congruent, hence $AD = CB$.
6. Prove triangles AGE and BEH congruent, hence $GE = EH$ and $AG = BH$.
7. From the right triangles obtain $FA = FB$ and $FC = FD$.
8. Using these equalities show $FG = FH$.
9. Compare triangles FGE and FHE to obtain $\angle GEF = \angle HEF$.
10. Both are right angles, so $EF \perp GEH$.
11. As GEH was arbitrary, EF is perpendicular to the plane.

Complete Cinematic Logic



PROPOSITION 4

If a straight-line is set up at right-angles to two intersecting lines at their point of intersection, then it is also perpendicular to the plane through those two lines.

1 Let EF stand at right-angles to AB and CD at their intersection E. 	2 Cut off equal lengths $AE = EB = CE = ED$. 	3 Draw an arbitrary line GEH through E in the plane of AB and CD. 	4 Join AD, CB, and also join FA, FG, FD, FC, FH, FB (with F any point on EF). 
5 Prove triangles AED and CEB are congruent (SAS) $\rightarrow AD = CB$ and angles equal. 	6 Prove triangles AGE and BEH are congruent $\rightarrow GE = EH$ and $AG = BH$. 	7 Since $AE = EB$ and FE is common and perpendicular, show $FA = FB$; similarly $FC = FD$. 	8 Using these equalities and the previously found equalities, show $FG = FH$. 
9 Compare triangles FGE and FHE: they have $GE = EH$, FE common, and $FG = FH$, hence angle $GEF = HEF$. 	10 Since both are right angles, EF is perpendicular to the arbitrary line GEH in the plane. 	11 As GEH was arbitrary, EF is perpendicular to every line in the plane through $E \rightarrow EF$ is perpendicular to the plane. 	

2.5 Proposition 5

Statement

If three straight-lines meet at one point, and a fourth is set up at right-angles to all of them at that point, then the three lines all lie in one plane.

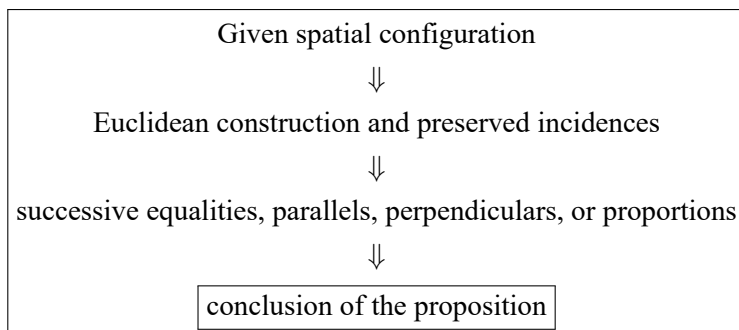
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let AB, BC, BE meet at B , and let BD be perpendicular to all of them.
2. Suppose the three lines do not all lie in the same plane.
3. Draw the plane through AB and BC ; its intersection with the plane through BD and BE is a straight line.
4. This produces two distinct perpendiculars from B to the same plane, contrary to Proposition 13.
5. Hence all three lines lie in one plane.

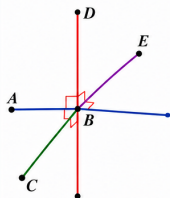
Complete Cinematic Logic



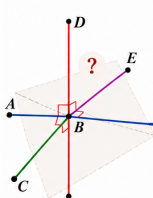
PROPOSITION 5

If three straight-lines meet at one point, and a fourth is set up at right-angles to all of them at that point, then the three lines all lie in one plane.

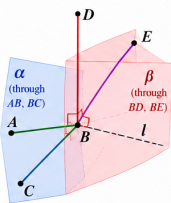
- 1** Let three straight-lines AB , BC , BE meet at B , and let BD be perpendicular to all of them at B .



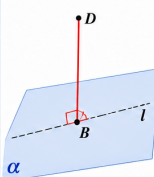
- 2** Suppose the three lines AB , BC , BE do not all lie in the same plane.



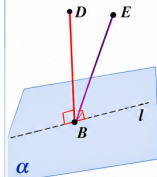
- 3** Take the plane α through AB and BC (blue plane). Let β be the plane through BD and BE (pink plane). Their intersection is a line l .



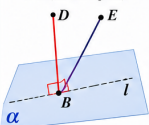
- 4** Since BD is perpendicular to AB and BC , it is perpendicular to the plane α (by definition). Thus BD is a perpendicular from B to plane α along the line l .



- 5** But BE also lies in plane β and is perpendicular to AB and BC (which lie in α), so BE is also perpendicular to plane α at B along the same line l .



- 6** Thus there are two distinct straight-lines BD and BE , both perpendicular to the same plane α and meeting at B (the same point).

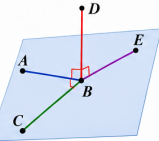


- 7** This contradicts Proposition 13: from a point to a given plane, there is only one perpendicular.

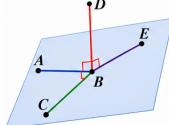


From a point outside a plane, there is only one straight-line perpendicular to the plane.

- 8** Therefore our supposition is false: the three lines AB , BC , BE must all lie in one plane.



- 9** Conclusion: The three lines AB , BC , BE are coplanar, all lying in the plane perpendicular to BD at B .



- 10** Hence: If three straight-lines meet at one point, and a fourth is set up at right-angles to all of them at that point, then the three lines all lie in one plane.

Therefore AB , BC , and BE all lie in one plane.



2.6 Proposition 6

Statement

If two straight-lines are perpendicular to the same plane, they are parallel.

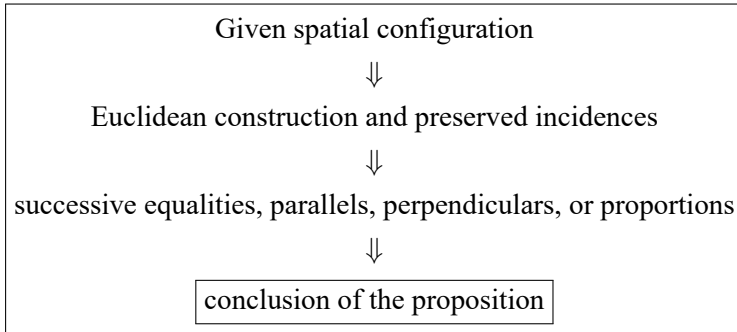
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

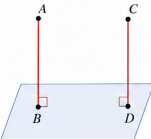
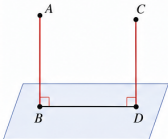
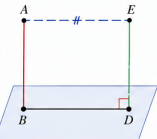
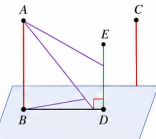
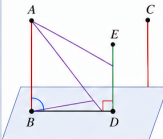
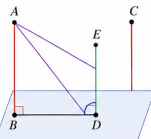
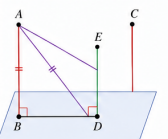
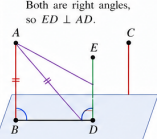
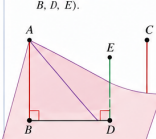
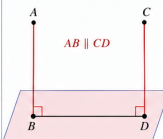
1. Let AB and CD both be perpendicular to the reference plane, meeting it at B and D .
2. Join BD .
3. In the plane draw $DE \perp BD$ and make $DE = AB$.
4. Join BE, AE, AD .
5. Since AB is perpendicular to the plane, $\angle ABD$ and $\angle ABE$ are right.
6. Use the right-angle relations at D .
7. Show triangles ABD and EDB congruent, hence $AD = BE$.
8. Then show triangles ABE and EDA congruent, so the required right-angle relations follow.
9. Apply the coplanarity criterion to the relevant lines.
10. Thus AB and CD make equal right angles with transversal BD in one plane, and are parallel.

Complete Cinematic Logic



PROPOSITION 6

If two straight-lines are perpendicular to the same plane, they are parallel.

1 Let AB and CD both be perpendicular to the reference plane. 	2 Let them meet the plane at B and D. Join BD. 	3 In the plane, draw DE perpendicular to BD and make $DE = AB$. 	4 Join BE, AE, AD. 	5 Since AB is perpendicular to the plane, angles ABD and ABE are right. 
6 Since $AB \parallel CD$ (given or to prove), angle CDB is also right (alternate interior). 	7 Triangles ABD and EDB are right triangles with $AB = DE$ (given), BD common, and both right angles $\rightarrow$ congruent (HL). Hence $AD = BE$. 	8 Then triangles ABE and EDA are right triangles with $AB = DE$ (given), AE common, $BE = AD$ (from step 7) $\rightarrow$ congruent (HL). Hence $\angle ABE = \angle EDA$. Both are right angles, so $ED \perp AD$. 	9 Now ED is perpendicular to BD (by construction), to DC (since $\angle CDB$ is right), and to AD (step 8). Thus BD, DA, DC lie in one plane (the plane of B, D, E). 	10 Hence AB and CD are both in this plane and each makes a right angle with the transversal BD. Therefore $AB \parallel CD$. 

2.7 Proposition 7

Statement

If two lines are parallel, and points are taken on each, the line joining those points lies in the same plane as the parallels.

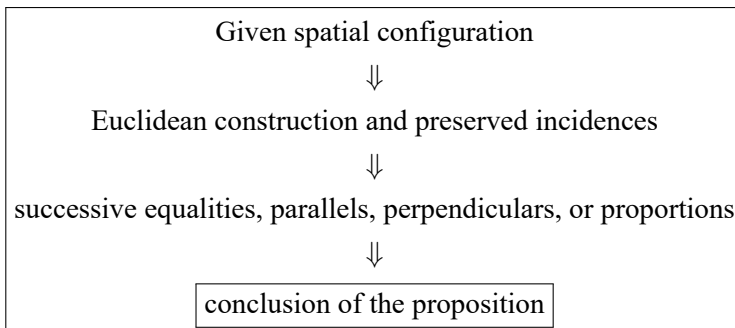
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let $AB \parallel CD$; take E on AB and F on CD .
2. Suppose EF is not in the plane of the parallels but in an elevated plane.
3. Draw a plane through EF ; by Proposition 3 it cuts the reference plane in a straight line.
4. That common section must pass through E and F .
5. Otherwise two distinct straight lines with the same endpoints would enclose an area, impossible.
6. Therefore EF lies in the same plane as AB and CD .

Complete Cinematic Logic



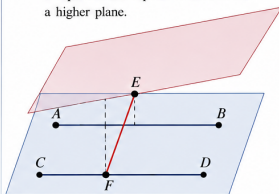
PROPOSITION 7

If two lines are parallel, and points are taken on each, the line joining those points lies in the same plane as the parallels.

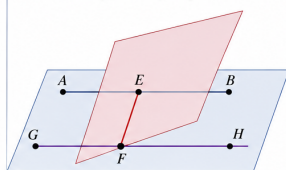
- 1** Let $AB \parallel CD$; take E on AB , F on CD .



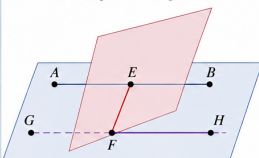
- 2** Suppose the joining line EF is not in the plane of the parallels but lies in a higher plane.



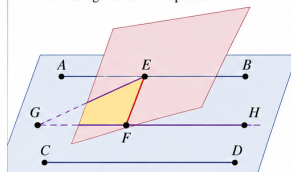
- 3** Draw a plane through EF ; it will cut the reference plane in a straight line (by Prop. 3).



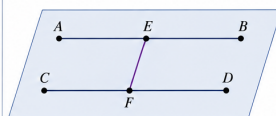
- 4** That cut must be EF itself, because it passes through E and F .



- 5** Then we have two distinct straight lines EGF and EF with the same endpoints, enclosing an area – impossible.



- 6** Therefore EF lies in the same plane as AB and CD .



2.8 Proposition 8

Statement

If two lines are parallel, and one is perpendicular to a plane, the other is also perpendicular to that plane.

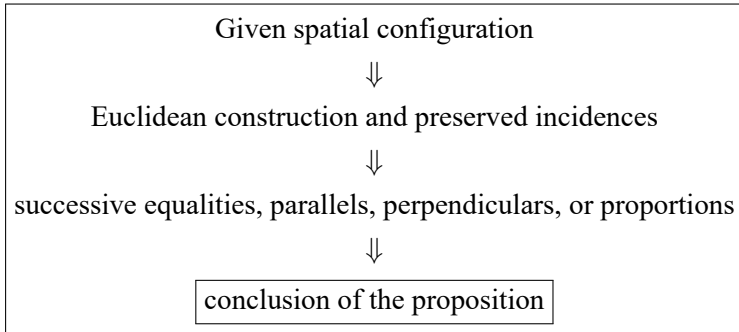
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

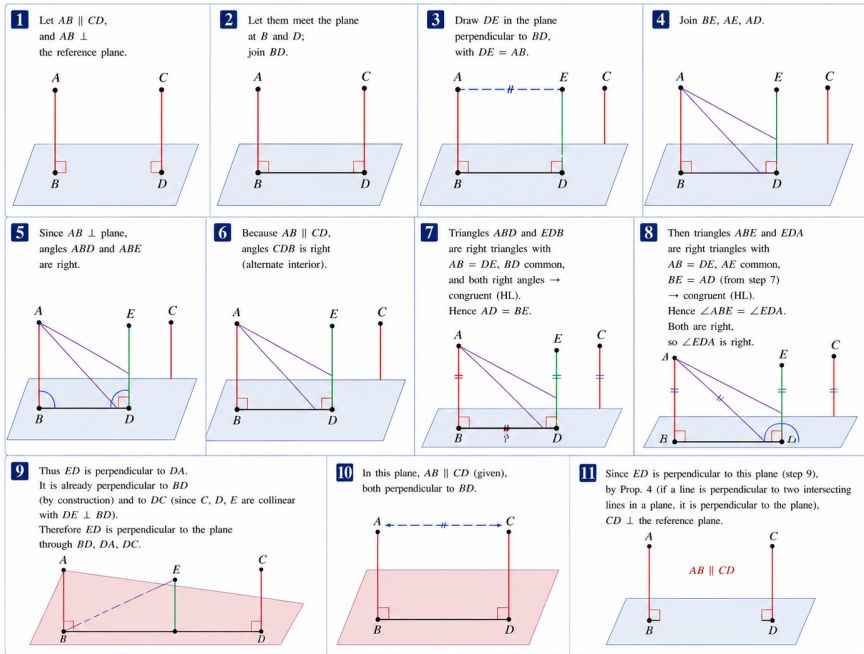
Proof Actions

1. Let $AB \parallel CD$ and AB be perpendicular to the reference plane.
2. Let them meet the plane at B, D and join BD .
3. Draw DE in the plane perpendicular to BD , with $DE = AB$.
4. Join BE, AE, AD .
5. Since AB is perpendicular to the plane, $\angle ABD$ and $\angle ABE$ are right.
6. Using $AB \parallel CD$, obtain the corresponding right angle at D .
7. Show triangles ABD and EDB congruent, hence $AD = BE$.
8. Then triangles ABE and EDA are congruent, hence $ED \perp AD$.
9. Thus ED is perpendicular to the relevant intersecting lines.
10. Apply Proposition 4 to the plane through them.
11. Therefore CD is perpendicular to the reference plane.

Complete Cinematic Logic

**PROPOSITION 8**

If two lines are parallel, and one is perpendicular to a plane,
the other is also perpendicular to that plane.



2.9 Proposition 9

Statement

Lines parallel to the same line, but not in the same plane as it, are parallel to each other.

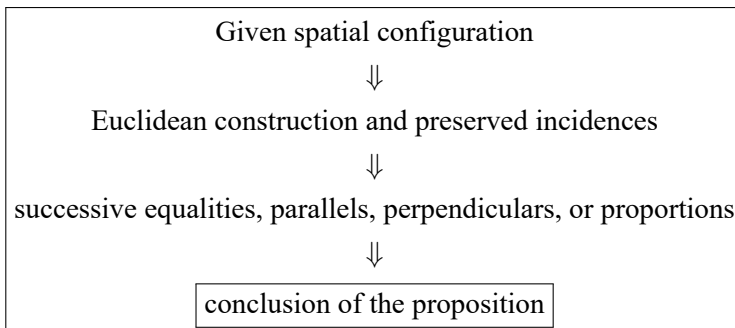
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let $AB \parallel EF$ and $CD \parallel EF$, with the lines not all coplanar.
2. Take G on EF .
3. In the plane through EF and AB , draw $GH \perp EF$.
4. In the plane through EF and CD , draw $GK \perp EF$.
5. By Proposition 4, EF is perpendicular to the plane through GH, GK .
6. By Proposition 8, AB is perpendicular to that plane.
7. Similarly CD is perpendicular to that plane.
8. By Proposition 6, $AB \parallel CD$.

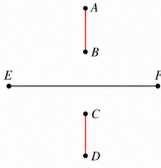
Complete Cinematic Logic



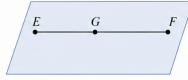
PROPOSITION 9

Lines parallel to the same line, but not in the same plane as it, are parallel to each other.

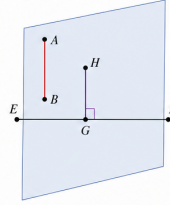
- 1** Let $AB \parallel EF$ and $CD \parallel EF$, with AB and CD not in the plane of EF .



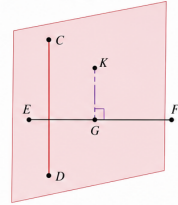
- 2** Take a point G on EF .



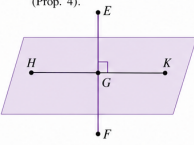
- 3** In the plane through EF and AB , draw $GH \perp EF$.



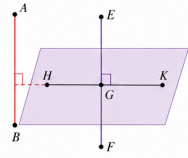
- 4** In the plane through EF and CD , draw $GK \perp EF$.



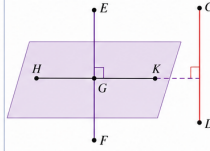
- 5** Since EF is perpendicular to both GH and GK , it is perpendicular to the plane through GH and GK (Prop. 4).



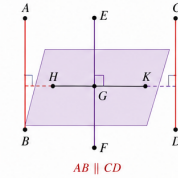
- 6** Since $AB \parallel EF$, AB is also perpendicular to that plane (Prop. 8).



- 7** Similarly, CD is perpendicular to that same plane (Prop. 8).



- 8** Therefore AB and CD are both perpendicular to the same plane $\Rightarrow$ they are parallel (Prop. 6).



2.10 Proposition 10

Statement

If two lines meeting at a point are respectively parallel to two other lines meeting at a point, but not in the same plane, the angles they contain are equal.

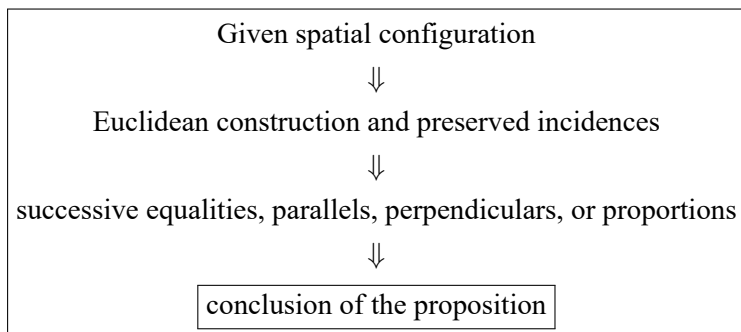
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

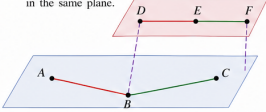
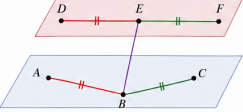
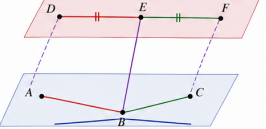
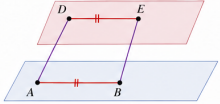
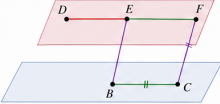
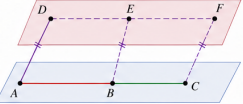
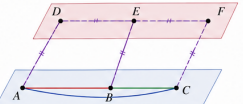
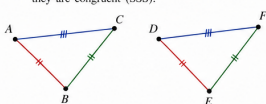
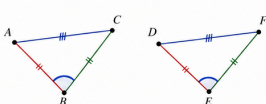
1. Let AB, BC meet at B and DE, EF meet at E , with $AB \parallel DE$ and $BC \parallel EF$.
2. Cut off $BA = ED$ and $BC = EF$.
3. Join AD, CF, BE, AC, DF .
4. Since $BA = ED$ and $BA \parallel ED$, $ABED$ is a parallelogram; hence $AD = BE$ and $AD \parallel BE$.
5. Similarly $CF = BE$ and $CF \parallel BE$.
6. Hence $AD \parallel CF$ and $AD = CF$.
7. Thus $AC \parallel DF$ and $AC = DF$.
8. Triangles ABC and DEF are congruent by SSS.
9. Therefore $\angle ABC = \angle DEF$.

Complete Cinematic Logic



PROPOSITION 10

If two lines meeting at a point are respectively parallel to two other lines meeting at a point, but not in the same plane, the angles they contain are equal.

1 Let AB and BC meet at B; let DE and EF meet at E, with $AB \parallel DE$, $BC \parallel EF$, and the two pairs not in the same plane. 	2 Cut off equal lengths: $BA = ED$, $BC = EF$. 	3 Join AD, CF, BE, AC, DF. 
4 Since $BA = ED$ and $BA \parallel ED$, quadrilateral $ABED$ is a parallelogram $\Rightarrow AD = BE$ and $AD \parallel BE$. 	5 Similarly, $CF = BE$ and $CF \parallel BE$. 	6 Hence $AD \parallel CF$ and $AD = CF$ (both parallel to BE). 
7 Join AC and DF; since $AD \parallel CF$ and equal, $AC \parallel DF$ and $AC = DF$. 	8 Now triangles ABC and DEF have $AB = DE$, $BC = EF$, and $AC = DF$; they are congruent (SSS). 	9 Therefore $\angle ABC = \angle DEF$. 

2.11 Proposition 11

Statement

To draw a perpendicular from a given elevated point to a given plane.

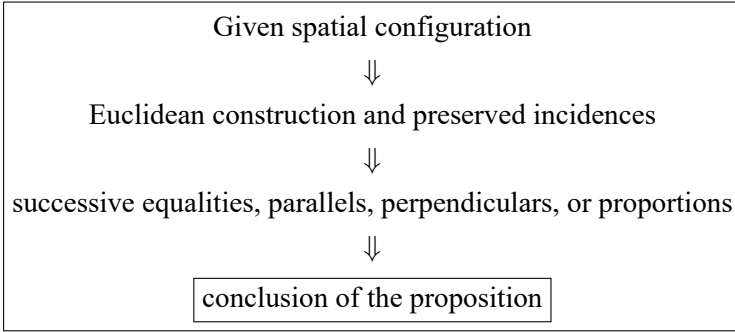
Given / Construction

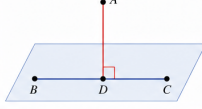
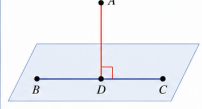
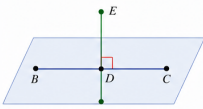
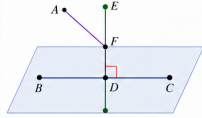
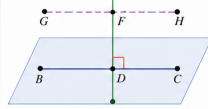
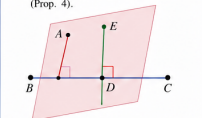
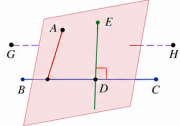
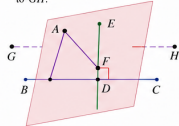
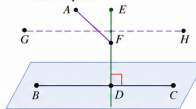
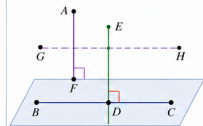
Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let A be the given point and let the reference plane be given.
2. In the plane draw any straight line BC .
3. From A draw $AD \perp BC$.
4. If AD is perpendicular to the plane, the construction is complete.
5. Otherwise draw DE through D in the reference plane with $DE \perp BC$.
6. From A drop $AF \perp DE$.
7. Through F draw $GH \parallel BC$.
8. By Proposition 4, BC is perpendicular to the plane through DA, DE .
9. By Proposition 8, GH is also perpendicular to that plane.
10. Hence $AF \perp GH$.
11. Also $AF \perp DE$.
12. By Proposition 4, AF is perpendicular to the reference plane.

Complete Cinematic Logic



PROPOSITION 11		To draw a perpendicular from a given elevated point to a given plane.	
1 Let A be the given point, and the given plane be the reference plane. 	2 In the reference plane, draw any straight line BC . 	3 From A , draw AD perpendicular to BC (using plane geometry). 	4 If AD is perpendicular to the reference plane, we are done. 
5 Otherwise, in the reference plane, draw DE through D perpendicular to BC . 	6 From A , drop AF perpendicular to DE . 	7 Through F , draw GH parallel to BC . 	8 Since BC is perpendicular to both DA and DE , it is perpendicular to the plane through DA and DE (Prop. 4). 
9 Since $GH \parallel BC$, GH is also perpendicular to that same plane (Prop. 8). 	10 But AF lies in that plane (through DA and DE) and GH is perpendicular to that plane, so AF is perpendicular to GH . 	11 Also, AF is perpendicular to DE by construction. Thus AF is perpendicular to two intersecting lines (DE and GH) in the reference plane. 	12 Therefore AF is perpendicular to the reference plane (Prop. 4). So AF is the required perpendicular. 

2.12 Proposition 12

Statement

To set up a perpendicular to a given plane from a given point in the plane.

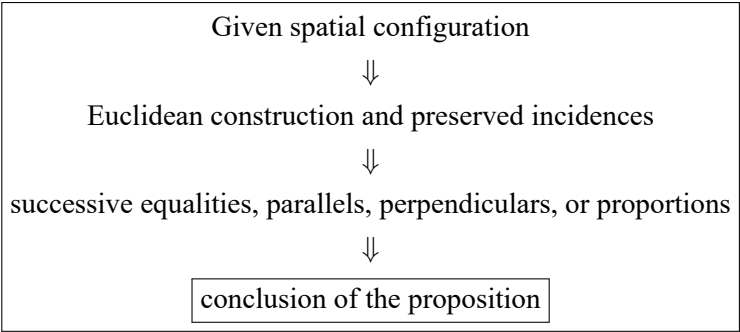
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

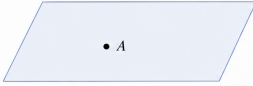
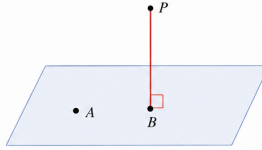
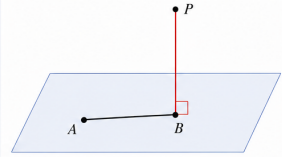
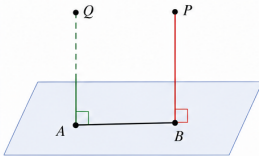
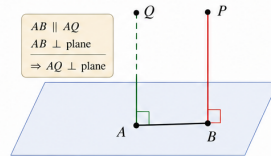
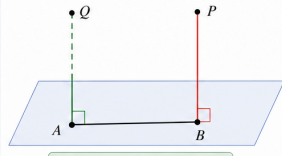
1. Let A be the given point in the plane.
2. From an elevated point draw a perpendicular to the plane by Proposition 11, meeting it at B .
3. Join AB .
4. Through A draw a line parallel to that perpendicular.
5. By Proposition 8 the parallel is also perpendicular to the plane.
6. Thus a perpendicular has been erected at A .

Complete Cinematic Logic



PROPOSITION 12

To set up a perpendicular to a given plane from a given point in the plane.

1 Let A be the point in the plane.**2** From some elevated point (constructible by Prop. 11), draw a perpendicular to the plane; let it meet the plane at B .**3** Join AB .**4** Through A , draw a line parallel to that perpendicular (using plane geometry).**5** By Prop. 8, this parallel is also perpendicular to the plane.
$$\begin{array}{l} AB \parallel AQ \\ AB \perp \text{plane} \\ \Rightarrow AQ \perp \text{plane} \end{array}$$
**6** Hence we have erected a perpendicular at A .

2.13 Proposition 13

Statement

Two different perpendiculars cannot be set up at the same point on the same side of a plane.

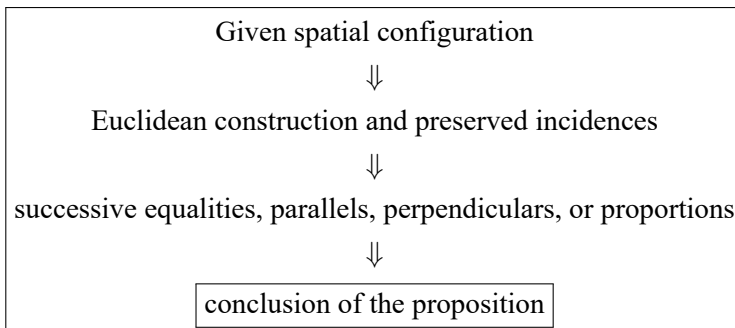
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Suppose AB and AC are both perpendicular to the reference plane at A on the same side.
2. The plane through BA, CA cuts the reference plane in DAE .
3. Since AC is perpendicular to the reference plane, $\angle CAE$ is right.
4. Similarly $\angle BAE$ is right.
5. Thus two distinct lines in the same plane make equal right angles with AE on the same side.
6. This is impossible.
7. Therefore only one perpendicular can be erected at that point on that side.

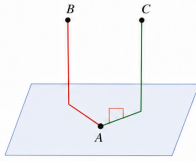
Complete Cinematic Logic



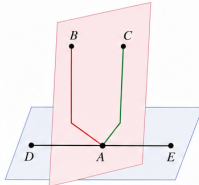
PROPOSITION 13

Two different perpendiculars cannot be set up at the same point on the same side of a plane.

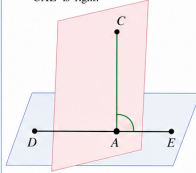
1 Suppose two lines AB and AC are both perpendicular to the reference plane at A , on the same side.



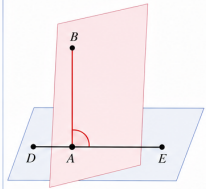
2 Draw the plane through BA and CA ; it cuts the reference plane in a line DAE (Prop. 3).



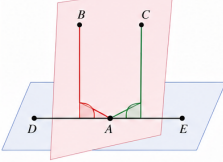
3 Since AC is perpendicular to the reference plane, it makes right angles with all lines in that plane through A ; in particular, angle CAE is right.



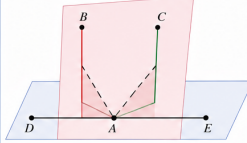
4 Similarly, AB is perpendicular, so angle BAE is right.



5 Thus CAE and BAE are equal right angles and lie in the same plane.

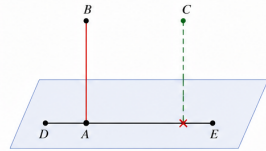


6 This is impossible because two distinct lines from A cannot both make the same angle with AE on the same side.



Two different lines through A cannot both be perpendicular to AE on the same side.

7 Therefore only one perpendicular is possible.



Hence two different perpendiculars cannot be set up at the same point on the same side of a plane.

2.14 Proposition 14

Statement

Planes to which the same straight-line is perpendicular are parallel.

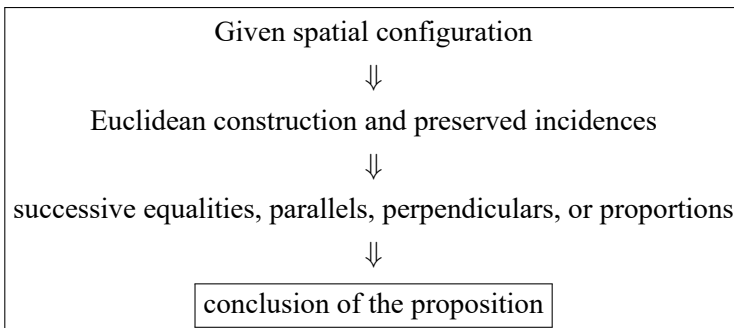
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let AB be perpendicular to planes CD and EF .
2. Suppose the planes meet when extended, with common section GH .
3. Take K on GH and join AK, BK .
4. Since AB is perpendicular to plane EF , $\angle ABK$ is right.
5. Since AB is perpendicular to plane CD , $\angle BAK$ is right.
6. Triangle ABK would have two angles summing to two right angles, impossible by I.17.
7. Therefore the planes do not meet and are parallel.

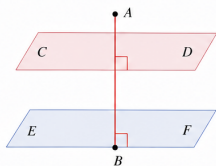
Complete Cinematic Logic



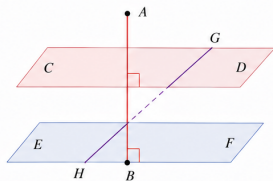
PROPOSITION 14

Planes to which the same straight-line is perpendicular are parallel.

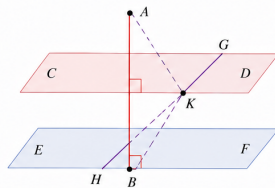
1 Let AB be perpendicular to both planes CD and EF .



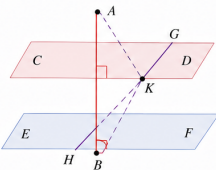
2 Suppose the planes, when extended, meet; let their common section be GH .



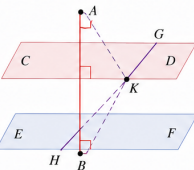
3 Take a point K on GH ; join AK and BK .



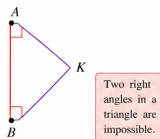
4 Since AB is perpendicular to plane EF , it is perpendicular to BK (a line in EF through K). Thus angle ABK is right.



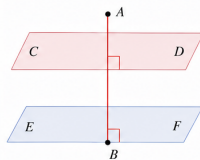
5 Similarly, because AB is perpendicular to plane CD , it is perpendicular to AK ; angle BAK is right.



6 In triangle ABK , the sum of angles ABK and BAK is two right angles, which is impossible (Prop. I.17).



7 Therefore the planes do not meet; they are parallel.



Planes CD and EF are parallel.

2.15 Proposition 15

Statement

If two intersecting lines are respectively parallel to two other intersecting lines not in the same plane, then the planes through them are parallel.

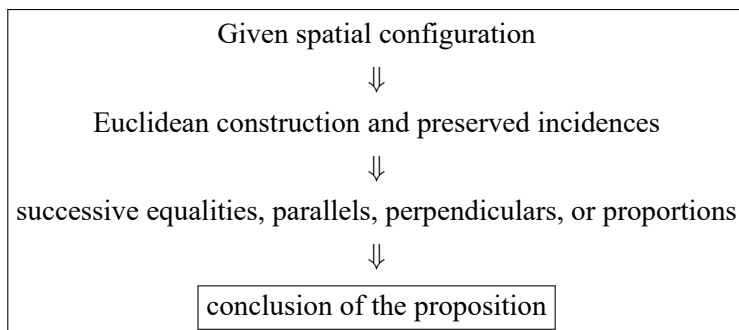
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

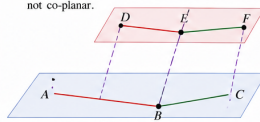
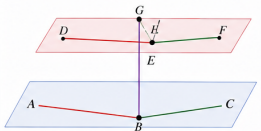
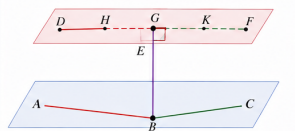
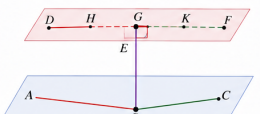
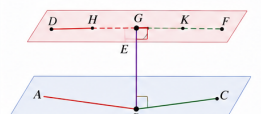
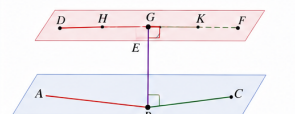
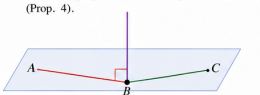
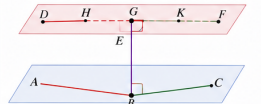
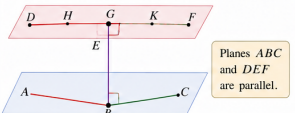
1. Let AB, BC meet at B and DE, EF meet at E , with corresponding lines parallel.
2. From B draw BG perpendicular to plane DEF , meeting it at G .
3. Through G draw $GH \parallel ED$ and $GK \parallel EF$.
4. Then $BG \perp GH$ and $BG \perp GK$.
5. Since $BA \parallel GH$, $BG \perp BA$.
6. Similarly $BG \perp BC$.
7. By Proposition 4, BG is perpendicular to plane ABC .
8. Thus BG is perpendicular to both planes.
9. By Proposition 14, planes ABC and DEF are parallel.

Complete Cinematic Logic



PROPOSITION 15

If two intersecting lines are respectively parallel to two other intersecting lines not in the same plane, then the planes through them are parallel.

1 Let AB and BC meet at B; let DE and EF meet at E, with $AB \parallel DE$, $BC \parallel EF$, and the two pairs not co-planar. 	2 From B, draw BG perpendicular to the plane through DE and EF (Prop. 11); let it meet that plane at G. 	3 Through G, draw $GH \parallel ED$ and $GK \parallel EF$. 
4 Since BG is perpendicular to the plane DEF, it is perpendicular to GH and GK; so angles BGH and BGK are right. 	5 Since $BA \parallel GH$, angle $GBA = \angle BGH$ (alternate interiors) $\Rightarrow \angle GBA$ is right. Thus $BG \perp BA$. 	6 Similarly, since $BC \parallel GK$, $BG \perp BC$. 
7 Hence BG is perpendicular to two intersecting lines BA and BC in the plane ABC; so BG is perpendicular to the plane ABC (Prop. 4). 	8 Thus the same line BG is perpendicular to both planes ABC and DEF. 	9 By Prop. 14, these two planes are parallel.  Planes ABC and DEF are parallel.

2.16 Proposition 16

Statement

If two parallel planes are cut by a third plane, their common sections with it are parallel.

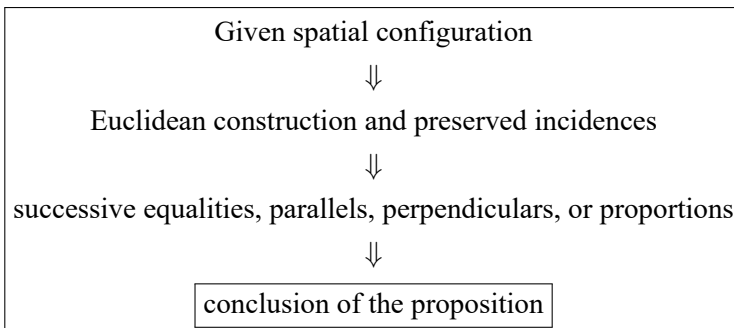
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let parallel planes AB, CD be cut by plane $EFGH$, with sections EF, GH .
2. Suppose EF and GH are not parallel.
3. Let their extensions meet at K .
4. Then K lies in plane AB and plane CD .
5. Thus the parallel planes have a common point and meet.
6. Contradiction.
7. They cannot meet on either side; therefore $EF \parallel GH$.

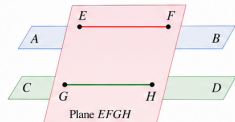
Complete Cinematic Logic



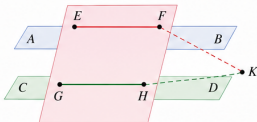
PROPOSITION 16

If two parallel planes are cut by a third plane, their common sections with it are parallel.

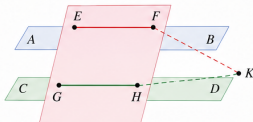
- 1** Let parallel planes AB and CD be cut by plane $EFGH$; let the common sections be EF and GH .



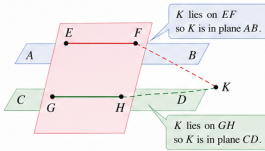
- 2** Suppose EF and GH are not parallel; then they meet when extended.



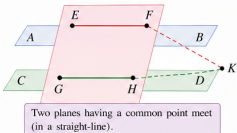
- 3** Let them meet at K (say on the side of F, H).



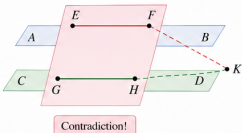
- 4** Since K lies on EF , it is in plane AB ; since it lies on GH , it is in plane CD .



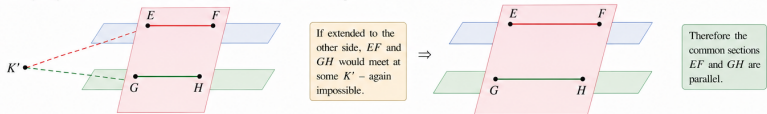
- 5** Thus K is a point common to both planes AB and CD ; hence the planes AB and CD meet.



- 6** But they are parallel by assumption – contradiction.



- 7** Similarly, they cannot meet on the other side. Therefore EF and GH are parallel.



2.17 Proposition 17

Statement

If two straight lines are cut by parallel planes, the segments are proportional.

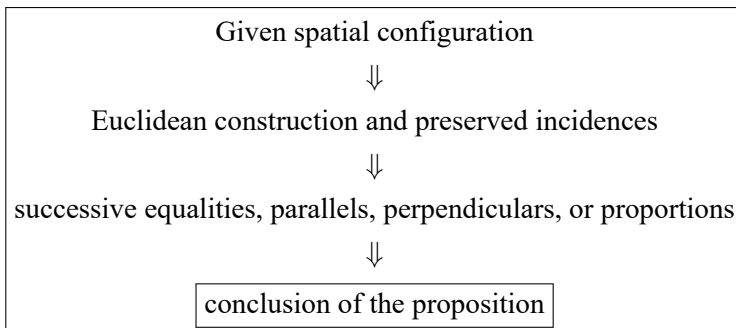
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let lines AB, CD be cut by parallel planes GH, KL, MN at A, E, B and C, F, D .
2. Join AC, BD, AD ; let AD meet plane KL at O .
3. Join EO, OF .
4. By Proposition 16, $EO \parallel BD$.
5. In triangle ABD , $AE/EB = AO/OD$.
6. Similarly $AC \parallel OF$.
7. In triangle ADC , $AO/OD = CF/FD$.
8. Hence $AE/EB = CF/FD$.

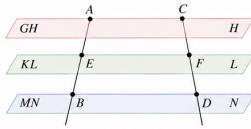
Complete Cinematic Logic



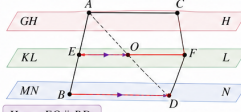
PROPOSITION 17

If two straight lines are cut by parallel planes, the segments are proportional.

- 1** Let lines AB and CD be cut by parallel planes GH , KL , MN at points A , E , B and C , F , D .



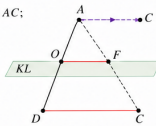
- 4** Since planes KL and MN are parallel, and plane $EBDO$ cuts them, the sections EO and BD are parallel (Prop. 16).



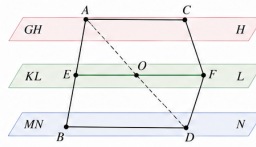
Hence $EO \parallel BD$.

- 7** In triangle ADC , $OF \parallel AC$; by similar triangles,

$$\frac{AO}{OD} = \frac{CF}{FD}$$

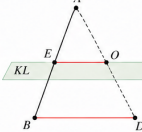


- 2** Join AC , BD , AD .
Let AD meet plane KL at O .

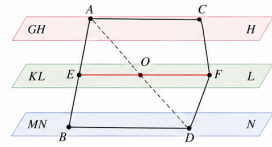


- 5** In triangle ABD , $EO \parallel BD$; by similar triangles,

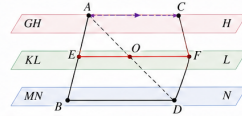
$$\frac{AE}{EB} = \frac{AO}{OD}$$



- 3** Join EO and OF .



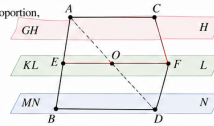
- 6** Similarly, since planes GH and KL are parallel and cut by plane $AOFC$, sections AC and OF are parallel (Prop. 16).



Hence $AC \parallel OF$.

- 8** Hence, by transitivity of proportion,

$$\frac{AE}{EB} = \frac{AO}{OD} = \frac{CF}{FD}$$



Therefore the segments are proportional.
 $\frac{AE}{EB} = \frac{CF}{FD}$

2.18 Proposition 18

Statement

If a line is perpendicular to a plane, then every plane through that line is perpendicular to that plane.

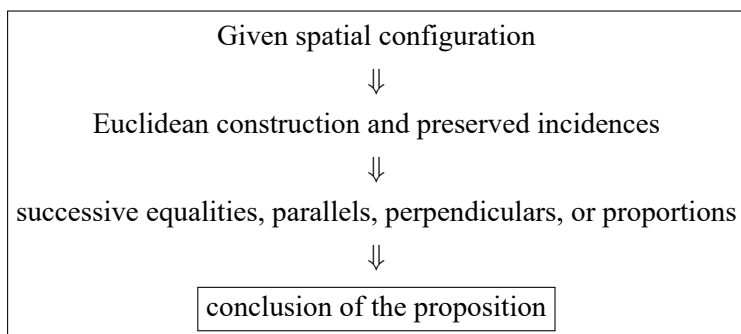
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

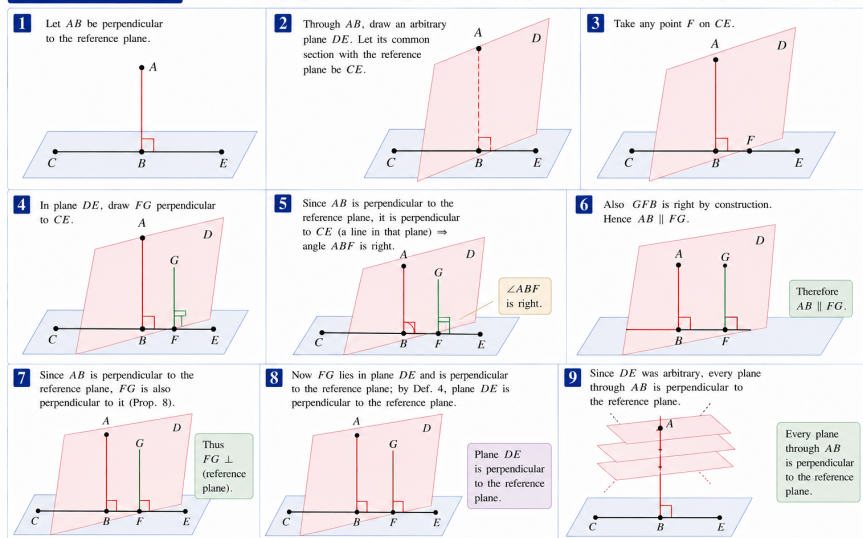
1. Let AB be perpendicular to the reference plane.
2. Through AB draw arbitrary plane DE ; let its common section with the reference plane be CE .
3. Take F on CE .
4. In plane DE draw $FG \perp CE$.
5. Since AB is perpendicular to the reference plane, it is perpendicular to CE .
6. Thus $AB \parallel FG$.
7. By Proposition 8, FG is perpendicular to the reference plane.
8. Since FG lies in plane DE , Definition 4 gives that plane DE is perpendicular to the reference plane.
9. As DE was arbitrary, every plane through AB is perpendicular to the reference plane.

Complete Cinematic Logic



PROPOSITION 18

If a line is perpendicular to a plane, then every plane through that line is perpendicular to that plane.



2.19 Proposition 19

Statement

If two intersecting planes are both perpendicular to a third plane, then their common section is also perpendicular to that third plane.

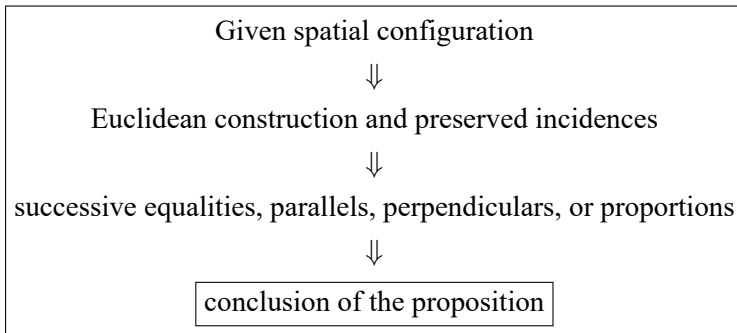
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

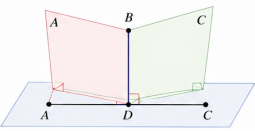
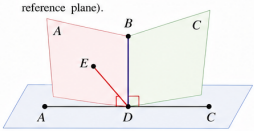
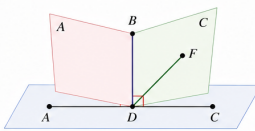
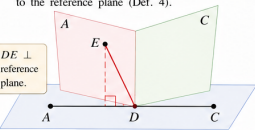
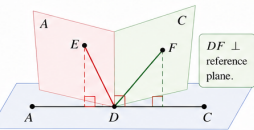
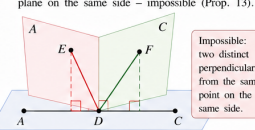
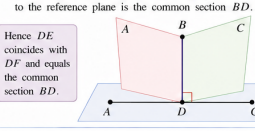
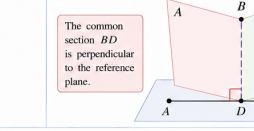
1. Let planes AB , BC be perpendicular to the reference plane and let their common section be BD .
2. From D draw DE in plane AB perpendicular to the section of AB with the reference plane.
3. Draw DF in plane BC perpendicular to the corresponding section.
4. By Definition 4, DE is perpendicular to the reference plane.
5. Similarly DF is perpendicular to the reference plane.
6. Two distinct perpendiculars from the same point on the same side are impossible by Proposition 13.
7. Hence the perpendiculars coincide with the common section BD .
8. Therefore BD is perpendicular to the reference plane.

Complete Cinematic Logic



PROPOSITION 19

If two intersecting planes are both perpendicular to a third plane, then their common section is also perpendicular to that third plane.

1 Let planes AB and BC be perpendicular to the reference plane; let their common section be BD. 	2 From a point D on BD, draw DE in plane AB perpendicular to AD (the common section of AB and the reference plane). 	3 Draw DF in plane BC perpendicular to CD (the common section of BC and the reference plane). 
4 Since plane AB is perpendicular to the reference plane, DE is perpendicular to the reference plane (Def. 4).  $DE \perp$ reference plane.	5 Similarly, DF is perpendicular to the reference plane.  $DF \perp$ reference plane.	6 Thus from the same point D, two distinct lines (DE and DF) are perpendicular to the reference plane on the same side – impossible (Prop. 13).  Impossible: two distinct perpendiculars from the same point on the same side.
7 Therefore DE and DF must coincide; the only line through D in the planes that is perpendicular to the reference plane is the common section BD.  Hence DE coincides with DF and equals the common section BD.	8 Hence BD is perpendicular to the reference plane.  The common section BD is perpendicular to the reference plane. Therefore: The common section BD is perpendicular to the reference plane.	

2.20 Proposition 20

Statement

In a solid angle formed by three plane angles, the sum of any two is greater than the third.

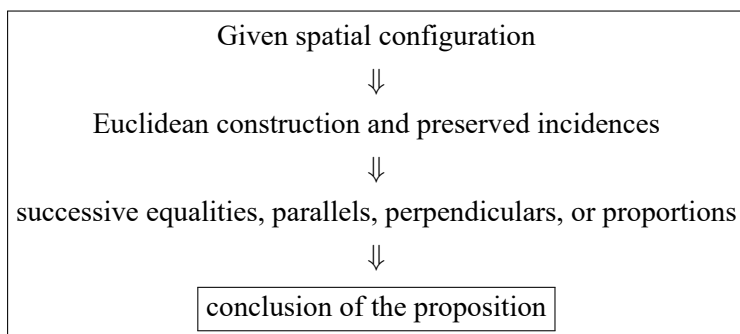
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

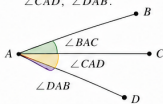
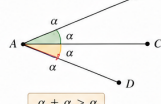
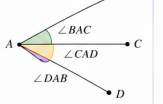
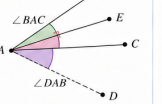
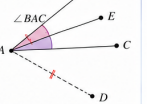
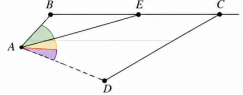
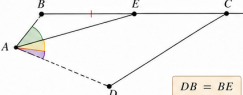
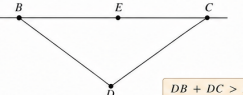
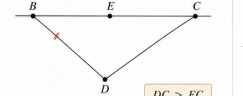
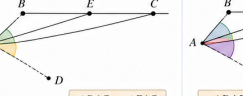
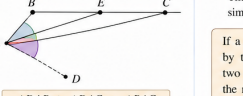
1. Let the solid angle at A be formed by $\angle BAC$, $\angle CAD$, $\angle DAB$.
2. If the three are equal, the result is immediate.
3. Otherwise let $\angle BAC$ be largest.
4. In the plane of BAC , construct $\angle BAE = \angle DAB$.
5. Make $AE = AD$.
6. Draw the single straight line BEC , with B, E, C collinear in that order and E between B and C ; join DB, DC .
7. Triangles DAB and EAB are congruent, hence $DB = BE$.
8. In triangle DBC , $DB + DC > BC$.
9. Since $BC = BE + EC$ and $DB = BE$, obtain $DC > EC$.
10. In triangles DAC, EAC , Proposition I.25 gives $\angle DAC > \angle EAC$.
11. Add $\angle DAB = \angle BAE$ to obtain $\angle DAB + \angle DAC > \angle BAC$.
12. The other cases follow similarly.

Complete Cinematic Logic



PROPOSITION 20

If a solid angle is contained by three plane angles, any two of them are greater than the remaining one.

1 Let solid angle at A be contained by the three plane angles $\angle BAC$, $\angle CAD$, $\angle DAB$. 	2 If the three angles are equal, the result is immediate. 	3 If not, let $\angle BAC$ be the largest. 	4 In the plane of $\angle BAC$, build $\angle BAE = \angle DAB$. 	5 Make $AE = AD$. 
6 Draw the straight line BEC through E so that E lies on BC (between B and C); join DB and DC. 	7 Since $AD = AE$, AB is common, and $\angle DAB = \angle BAE$, triangles DAB and EAB are congruent $\Rightarrow DB = BE$. 	8 In triangle DBC, $DB + DC > BC$ (triangle inequality). 		
9 Since E is between B and C, $BC = BE + EC$. With $DB = BE$ from (7): $BE + DC > BE + EC \Rightarrow DC > EC$. 	10 In triangles DAC and EAC: $AD = AE$, AC is common, and $DC > EC$; by 1.25, $\angle DAC > \angle EAC$. 	11 And $\angle DAB = \angle BAE$ (by construction). Therefore $\angle DAB + \angle DAC > \angle BAE + \angle EAC = \angle BAC$. 	12 Thus the sum of the two angles containing the largest exceeds the largest. The other cases follow similarly. If a solid angle is contained by three plane angles, any two of them are greater than the remaining one.	

2.21 Proposition 21

Statement

The sum of the plane angles forming any solid angle is less than four right angles.

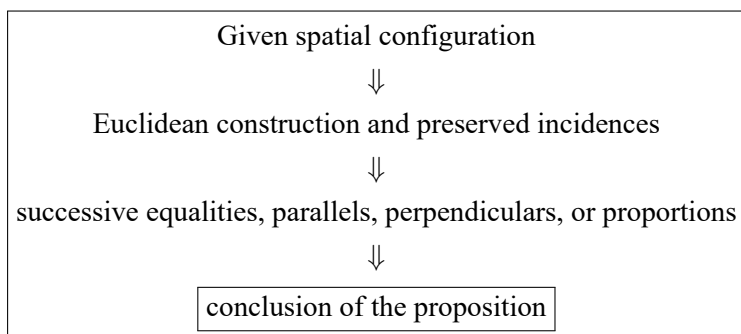
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

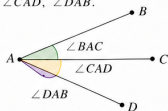
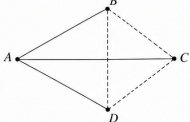
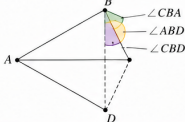
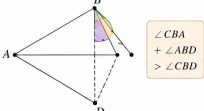
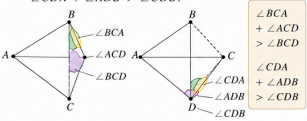
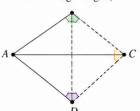
1. Let the solid angle at A be formed by BAC, CAD, DAB .
2. Take points B, C, D on the edges and join BC, CD, DB .
3. At B , apply Proposition 20 to the solid angle there.
4. Thus $\angle CBA + \angle ABD > \angle CBD$.
5. Similarly at C and D obtain the corresponding inequalities.
6. Adding them, the six base-adjacent face angles exceed the three angles of triangle BCD , i.e. two right angles.
7. The nine angles of triangles ABC, ACD, ADB total six right angles.
8. Subtracting shows the three plane angles at A total less than four right angles.
9. Therefore the plane angles forming the solid angle sum to less than four right angles.

Complete Cinematic Logic



PROPOSITION 21

The sum of the plane angles forming any solid angle is less than four right angles.

1 Let solid angle at A be formed by plane angles $\angle BAC$, $\angle CAD$, $\angle DAB$. 	2 Take points B, C, D on the edges and join BC, CD, DB. 	3 Consider the solid angle at B formed by plane angles $\angle CBA$, $\angle ABD$, $\angle CBD$. 
4 By Prop. 20 (at B), $\angle CBA + \angle ABD > \angle CBD$. 	5 Similarly, at C and at D: $\angle BCA + \angle ACD > \angle BCD$, $\angle CDA + \angle ADB > \angle CDB$. 	6 Add the three inequalities: (sum of six angles from the faces) $> \angle CBD + \angle BCD + \angle CDB$. But in $\triangle BCD$, $\angle CBD + \angle BCD + \angle CDB = 2$ right angles. (sum of the six face angles) > 2 right angles.
7 The sum of all nine angles of triangles ABC, ACD, ADB is six right angles (each triangle has two right angles). (sum of nine angles of the three triangles) $= 6$ right angles. 	8 Subtract the six face angles on the sides from the total nine angles. The remaining three are $\angle BAC$, $\angle CAD$, $\angle DAB$. Hence $\angle BAC + \angle CAD + \angle DAB$ $=$ (sum of nine angles) $=$ (sum of six face angles) $< 6 - 2 = 4$ right angles. 	9 Therefore the sum of the three plane angles of the solid angle at A is less than four right angles. $\angle BAC + \angle CAD + \angle DAB$ < 4 right angles.

2.22 Proposition 22

Statement

Given three plane angles such that any two sum to more than the third, and with equal sides, it is possible to form a triangle from the joining lines.

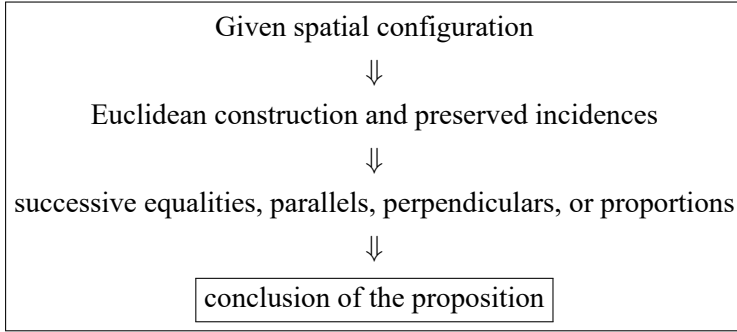
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

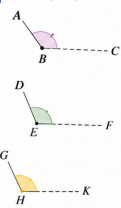
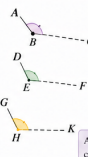
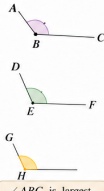
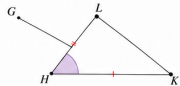
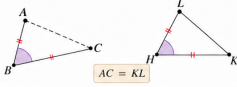
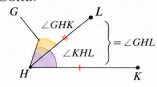
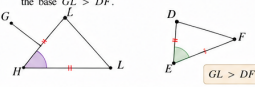
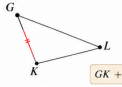
Proof Actions

1. Let the angles be ABC, DEF, GHK with equal adjacent sides.
2. Join AC, DF, GK .
3. If the angles are equal, then $AC = DF = GK$ and the result is immediate.
4. Otherwise suppose ABC is largest.
5. Construct $\angle KHL = \angle ABC$ on HK and make HL equal to a given equal side.
6. Join KL, GL .
7. Congruence gives $AC = KL$.
8. From the angle-sum hypothesis obtain $\angle GHL > \angle DEF$.
9. By I.24, $GL > DF$.
10. In triangle GKL , $GK + KL > GL$, hence $GK + AC > DF$.
11. Similarly the other triangle inequalities hold; therefore AC, DF, GK form a triangle.

Complete Cinematic Logic



PROPOSITION 22 Given three plane angles such that any two sum to more than the third, and with equal sides, it is possible to form a triangle from the joining lines.

1 Let the three plane angles be $\angle ABC$, $\angle DEF$, $\angle GHK$, such that any two sum to more than the third. Let the equal sides be $AB = BC = DE = EF = GH = HK$.  $\angle ABC + \angle DEF > \angle GHK$ $\angle ABC + \angle GHK > \angle DEF$ $\angle DEF + \angle GHK > \angle ABC$	2 Join AC, DF, GK. 	3 If the three angles are equal, then $AC = DF = GK$ and a triangle is trivial.  $AC = DF = GK$ A triangle can clearly be formed.	4 If not, suppose $\angle ABC$ is the largest.  $\angle ABC$ is largest.
6 Join KL and GL. 	7 Since $AB = BC$ and $KH = HL$, and $\angle ABC = \angle KHL$, triangles ABC and KHL are congruent, hence $AC = KL$.  $AC = KL$	8 Since $\angle ABC + \angle GHK > \angle DEF$ and $\angle ABC = \angle KHL$, we get $\angle GHK + \angle KHL > \angle DEF$. But $\angle GHK + \angle KHL = \angle GHL$. Therefore $\angle GHL > \angle DEF$ 	
9 In triangles GHL and DEF, we have $GH = DE$, $HL = EF$ and included angle $\angle GHL > \angle DEF$. Thus, by Prop. 1.24 (SAS inequality), the base $GL > DF$.  $GL > DF$	10 In triangle GKL, $GK + KL > GL$ (triangle inequality). Since $KL = AC$ and $GL > DF$, we have $GK + AC > DF$.  $GK + AC > DF$ Similarly, $AC + DF > GK$ $DF + GK > AC$		11 Thus AC, DF, GK satisfy the three triangle inequalities, so a triangle can be formed from the joining lines.  A triangle can be formed.

2.23 Proposition 23

Statement

To construct a solid angle from three given plane angles, provided any two exceed the third and their total is less than four right angles.

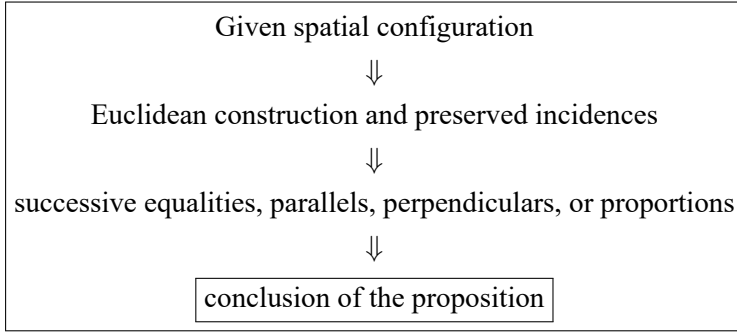
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

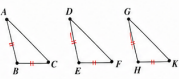
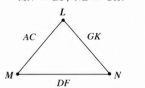
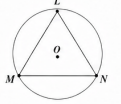
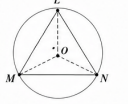
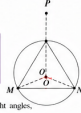
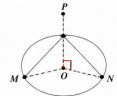
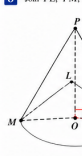
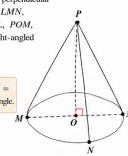
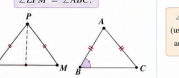
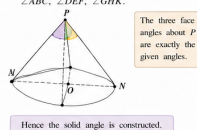
1. Let the given angles be ABC, DEF, GHK .
2. Cut off equal sides and join AC, DF, GK .
3. By Proposition 22 construct triangle LMN with $LM = AC, MN = DF, NL = GK$.
4. Circumscribe a circle about LMN with centre O .
5. Join LO, MO, NO .
6. Show $AB > LO$ by reductio using the hypothesis that the three angles total less than four right angles.
7. Erect OP perpendicular to plane LMN with $OP^2 = AB^2 - LO^2$.
8. Join PL, PM, PN .
9. The triangles POL, POM, PON are right.
10. By Pythagoras, $PL = PM = PN = AB$.
11. By SSS, $\angle LPM = \angle ABC$.
12. Similarly $\angle MPN = \angle DEF$ and $\angle LPN = \angle GHK$.
13. Thus the solid angle at P has the three required plane angles.

Complete Cinematic Logic



PROPOSITION 23

To construct a solid angle from three given plane angles (with the sum condition and total less than four right angles).

1 Let the given plane angles be $\angle ABC$, $\angle DEF$, $\angle GHK$.  $\angle ABC + \angle DEF > \angle GHK$ $\angle ABC + \angle GHK > \angle DEF$ $\angle DEF + \angle GHK > \angle ABC$ and $\angle ABC + \angle DEF + \angle GHK < 4$ right angles.	2 Cut off equal sides $AB = BC$, $DE = EF$, $GH = HK$. Join AC, DF, GK. 	3 By Prop. 22, AC, DF, GK can form a triangle. Construct triangle LMN with $LM = AC$, $MN = DF$, $NL = GK$. 	4 Circumscribe a circle about triangle LMN; let its centre be O. 	5 Join LO, MO, NO. 
6 Show that $AB > LO$. Suppose $AB \leq LO$. If $AB = LO$, then in $\triangle POL$, $PL = PM = AB = LO$, so $\triangle POL$ is equilateral and $\angle LOP = 2$ right angles. Then $\angle LPM = \angle LOP = 2$ right angles, so $\angle ABC = 2$ right angles. Either case makes the sum of the three given angles ≥ 4 right angles, contrary to hypothesis. Therefore $AB > LO$. 	7 Erect OP perpendicular to the plane of circle LMN at O, with OP chosen so that $OP^2 = AB^2 - LO^2$ (possible since $AB > LO$). 	8 Join PL, PM, PN. 	9 Since OP is perpendicular to the plane LMN, triangles POL, POM, PON are right-angled at O. $\angle POL = \angle POM = \angle PON = 1$ right angle. 	
10 By Pythagoras in right triangles: $PL^2 = OP^2 + LO^2 = AB^2$ $PM^2 = OP^2 + MO^2 = AB^2$ $PN^2 = OP^2 + NO^2 = AB^2$ Hence $PL = PM = PN = AB$. 	11 In triangles PLM and ABC: $PL = PM = AB = BC$, and $LM = AC$. By SSS, the included angles are equal: $\angle LPM = \angle ABC.$ 	12 Similarly, $\angle MPN = \angle DEF$ (using $PM = PN = DE = EF$ and $MN = DF$). $\angle LPN = \angle GHK$ (using $PL = PN = GH = HK$ and $LN = GK$). 	13 Thus the solid angle at P is formed by the three required plane angles $\angle ABC$, $\angle DEF$, $\angle GHK$. The three face angles about P are exactly the given angles. Hence the solid angle is constructed. 	

2.24 Proposition 24

Statement

If a solid is contained by parallel planes, its opposite faces are equal parallelograms.

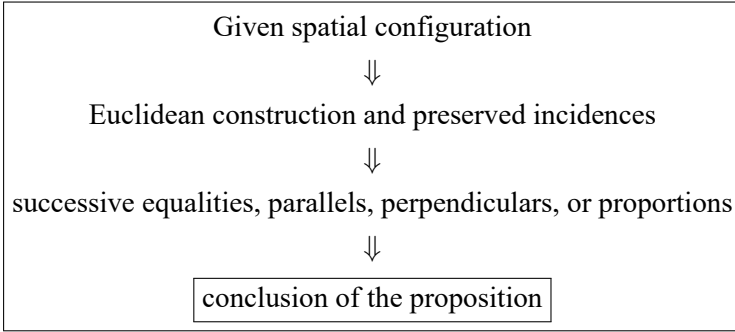
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let solid $CDHG$ be bounded by the stated pairs of parallel planes.
2. By Proposition 16, sections of parallel planes by each face-plane are parallel.
3. Hence face AC is a parallelogram.
4. Similarly every face is a parallelogram.
5. Join AH and DF .
6. Use Proposition 10 on corresponding parallel edges to obtain $\angle ABH = \angle DCF$.
7. With $AB = DC$ and $BH = CF$, triangles ABH and DCF are congruent, hence $AH = DF$.
8. Each opposite parallelogram is twice its corresponding congruent triangle, so $BG = CE$.
9. Similarly $AC = GF$ and $AE = BF$.
10. Therefore opposite faces are equal parallelograms.

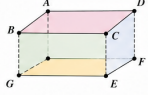
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PROPOSITION 24

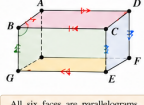
If a solid is contained by parallel planes, the opposite faces are equal parallelograms.

1 Let the solid $CDHG$ be contained by the six parallel planes AC , GF , AH , DF , BF , AE .



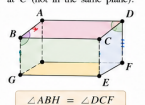
Opposite bounding planes:
 $AC \parallel GF$ (top $\parallel$ bottom) $BG \parallel CE$ (left $\parallel$ right) $AH \parallel DF$ (back $\parallel$ front)
 Thus the solid has opposite faces in parallel planes.

2 By XI.16: If two parallel planes are cut by a third plane, their sections are parallel lines. Each face is cut by the parallel planes meeting it in the other pair, so each face has both pairs of opposite edges parallel and is therefore a parallelogram.



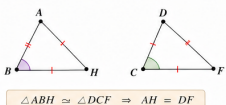
All six faces are parallelograms.

3 By XI.10: If two straight lines which intersect one another are respectively parallel to two straight lines which do not lie in the same plane, the straight lines make equal angles. Here $AB \parallel DC$ and $BH \parallel CF$, where AB and BH intersect at B , and DC and CF intersect at C (not in the same plane).



$\angle ABH = \angle DCF$

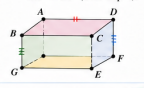
4 Also, in the parallelograms of the faces, $AB = DC$ and $BH = CF$ (opposite sides of parallelograms). Hence in triangles ABH and DCF we have $AB = DC$, $BH = CF$, $\angle ABH = \angle DCF$. By I.4 (SAS), the triangles are congruent.



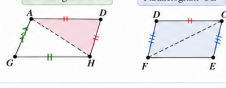
$\triangle ABH \cong \triangle DCF \Rightarrow AH = DF$

5 By the same reasoning on the other pairs of parallel edges:
 $AD \parallel BC$ and $AH \parallel DE \Rightarrow \angle ADH = \angle BCE$
 $AD = BC$, $AH = DE \Rightarrow \triangle ADH \cong \triangle BCE$.
 Also,
 $AB \parallel DC$ and $AE \parallel BF \Rightarrow \angle BAE = \angle CDF$
 $AB = DC$, $AE = BF \Rightarrow \triangle BAE \cong \triangle CDF$.
 Hence also $AD = BC$ and $AE = BF$.

6 Thus opposite edges are equal:
 $AB = DC$, $BC = AD$,
 $AE = BF$
 and
 $AH = DF$ (from step 4).

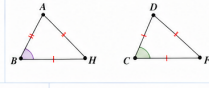


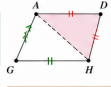
7 Face BG is the parallelogram $ABHG$. Its diagonal is BH . Face CE is the opposite parallelogram $CDEF$. Its diagonal is CF .



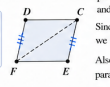
Parallelogram BG Parallelogram CE

8 From steps 3–4,
 $\triangle ABH \cong \triangle DCF$.
 Therefore
 $\{ABH\} = \{DCF\}$.



9 Each parallelogram is twice the triangle on one of its diagonals:
 Parallelogram BG
 $=$ twice $\triangle ABH$
 $= 2\{ABH\}$

 Hence $BG = CE$.

10 Similarly, parallelogram AC is twice $\triangle ABC$ and parallelogram GF is twice $\triangle DCF$. Since $\triangle ABC \cong \triangle DCF$ (by step 8 with $AE \parallel BF$), we have $AC = GF$. Also, parallelogram AE is twice $\triangle AEB$ and parallelogram BF is twice $\triangle CFB$. Since $\triangle AEB \cong \triangle CFB$, we have $AE = BF$.



Parallelogram CE
 $=$ twice $\triangle DCF$
 $= 2\{DCF\}$

$BG = CE$
 $AC = GF$
 $AE = BF$
 (all opposite faces equal).

A solid contained by parallel planes has opposite faces that are equal parallelograms. (Q.E.D.)

2.25 Proposition 25

Statement

If a parallelepiped is cut by a plane parallel to its opposite faces, then the volumes are proportional to the bases.

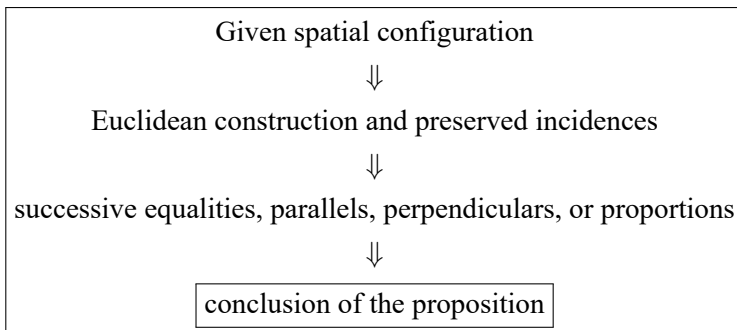
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let parallelepiped $ABCD$ be cut by plane FG parallel to opposite faces RA, DH .
2. Extend AH both ways and mark arbitrary equal segments on the extensions.
3. Complete the corresponding parallelograms and solids.
4. Equal bases at equal height give equal corresponding solids.
5. Establish correspondence between multiples of base AF and solid AC , and multiples of base FH and solid HC .
6. Apply Definition V.5 to obtain equality of ratios.
7. Hence $\text{area}(AEFV)/\text{area}(EHCF) = \text{vol}(ABFV)/\text{vol}(EGCD)$.

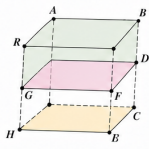
Complete Cinematic Logic



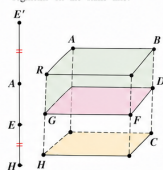
PROPOSITION 25

If a parallelepiped is cut by a plane parallel to its opposite faces, then the volumes are proportional to the bases.

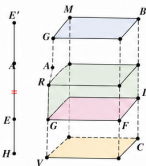
- 1 Let the parallelepiped $ABCD$ be cut by the plane FG which is parallel to the opposite faces RA and DH .

Plane $FG \parallel RA$ and DH .

- 2 Extend AH both ways and mark points E and E' such that $AE = EH$ (arbitrary multiple). Thus AE and EH are equal segments on the same line.



- 3 Complete the parallelepiped below $ABCD$ on the equal base $AEFV$, and above it on the equal base $E'GBM$.



- 4 By Prop. 31 (lemma): Solids on equal bases and between the same parallels are equal.

Solids on bases $E'GBM$ and $AEFV$ between the same parallels RA and DH are equal.

The middle solid $ABFG$ also has the same height between RA and DH .

Solids on bases $AEFV$ and $EHCF$ between the same parallels RA and DH are equal.

- 5 The segments on AH are in equal ratio: $AE : EH = 1 : 1$.

Hence

- The height of $ABFG$ is 1 part, so its volume is 1 part.
- The height of the top solid $E'GBM$ is 1 part, so its volume is 1 part.
- The height of the bottom solid $EHCF$ is 1 part, so its volume is 1 part.

Therefore,

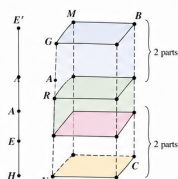
$$\text{volume}(ABFG) = 1 \text{ part}$$

$$\text{volume}(E'GBM) = 1 \text{ part}$$

$$\text{volume}(EHCF) = 1 \text{ part}$$

$$\text{volume}(ABFV) \text{ (top two parts)} = 2 \text{ parts}$$

$$\text{volume}(EGCD) \text{ (bottom two parts)} = 2 \text{ parts}$$



- 6 Thus the multiples of the base AF correspond to the multiples of the solid AC :

$$\text{Base } AF \text{ (one part)} \Leftrightarrow \text{Solid } AC \text{ (one part)}$$

$$\text{Base } AE \text{ (two parts)} \Leftrightarrow \text{Solid } ABFV \text{ (two parts)}$$

Similarly,

$$\text{Base } FH \text{ (one part)} \Leftrightarrow \text{Solid } HC \text{ (one part)}$$

$$\text{Base } EH \text{ (two parts)} \Leftrightarrow \text{Solid } EGCD \text{ (two parts)}$$



$$AF : FH = 1 : 1$$

$$AE : EH = 2 : 1$$

$$ABFV : HC = 2 : 1$$

$$ABFV : EGCD = 2 : 2$$

- 7 By Definition V.5 of proportion:

If equals (multiples) correspond to equals, then the magnitudes are proportional.

Therefore,

$$\frac{\text{area}(AEFV)}{\text{area}(EHCF)} = \frac{\text{volume}(ABFV)}{\text{volume}(EGCD)}$$

Hence, in a parallelepiped cut by a plane parallel to opposite faces, the volumes are proportional to the bases.

2.26 Proposition 26

Statement

To construct a solid angle equal to a given solid angle on a given line at a given point.

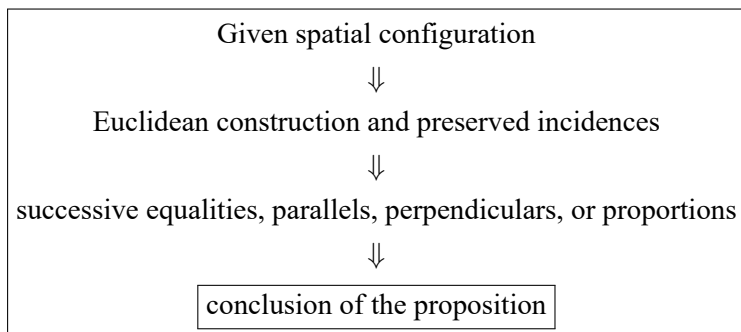
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

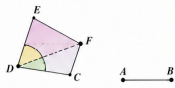
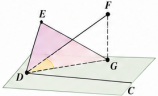
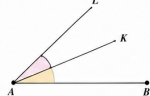
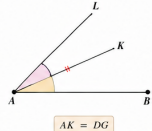
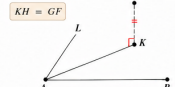
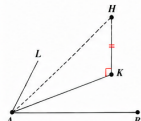
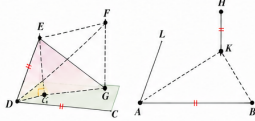
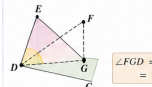
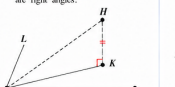
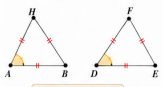
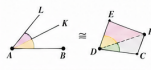
1. Let the given solid angle at D be formed by EDC , EDF , FDC , and let AB with point A be given.
2. Take F on DF ; drop FG perpendicular to plane EDC and join DG .
3. At A construct $\angle BAL = \angle EDC$ and $\angle BAK = \angle EDG$.
4. Make $AK = DG$.
5. At K erect KH perpendicular to plane BAL and make $KH = GF$.
6. Join HA .
7. Cut off $AB = DE$ and join HB , KB , FE , GE .
8. Use the right angles at G and K .
9. Establish the needed congruent right triangles.
10. Obtain $KB = GE$, $HB = FE$, and $AH = FD$.
11. Then triangles HAB , FDE are congruent, so $\angle BAH = \angle EDF$.
12. Similarly $\angle HAL = \angle FDC$, while $\angle BAL = \angle EDC$.
13. Thus the solid angle at A equals the one at D .

Complete Cinematic Logic



PROPOSITION 26

To construct a solid angle equal to a given solid angle on a given line at a given point.

1 Let the given solid angle at D be formed by the three plane angles EDC, EDF, FDC. Let the given line be AB, with point A on it. 	2 Take an arbitrary point F on DF. From F drop the perpendicular FG to the plane of EDC. Let it meet the plane at G. Join DG. 	3 At A, construct the plane angle BAL equal to the given angle EDC, and the plane angle BAK equal to the given angle EDG. 	4 On the ray AK make $AK = DG$.  $AK = DG$	
5 At K erect KH perpendicular to the plane through BAL (that is, to the plane of the page), and make $KH = GF$.  $KH = GF$	6 Join HA. 	7 Now cut off $AB = DE$ on the given line. Join HB, KB, FE, GE. 	8 Since FG is perpendicular to the plane EDC, $\angle FGD$ and $\angle FGE$ are right angles.  $\angle FGD = \angle FGE = 90^\circ$	
9 Similarly, since KH is perpendicular to the plane through BAL, $\angle HKA$ and $\angle HKB$ are right angles.  $\angle HKA = \angle HKB = 90^\circ$	10 From the right triangles FGD and HKA, $FG = KH$ and $DG = AK$ (given), so $FD = AH$ (RHS). From the right triangles FGE and HKB, $GF = KH$ and $GE = KB$, so $FE = HB$ (RHS). Also, $GE = KB$. $FD = AH$ $FE = HB$ $GE = KB$	11 In triangles HAB and FDE, $HA = FD$, $AB = DE$ (cut off), $HB = FE$, so $\triangle HAB \cong \triangle FDE$ (SSS). Hence $\angle BAH = \angle EDF$.  $\angle BAH = \angle EDF$	12 By construction, $\angle BAL = \angle EDC$ and $\angle BAK = \angle EDG$. Therefore $\angle HAL = \angle FDC$ (the remaining angle at H), and we already have $\angle BAH = \angle EDF$. $\angle HAL = \angle FDC$ $\angle BAL = \angle EDC$ $\angle BAH = \angle EDF$	13 Thus the three plane angles of the solid angle at A are equal respectively to the three plane angles of the given solid angle at D.  The solid angle at A is equal to the given solid angle at D.

2.27 Proposition 27

Statement

To describe a parallelepiped similar to a given one on a given line.

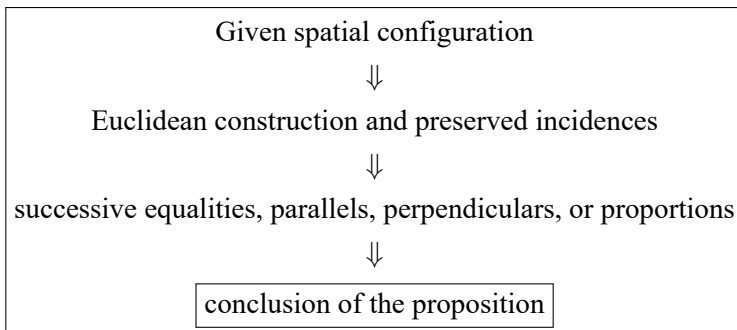
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

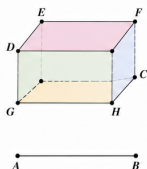
1. Let AB be the given line and CD the given parallelepiped with vertex C .
2. At A construct a solid angle equal to that at C by Proposition 26.
3. Set $BA/AK = EC/CG$ and $AK/AH = GC/CF$.
4. Hence $BA/AH = EC/CF$.
5. Complete parallelepiped AL on these edges.
6. Corresponding sides about equal angles are proportional, so corresponding parallelograms are similar.
7. Hence all corresponding faces are similar.
8. Therefore the solids are similar.

Complete Cinematic Logic



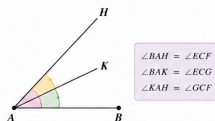
PROPOSITION 27**To describe a parallelepiped similar to a given one on a given line.**

- 1** Let AB be the given line.
Let $CDEFG$ be the given parallelepiped, with vertex C .



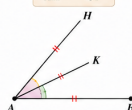
- 2** Construct at A a solid angle equal to the solid angle at C (Prop. 26), so that

$$\begin{aligned}\angle BAH &= \angle ECF, \\ \angle BAK &= \angle ECG, \\ \angle KAH &= \angle GCF.\end{aligned}$$



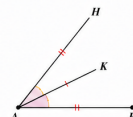
- 3** On the edges about A set the sides proportional to the corresponding edges about C :

$$\begin{aligned}\frac{BA}{AK} &= \frac{EC}{CG}, \\ \frac{AK}{AH} &= \frac{GC}{CF}.\end{aligned}$$

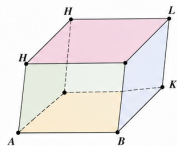


- 4** Hence, by equality of ratios (Prop. V.5),

$$\frac{BA}{AH} = \frac{EC}{CF}$$



- 5** Complete the parallelepiped $ABHK$ on these edges.
Through B draw a line parallel to AH , through H parallel to AK , and through K parallel to AB .
Let their intersections be L .

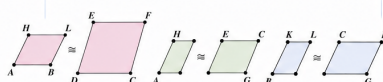


- 6** In triangles about A the two sides including equal angles are proportional:

$$\text{From } \angle BAH = \angle ECF \text{ and } \frac{BA}{AH} = \frac{EC}{CF}, \text{ the parallelograms on these sides are similar. (Prop. VI.6)}$$

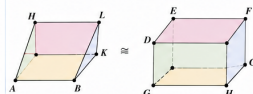
$$\text{From } \angle BAK = \angle ECG \text{ and } \frac{BA}{AK} = \frac{EC}{CG}, \text{ the parallelograms on these sides are similar.}$$

$$\text{From } \angle KAH = \angle GCF \text{ and } \frac{AK}{AH} = \frac{GC}{CF}, \text{ the parallelograms on these sides are similar.}$$



- 7** Thus each face of $ABHK$ is similar to the corresponding face of $CDEFG$.

Therefore the parallelepipeds $ABHK$ and $CDEFG$ are similar.



The parallelepiped $ABHK$ on the given line AB is similar to the given parallelepiped $CDEFG$.

2.28 Proposition 28

Statement

A parallelepiped cut by a plane through the diagonals of opposite faces is divided into two equal prisms.

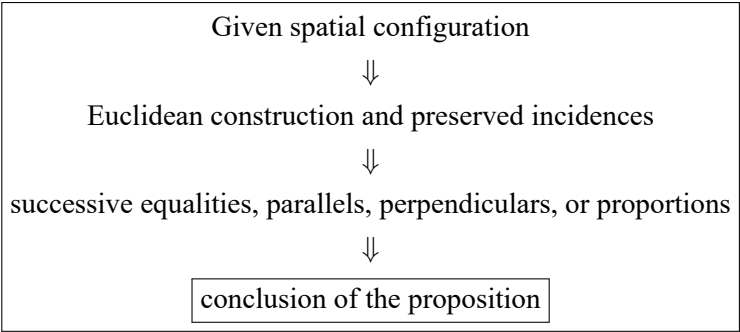
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let parallelepiped AB be cut by plane $CDEF$ through diagonals CF, DE of opposite faces.
2. The plane divides it into two prisms with triangular bases CGF, ADE and CFB, DEH .
3. The triangular bases are equal, being halves of equal parallelograms.
4. The three remaining faces of one prism are respectively equal to those of the other.
5. Thus the prisms are contained by equal planes in number and magnitude.
6. Therefore the two prisms are equal.
7. Hence the original solid is bisected.

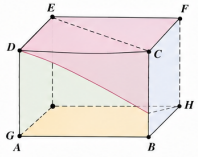
Complete Cinematic Logic



PROPOSITION 28

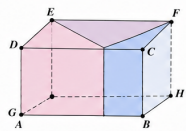
A parallelepiped cut by a plane through the diagonals of opposite faces is divided into two equal prisms.

1 Let the parallelepiped AB be cut by the plane $CDEF$ through the diagonals CF and DE of opposite faces.

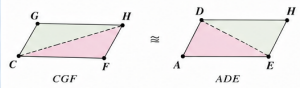


2 The plane $CDEF$ divides the solid into two prisms:

- prism $C-G-F-D-E$; with triangular bases CGF and ADE .
- prism $C-F-B-D-E-H$; with triangular bases CFB and DEH .



3 The two triangular bases are equal:
Triangle CGF is half of parallelogram $CGFH$; triangle ADE is half of parallelogram $ADEH$. But opposite faces of a parallelepiped are equal parallelograms (Prop. 24).
Therefore, $CGF = ADE$.

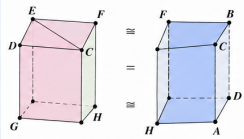


$CGF = ADE$

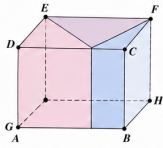
4 The three rectangular faces of one prism are respectively equal to the three of the other, since opposite faces of a parallelepiped are equal.

Prism $CGFED$ $CG = AD$ (faces $GCDH$ and $ADHE$) $GF = DE$ (faces $CGFH$ and $ADEH$) $FE = CD$ (faces $FEHG$ and $CDAB$)	Prism $CFBEDH$ $CB = HG$ (faces $CBHG$ and $ADHE$) $FB = EH$ (faces $FEBA$ and $GCDH$) $BD = CF$ (faces $BDCA$ and $FEHG$)
--	---

5 Thus the two prisms are contained by equal planes in number and in magnitude (Def. XI.10).



6 Therefore, the two prisms are equal.
Hence the parallelepiped $ABCD-EFGH$ is cut into two equal parts.



The plane through the diagonals of opposite faces divides the parallelepiped into two equal prisms.

2.29 Proposition 29

Statement

Parallelepipeds on the same base, same height, with standing edges on the same straight lines, are equal.

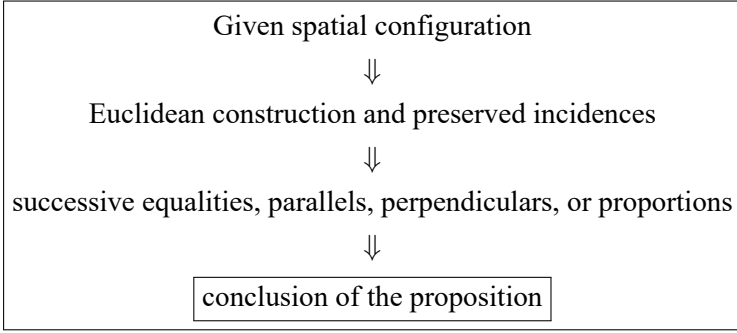
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

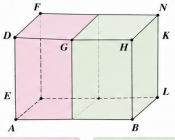
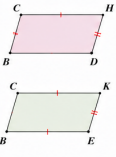
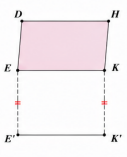
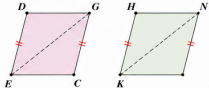
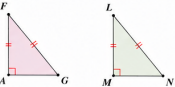
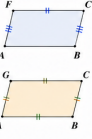
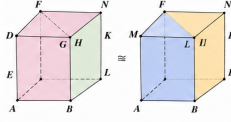
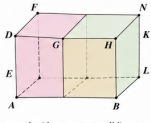
1. Let CM, CN be parallelepipeds on base AB with standing edges on the same straight lines.
2. From the face parallelograms obtain $CB = DH = EK$.
3. Subtract the common part to get $DE = HK$.
4. Hence triangles DCE and HBK are equal and corresponding parallelograms are equal.
5. Similarly triangle AFG equals MLN .
6. Use equality of opposite edges such as $CF = BM$ and $CG = BN$.
7. Therefore the corresponding triangular prisms are equal.
8. Add the common solid to both.
9. Hence the whole solids CM and CN are equal.

Complete Cinematic Logic



PROPOSITION 29

Parallelepipeds on the same base, same height, with standing edges on the same straight lines, are equal.

1 Let the parallelepiped CM and the parallelepiped CN be on the same base AB, of the same height, and let their standing edges lie on the same straight lines FN and DK.  Parallelepiped CM (built on AB) Parallelepiped CN (built on AB) Standing edges on the same lines FN (at A) and DK (at B).	2 Since CH and CK are parallelograms, $CB = DH = EK$ (opposite sides of parallelograms).  $CB = DH = EK$	3 Take the equal segment EH from each equal. Therefore, $DE = HK$.  $DE = HK$	4 In triangles DCE and HBK we have $DC = HB$ (opposite faces), $DE = HK$ (from step 3), and $\angle CDE = \angle KHB = 90^\circ$. Therefore, $\triangle DCE \cong \triangle HBK$ (I.4). Hence $DC = HB$, $CE = BK$, $DE = HK$. Therefore, parallelograms DG and HN are equal (Opposite sides equal and included angle equal).  Parallelogram $DG =$ Parallelogram HN
5 Similarly, in $\triangle AFG$ and $\triangle MLN$ we have $AF = ML$ (opposite faces), $AG = MN$ (from step 4), and $\angle FAG = \angle LMN = 90^\circ$. Therefore, $\triangle AFG \cong \triangle MLN$ (I.4). Hence $AF = ML$, $AG = MN$, $FG = LN$.  $\triangle AFG = \triangle MLN$	6 Also, opposite faces of the parallelepipeds are equal. Therefore, $CF = BM$ and $CG = BN$. 	7 From steps 4, 5, and 6: The prism with triangular bases AFG and DCE is equal to the prism with triangular bases MLN and HBK (Def. XI.10).  Prism $(AFG, DCE) =$ Prism (MLN, HBK)	8 The solids CM and CN have in common the parallelepiped on base AB and opposite faces GE and HM.  Adding equals (the common solid) to equals (the two prisms) gives equals. Therefore, the parallelepiped CM is equal to the parallelepiped CN.

2.30 Proposition 30

Statement

Parallelepipeds on the same base and of the same height, though their standing edges are not on the same straight lines, are equal.

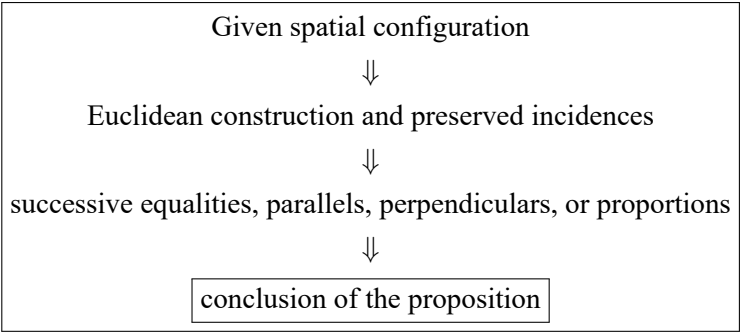
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Construct auxiliary parallelepipeds whose standing edges are aligned on the same straight lines.
2. Apply Proposition 29 to the aligned auxiliary solids.
3. Relate the auxiliary solids to the original solids by the equal-base, equal-height construction.
4. By transitivity, the two original parallelepipeds are equal.

Complete Cinematic Logic



PROPOSITION 30

Parallelepipeds on the same base, same height, with standing edges not on the same straight lines, are equal.

1 Let the parallelepipeds CM and CN be on the same base AB , of the same height, but with standing edges not on the same straight lines.

Parallelepiped CM

Parallelepiped CN

Same base AB and same height, but standing edges on different lines.

2 Through A draw the straight lines FN and DK . Through B draw the straight lines $F'N'$ and $D'K'$. On these lines construct the parallelepipeds CM_1 and CN_1 on the same base AB and of the same height as the given ones, with standing edges on the same straight lines FN and DK .

Parallelepiped CM_1

Parallelepiped CN_1

Standing edges of CM_1 and CN_1 lie on the same lines FN and DK .

3 By Prop. 29 (parallelepipeds on the same base, same height, with standing edges on the same straight lines are equal), we have

$CM_1 = CN_1$

(They are on the same base AB , of the same height, and their standing edges are on the same lines FN and DK .)

Parallelepiped CM_1

Parallelepiped CN_1

$CM_1 = CN_1$

4 The parallelepipeds CM and CM_1 are on the same base AB and between the same parallels AB and FN . By Prop. 31 (parallelepipeds on equal bases and between the same parallels are equal),

$CM = CM_1$

Parallelepiped CM

Parallelepiped CM_1

Same base AB and same height (AB and FN); hence $CM = CM_1$.

5 Similarly, CN and CN_1 are on the same base AB and between the same parallels AB and $F'N'$. By Prop. 31,

$CN = CN_1$

Parallelepiped CN

Parallelepiped CN_1

Same base AB and same height (AB and $F'N'$); hence $CN = CN_1$.

6 From steps 3, 4, and 5, by transitivity of equality,

$CM = CM_1$

$CM_1 = CN_1$

$CN_1 = CN$

$CM = CN$

Parallelepiped CM

Parallelepiped CN

Therefore the parallelepipeds CM and CN are equal.

Prop. 29. Parallelepipeds on the same base, same height, with standing edges on the same straight lines, are equal.

Prop. 31. Parallelepipeds on equal bases and between the same parallels are equal.

2.31 Proposition 31

Statement

Parallelepipeds on equal bases and of the same height are equal.

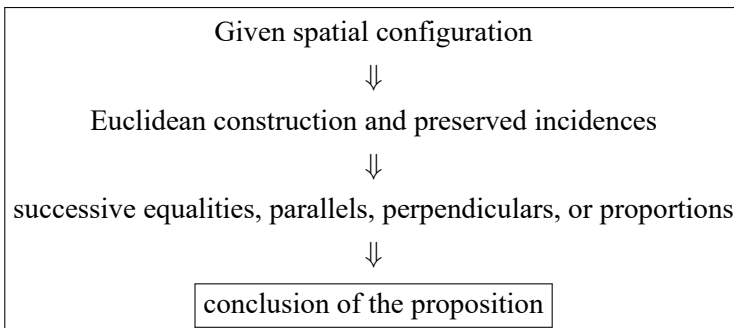
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

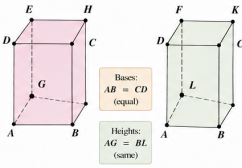
1. Let AE, CF be parallelepipeds on equal bases AB, CD and with the same height.
2. In the aligned/right case use the preceding propositions.
3. For the general case transform each to an equivalent right or aligned parallelepiped on the same base.
4. The transformed solids have equal bases and the same height.
5. Hence the transformed solids are equal.
6. Therefore the original solids are equal.

Complete Cinematic Logic

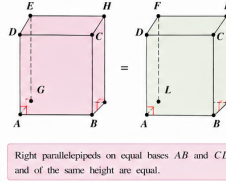


PROPOSITION 31**Parallelepipeds on equal bases and of the same height are equal.**

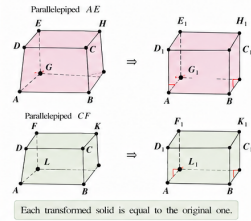
- 1** Let the parallelepiped AE and the parallelepiped CF be on equal bases AB and CD , and of the same height.



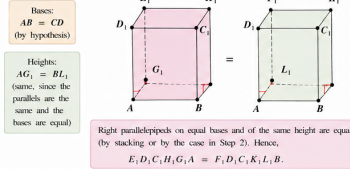
- 2** If the standing edges are perpendicular to the bases and stand on the same straight lines, the result follows (by stacking or by Prop. 30).



- 3** In the general case, by Prop. 29 or Prop. 30 we can transform each parallelepiped into an equal right parallelepiped on the same base and between the same parallels.



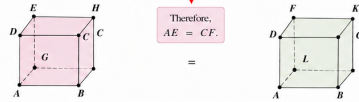
- 4** The transformed solids are right parallelepipeds on the same bases AB and CD and of same height.



- 5** By steps 3 and 4, we have

$$\begin{aligned}
 AE &= E_1D_1C_1H_1G_1A & E_1D_1C_1H_1G_1A &= F_1D_1C_1K_1L_1B & F_1D_1C_1K_1L_1B &= CF \\
 &(\text{by Prop. 29 or 30}) & &(\text{by Step 4}) & &(\text{by Prop. 29 or 30})
 \end{aligned}$$

Therefore,
 $AE = CF$.



Prop. 29. Parallelepipeds on the same base, same height, with standing edges on the same straight lines, are equal.

Prop. 30. The same, even when the standing edges are not on the same lines.

Prop. 31. Parallelepipeds on equal bases and between the same parallels are equal.

2.32 Proposition 32

Statement

Parallelepipeds of the same height are to one another as their bases.

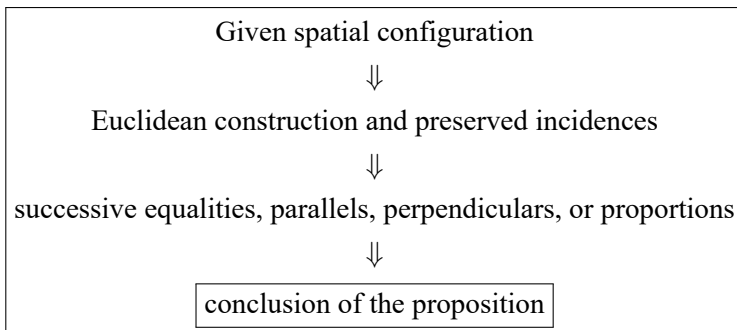
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

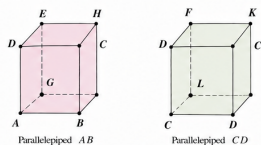
1. Let solids AB, CD have the same height.
2. Apply a parallelogram FH equal to base AE to FG at the same angle and complete solid GK with the same height as CD .
3. By Proposition 31, solid $AB = GK$.
4. Cut solid CK by plane DG parallel to its opposite faces; by Proposition 25, $\text{area}(CF)/\text{area}(FH) = \text{vol}(CD)/\text{vol}(DH)$.
5. Since $FH = AE$ and $GK = AB$, substitute.
6. Thus $\text{area}(AE)/\text{area}(CF) = \text{vol}(AB)/\text{vol}(CD)$.

Complete Cinematic Logic



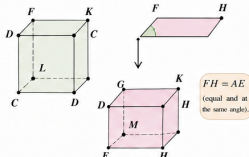
PROPOSITION 32**Parallelepipeds of the same height are to one another as their bases.**

- 1 Let the parallelepiped AB and the parallelepiped CD be of the same height.



Heights: $AG = CL$
(the same parallels).

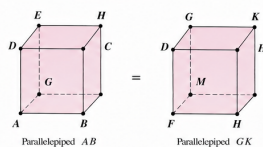
- 2 On FG take the parallelogram FH equal to the base AE and applied at the same angle. Complete the parallelepiped GK on FH with the same height as CD .



Parallelepiped GK has the same height as CD
(between the same parallels DK and FH).

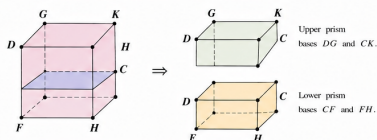
- 3 Since the parallelepipeds AB and GK are on equal bases ($AE = FH$) and of the same height, by Proposition 31,

$$AB = GK$$



- 4 Cut the parallelepiped GK by the plane DG parallel to its opposite faces.

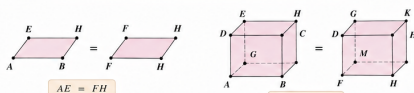
By Proposition 25 (if a parallelepiped is cut by a plane parallel to opposite faces, the volumes are proportional to the bases),



$$\text{By Prop. 25, } \frac{\text{area}(CF)}{\text{area}(FH)} = \frac{\text{vol}(CD)}{\text{vol}(DH)}$$

- 5 Since $FH = AE$ and, by step 3, $GK = AB$,
 $\text{area}(FH) = \text{area}(AE)$ and $\text{vol}(GK) = \text{vol}(AB)$.
From the proportion in step 4,

$$\frac{\text{area}(CF)}{\text{area}(FH)} = \frac{\text{vol}(CD)}{\text{vol}(DH)} \Rightarrow \frac{\text{area}(AE)}{\text{area}(CF)} = \frac{\text{vol}(AB)}{\text{vol}(CD)}$$



$$AE = FH$$

$$AB = GK$$

Therefore, parallelepipeds of the same height
are to one another as their bases.

2.33 Proposition 33

Statement

Similar parallelepipeds are in the cubed ratio of their corresponding sides.

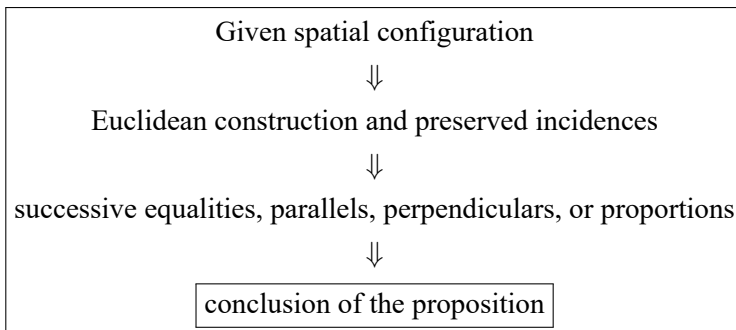
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let AB, CD be similar with corresponding sides AE, CF .
2. Extend the three corresponding edges and set $EK = CF$, $EL = FN$, $EM = FR$.
3. Complete parallelogram KL and solid KP .
4. By matching corresponding faces, KP is equal and similar to CD .
5. Complete the auxiliary parallelograms and solids EO, LQ .
6. By Proposition 32, $\text{vol}(AB)/\text{vol}(EO) = AE/EK$.
7. Similarly $\text{vol}(EO)/\text{vol}(LQ) = AE/EK$ and $\text{vol}(LQ)/\text{vol}(KP) = AE/EK$.
8. Therefore $\text{vol}(AB)/\text{vol}(KP) = (AE/EK)^3$.
9. Since $KP = CD$ and $EK = CF$, $\text{vol}(AB)/\text{vol}(CD) = (AE/CF)^3$.

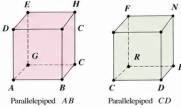
Complete Cinematic Logic



PROPOSITION 33

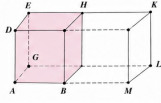
Similar parallelepipeds are in the cubed ratio of their corresponding sides.

- 1 Let the parallelepipeds AB and CD be similar, and let AE and CF be corresponding sides.



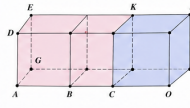
Similar solids:
 $AB \sim CD$ (corresponding faces are similar).
Corresponding sides are in the same ratio.

- 2 Produce the three edges from A through E, G, H beyond E, G, H to K, L, M , and make
 $EK = CF, EL = FN, EM = FR$.



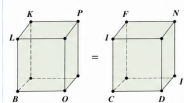
Thus $AE \parallel KF, AG \parallel KL, AH \parallel KM$
and $EK = CF, EL = FN, EM = FR$.

- 3 Complete the parallelogram KL on EK and EL , and complete the parallelepiped KP on the parallelogram KL with the same height as AB .



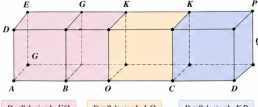
Parallelogram KL is applied in the same direction
on base AB and at the same angle.

- 4 By the construction and since $AB \sim CD$, all corresponding faces of KP and CD are equal and similarly placed.



Hence the parallelepipeds KP and CD
are equal and therefore similar.

- 5 Complete the parallelogram GK on EG and EK .
On the bases GK and KL complete the parallelepipeds EO and LQ with the same height as AB .



Parallelepiped EO
(on base GK)

Parallelepiped LQ
(on base KL)

Parallelepiped KP
(on base KL)

All three solids have the same height as AB
(between the same parallels).

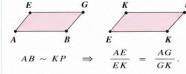
- 6 By Proposition 32, solids of the same height are proportional to their bases.
Hence, for the solids AB and EO ,

$$\frac{\text{vol}(AB)}{\text{vol}(EO)} = \frac{\text{area}(AG)}{\text{area}(GK)}.$$

But AG and GK correspond to AE and EK
in similar bases AB and KP .

Therefore,

$$\frac{\text{vol}(AB)}{\text{vol}(EO)} = \frac{AE}{EK}.$$



$$AB \sim KP \Rightarrow \frac{AE}{EK} = \frac{AG}{GK}.$$

- 7 Apply Proposition 32 to the pairs (EO, LQ) and (LQ, KP) .

First,

$$\frac{\text{vol}(EO)}{\text{vol}(LQ)} = \frac{\text{area}(GK)}{\text{area}(KL)} = \frac{GK}{KL}.$$

But in similar bases AB and KP ,

$$\frac{GK}{KL} = \frac{GE}{EL} = \frac{AE}{EK}.$$

Secondly,

$$\frac{\text{vol}(LQ)}{\text{vol}(KP)} = \frac{\text{area}(KL)}{\text{area}(KM)} = \frac{KL}{KM}.$$

Again, in similar bases AB and KP ,

$$\frac{KL}{KM} = \frac{HE}{EM} = \frac{AE}{EK}.$$

$$\begin{aligned} \text{Thus,} \\ \frac{\text{vol}(AB)}{\text{vol}(EO)} &= \frac{AE}{EK} \\ &= \frac{\text{vol}(EO)}{\text{vol}(LQ)} \\ &= \frac{\text{vol}(LQ)}{\text{vol}(KP)}. \end{aligned}$$

- 8 Since these magnitudes are continually proportional, the ratio of the first to the last is the cube of the common ratio (Def. V.14). Hence,

$$\frac{\text{vol}(AB)}{\text{vol}(KP)} = \left(\frac{AE}{EK} \right)^3.$$

Definition (V.14):

If $a : b = b : c = c : d$,
then $a : d = (a : b)^3$ (cubed ratio).

- 9 But by step 4, $KP = CD$, and by construction $EK = CF$.

Therefore,

$$\frac{\text{vol}(AB)}{\text{vol}(CD)} = \left(\frac{AE}{CF} \right)^3.$$

Hence similar parallelepipeds
are in the cubed ratio of their
corresponding sides.

2.34 Proposition 34

Statement

For equal parallelepipeds, bases are reciprocally proportional to heights; and conversely.

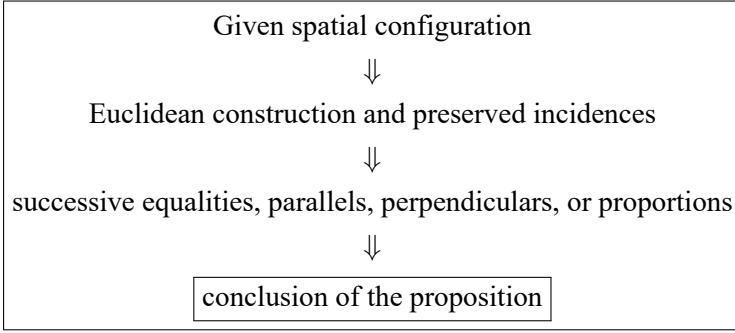
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

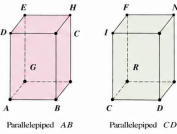
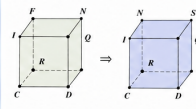
Proof Actions

1. Let AB, CD be equal parallelepipeds.
2. For perpendicular heights, construct an auxiliary solid on the base of one with the height of the other and apply Proposition 32.
3. Obtain the reciprocal proportion between bases and heights.
4. For oblique heights, reduce to right/aligned solids by Propositions 29–30 and apply the same argument.
5. Conversely assume the bases and heights are reciprocally proportional.
6. Construct an auxiliary solid with the base of CD and height of AB .
7. Use Proposition 32 and the given proportion to identify its volume with AB and CD .
8. Therefore the original parallelepipeds are equal.

Complete Cinematic Logic

**PROPOSITION 34**

For equal parallelepipeds, bases are reciprocally proportional to heights; and conversely.

I. FORWARD: If two parallelepipeds are equal, then their bases are reciprocally proportional to their heights.**1** Let the parallelepipeds AB and CD be equal.Given: $AB = CD$.**2** (Heights perpendicular to bases.)Let EH and NQ be the heights of AB and CD over their bases AB and CD , and let EH be perpendicular to its base.On the same base NQ construct the parallelepiped $NQRS$ with height EH (equal to the height of AB). $NQRS$ is on the same base NQ as CD , and has height EH .

Thus, when the heights are perpendicular to the bases, bases are reciprocally proportional to the heights.

3 By Proposition 32 (same height),

$$\frac{\text{vol}(AB)}{\text{vol}(NQRS)} = \frac{\text{area}(AB)}{\text{area}(NQ)}$$

But $AB = CD = NQRS$; hence

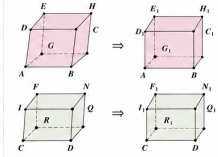
$$\text{vol}(AB) = \text{vol}(NQRS).$$

Therefore,

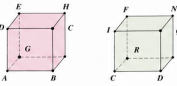
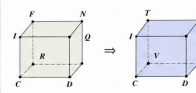
$$\frac{\text{area}(AB)}{\text{area}(NQ)} = 1 = \frac{\text{height } NQRS}{\text{height } AB}$$

Since height $NQRS = EH$ and height $AB = EH$, we get

$$\frac{\text{area}(AB)}{\text{area}(NQ)} = \frac{\text{height } CD}{\text{height } AB}$$

4 (Heights oblique.)If the heights are not perpendicular, by Propositions 29 or 30 we can transform AB and CD into right parallelepipeds AB_1 and CD_1 on the same bases and of the same heights as AB and CD , respectively.Right parallelepipeds AB_1 and CD_1 are equal to AB and CD respectively (Props. 29 or 30). Apply the perpendicular case to these right solids. Hence the result holds in all cases.**II. CONVERSE:** If the bases are reciprocally proportional to the heights, then the parallelepipeds are equal.**5** Let parallelepipeds AB and CD be such that

$$\frac{\text{area}(AB)}{\text{area}(CD)} = \frac{\text{height } CD}{\text{height } AB}.$$

Heights: EH (of AB), NQ (of CD).**6** On base CD construct the parallelepiped $CDTU$ with height equal to EH (the height of AB). $CDTU$ is on the same base CD as CD and has height EH .**7** By Proposition 32 (same height),

$$\frac{\text{vol}(AB)}{\text{vol}(CDTU)} = \frac{\text{area}(AB)}{\text{area}(CD)}$$

By the given proportion,

$$\frac{\text{area}(AB)}{\text{area}(CD)} = \frac{\text{height } CD}{\text{height } AB}.$$

But height $CD = \text{height } CDTU = NQ$, so

$$\frac{\text{vol}(AB)}{\text{vol}(CDTU)} = \frac{NQ}{EH}.$$

8 Again by Proposition 32 (same base CD),

$$\frac{\text{vol}(CDTU)}{\text{vol}(CD)} = \frac{\text{height } CDTU}{\text{height } CD} = \frac{EH}{NQ}.$$

Multiplying the relations in steps 7 and 8,

$$\frac{\text{vol}(AB)}{\text{vol}(CD)} = 1.$$

Hence $\text{vol}(AB) = \text{vol}(CD)$, so

$$AB = CD.$$

Therefore: (Forward) If $AB = CD$, then $\frac{\text{area}(AB)}{\text{area}(CD)} = \frac{\text{height } CD}{\text{height } AB}$.(Converse) If $\frac{\text{area}(AB)}{\text{area}(CD)} = \frac{\text{height } CD}{\text{height } AB}$, then $AB = CD$.

2.35 Proposition 35

Statement

If two equal plane angles have equal raised lines at their vertices making equal angles with the original sides, then the perpendiculars from the raised points to the planes are equal, and the angles formed by the raised lines and the lines to the footpoints are equal.

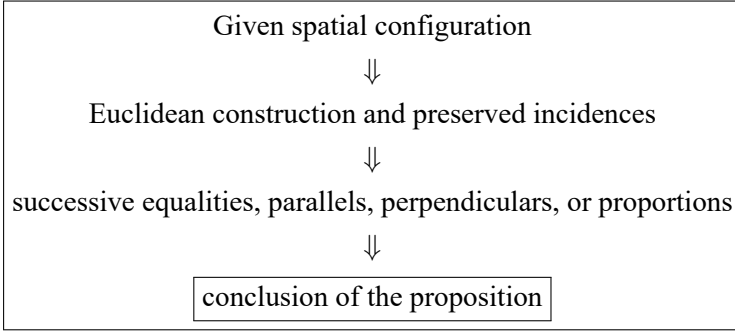
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

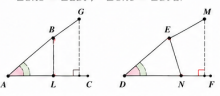
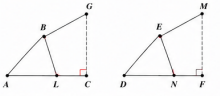
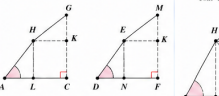
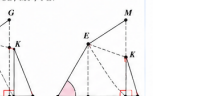
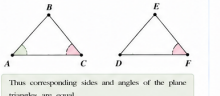
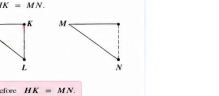
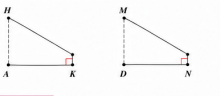
1. Let $\angle BAC = \angle EDF$ and let raised lines AG, DM make equal corresponding angles with the sides.
2. From G, M drop perpendiculars GL, MN to the respective planes and join LA, ND .
3. Set $AH = DM$ and draw $HK \parallel GL$.
4. From K, N drop perpendiculars to the corresponding sides and complete the required joins.
5. Use the right triangles to establish $AC = DF$ and $AB = DE$.
6. Then triangles ABC, DEF are congruent.
7. From corresponding right triangles obtain $CK = FN$ and then $AK = DN$.
8. Use Pythagoras again to obtain $HK = MN$.
9. Triangles HAK, MDN are congruent by SSS, so $\angle HAK = \angle MDN$.
10. Thus both the perpendiculars and the required angles are equal.

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PROPOSITION 35

If two equal plane angles have equal raised lines at their vertices making equal angles with the original sides, then the perpendiculars from the raised points to the planes are equal, and the angles formed by the raised lines and the lines to the footpoints are equal.

1 Let the plane angles BAC and EDF be equal. From A and D take the raised lines AG and DM making equal angles with the sides; that is, $\angle BAG = \angle EDF$, $\angle GAC = \angle DFE$.  Given: $\angle BAC = \angle EDF$, $\angle BAG = \angle EDF$, $\angle GAC = \angle DFE$.	2 At G and M draw perpendiculars $GL \perp$ (plane BAC) and $MN \perp$ (plane EDF). Join LA and ND.  GL and MN are perpendicular to the respective planes at L and N.	3 On AG take H such that $AH = DM$. Through H draw $HK \parallel GL$; hence $HK \perp$ (plane BAC) and $HK = GL$.  HK is perpendicular to the plane BAC.	4 From K and N drop perpendiculars to the sides of the angles: $KC \perp AC$, $KB \perp AB$; $HK \perp DF$, $NE \perp DE$. Join HC, CB, MF, FE. 
5 In right triangles KAC and NDE, $AC^2 = AK^2 + KC^2$, $DE^2 = DN^2 + NE^2$. But $AK = DN$ (since AG and DM make equal angles with the sides and $AH = DM$—equal right triangles AHK and MDN in the vertical planes), and $KC = NF$ (both perpendiculars from K and N to the corresponding sides making equal angles with the sides). Hence $AC = DE$. Similarly, in right triangles KAB and NDE, $AB^2 = AK^2 + KB^2$, $DE^2 = DN^2 + NE^2$, with $AK = DN$ and $KB = NE$. Hence $AB = DE$. Therefore $AC = DE$ and $AB = DE$.	6 Since the plane angles BAC and EDF are equal and their two sides about the equal angle are respectively equal ($AB = DE$, $AC = DF$), triangles ABC and DEF are congruent (Def. 14). Hence $BC = EF$ and $\angle ABC = \angle DEF$. Also $\angle ACB = \angle DFE$.  Thus corresponding sides and angles of the plane triangles are equal.	7 In right triangles KCB and NFE, $CB^2 = KB^2 + KC^2$, $EF^2 = NE^2 + NF^2$. But $CB = EF$ (step 6) and $KB = NE$ (perpendiculars from K and N to the corresponding sides which make equal angles with the other sides), therefore $KC = NF$. In right triangles KAC and NDE, $AC^2 = AK^2 + KC^2$, $DN^2 = DN^2 + NF^2$. But $AC = DE$ (step 5) and $KC = NF$, therefore $AK = DN$. Hence $KC = NF$ and $AK = DN$.	8 In right triangles HKL and MNN, $KL^2 = HL^2 + KL^2$, $MN^2 = NN^2 + NN^2$. But $HK = GL$, MN is corresponding, and $KL = LN$ (both are perpendiculars from K and N to the same sides AC and DF which are equal and subtend equal angles). Hence $HK = MN$.  Therefore $HK = MN$.
9 In triangles HAK and MDN we have $HA = MD$ (by construction), $AK = DN$ (step 7), $HK = MN$ (step 8). Therefore $\triangle HAK \cong \triangle MDN$ (Def. 14). Hence $\angle HAK = \angle MDN$. 	10 But $\angle HAK$ is the angle between the raised line AG and the line AL (in the plane BAC), and $\angle MDN$ is the angle between the raised line DM and the line DN (in the plane EDF). Therefore the angles which the raised lines make with the lines to the footpoints are equal. Also, from step 8, $HK = MN$; that is, the perpendiculars from the raised points to the planes are equal. Thus, if two equal plane angles have equal raised lines at their vertices making equal angles with the original sides, then the perpendiculars from the raised points to the planes are equal, and the angles formed by the raised lines and the lines to the footpoints are equal.		

2.36 Proposition 36

Statement

If three lines are proportional, the parallelepiped formed from them is equal to the equilateral parallelepiped on the middle line, with the same angles.

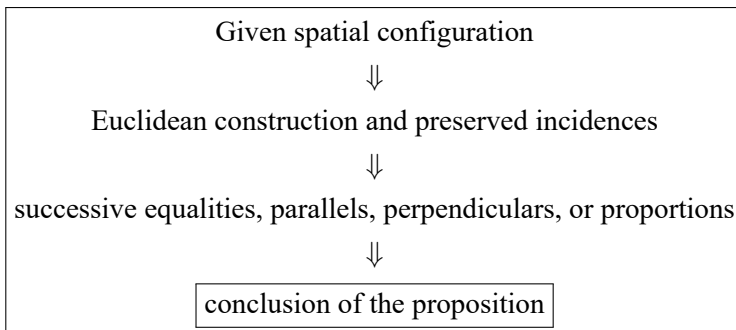
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let $A : B = B : C$ and construct at E an equilateral parallelepiped EK with edges $DE = GE = EF = B$.
2. Take $LM = A$ and at L construct an equal solid angle with $LO = B$ and $LN = C$; complete solid LH .
3. From the proportion obtain $LM/EF = DE/LN$.
4. By VI.14 the corresponding base parallelograms are equal.
5. The raised edges LO, EG are equal and make equal corresponding angles; by Proposition 35 their perpendicular heights are equal.
6. By Proposition 31, solids LH and EK are equal.
7. Thus the solid on A, B, C equals the equilateral solid on B with the same angles.

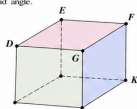
Complete Cinematic Logic



PROPOSITION 36

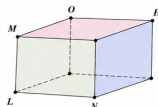
If three lines are proportional, the parallelepiped formed from them is equal to the equilateral parallelepiped on the middle line, with the same angles.

1 Let the three lines A, B, C be proportional, $A : B = B : C$.
At E form a solid angle whose three edges DE, GE, EF are equal to the middle line B , so that $DE = GE = EF = B$.
Complete the parallelepiped EK on this solid angle.



Thus $DE = GE = EF = B$, and EK is the equilateral parallelepiped on the line B , with the given angles.

2 On a straight line take $LM = A$.
At L construct a solid angle equal to that at E , so that $LO = B$ and $LN = C$ in the same directions as DE and EF .
Complete the parallelepiped LH on this solid angle.



Here $LM = A$, $LO = B$, $LN = C$, and the solid angle at L is equal to that at E .

3 By the proportion $A : B = B : C$, we have
$$\frac{LM}{EF} = \frac{DE}{LN}.$$

The included solid angles at L and E are equal (by construction).

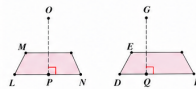
Therefore, by Proposition VI.14, the parallelograms MN and DF are equal.



Hence $\text{area}(MN) = \text{area}(DF)$.

4 The raised edges LO and EG are equal (both equal to B) and make equal angles with the corresponding sides of the base parallelograms.

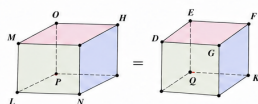
By Proposition 35 (corollary), the perpendiculars from O and G to the planes LMN and DEF are equal.



Thus the perpendicular heights OP and GQ are equal.

5 From steps 3 and 4, the two parallelepipeds LH and EK have equal bases (parallelograms MN and DF) and equal heights (perpendiculars OP and GQ).

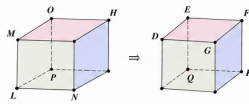
Therefore, by Proposition 31, the parallelepipeds LH and EK are equal.



Hence $LH = EK$.

6 But the solid LH is the parallelepiped formed from the three lines A, B, C (edges $LM = A$, $LO = B$, $LN = C$) with the given solid angle.

The solid EK is the equilateral parallelepiped on the middle line B (edges $DE = GE = EF = B$) with the same solid angle.



Therefore these two parallelepipeds, having the same angles, are equal.

7 Hence, if three lines are proportional ($A : B = B : C$), the parallelepiped formed from them is equal to the equilateral parallelepiped on the middle line, with the same angles.

The parallelepiped on A, B, C
(with the given solid angle)
 $=$
the equilateral parallelepiped on B
(with the same angles).

★ **Summary.** When $A : B = B : C$, the solid angle built on A, B, C gives the same solid as the equilateral solid on B built with the same angles.

2.37 Proposition 37

Statement

If four lines are proportional, the similar parallelepipeds on them are proportional; and conversely.

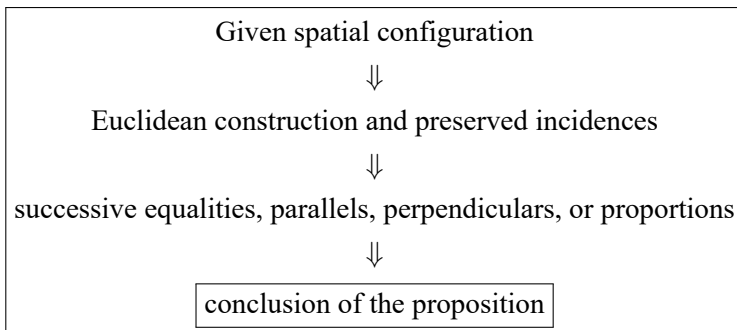
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

1. Let $AB : CD = EF : GH$ and let similar solids KA, LC, ME, NG be built on them.
2. By Proposition 33, $\text{vol}(KA)/\text{vol}(LC) = (AB/CD)^3$ and $\text{vol}(ME)/\text{vol}(NG) = (EF/GH)^3$.
3. Since the side ratios are equal, their cubes are equal.
4. Hence $KA/LC = ME/NG$.
5. Conversely suppose the volume ratios are equal.
6. By Proposition 33 the corresponding cubed side ratios are equal.
7. Taking positive cube roots gives $AB/CD = EF/GH$.

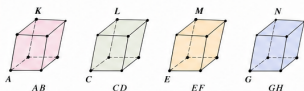
Complete Cinematic Logic



PROPOSITION 37 If four lines are proportional, the similar parallelepipeds on them are proportional; and conversely.**I. DIRECT:** If four lines are proportional, the similar parallelepipeds on them are proportional.**1** Let four lines be proportional:

$$AB : CD = EF : GH.$$

On these lines, construct similar parallelepipeds

 KA , LC , ME , NG respectively on the same solid angle.

Similar parallelepipeds on proportional lines with the same solid angle.

2 By Proposition 33 (similar parallelepipeds are in the cubed ratio of their corresponding sides), from the similar solids KA and LC ,

$$\frac{\text{vol}(KA)}{\text{vol}(LC)} = \left(\frac{AB}{CD} \right)^3.$$

and from the similar solids ME and NG ,

$$\frac{\text{vol}(ME)}{\text{vol}(NG)} = \left(\frac{EF}{GH} \right)^3.$$

3 Since $AB : CD = EF : GH$,

$$\text{we have } \frac{AB}{CD} = \frac{EF}{GH}.$$

Therefore, their cubes are equal:

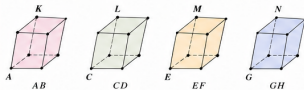
$$\left(\frac{AB}{CD} \right)^3 = \left(\frac{EF}{GH} \right)^3.$$

Hence

$$\frac{\text{vol}(KA)}{\text{vol}(LC)} = \frac{\text{vol}(ME)}{\text{vol}(NG)}.$$

Thus the similar parallelepipeds KA and LC , ME and NG are proportional.**II. CONVERSE:** If the similar parallelepipeds on four lines are proportional, then the lines are proportional.**1** Let similar parallelepipeds KA , LC , ME , NG (on the same solid angle) be proportional:

$$\frac{\text{vol}(KA)}{\text{vol}(LC)} = \frac{\text{vol}(ME)}{\text{vol}(NG)}.$$

The solids are similar, built on the lines AB , CD , EF , GH respectively.**2** By Proposition 33,

$$\frac{\text{vol}(KA)}{\text{vol}(LC)} = \left(\frac{AB}{CD} \right)^3 \quad \text{and}$$

$$\frac{\text{vol}(ME)}{\text{vol}(NG)} = \left(\frac{EF}{GH} \right)^3.$$

By the given proportion of the solids,

$$\left(\frac{AB}{CD} \right)^3 = \left(\frac{EF}{GH} \right)^3.$$

Thus the cubed ratios of the corresponding lines are equal.

3 Taking the cube roots (all magnitudes positive),

$$\frac{AB}{CD} = \frac{EF}{GH}.$$

Hence the four lines are proportional:

$$AB : CD = EF : GH.$$

★ **Summary.** If four lines are proportional, the similar parallelepipeds constructed on them (with the same solid angle) are proportional. Conversely, if the similar parallelepipeds on four lines are proportional, then the four lines are proportional.

2.38 Proposition 38

Statement

In a cube, if opposite faces are bisected and planes are drawn through the bisection points, the intersection of those planes and the cube's diameter bisect each other.

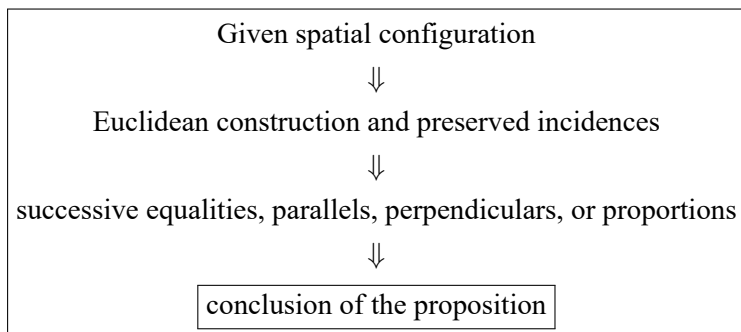
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

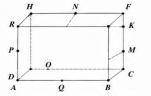
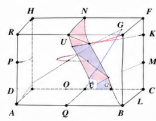
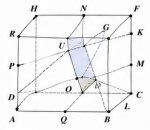
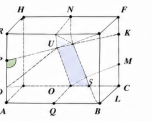
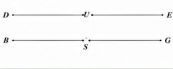
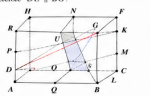
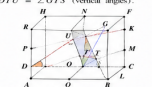
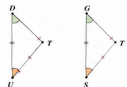
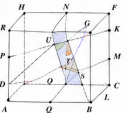
1. Let cube AF have the stated opposite faces bisected at K, L, M, N, O, P, Q, R .
2. Through the bisection points draw planes KN, OR ; let their common section be US and let the cube's diameter be DG .
3. Join DU, UE, BS, SG .
4. From the parallel edges obtain $\angle DOU = \angle UPE$.
5. With $DO = PE$ and $OU = UP$, triangles DOU, PUE are congruent; hence $DU = UE$.
6. Therefore D, U, E are collinear; similarly B, S, G are collinear and $BS = SG$.
7. From the cube's parallel edges obtain $DE \parallel BG$.
8. Let $T = DG \cap US$; use alternate and vertical angles in triangles DTU, GTS .
9. With the corresponding equal half-lines, the triangles are congruent, giving $DT = TG$ and $UT = TS$.
10. Thus T bisects both DG and US .

Complete Cinematic Logic



PROPOSITION 38

In a cube, if opposite faces are bisected and planes are drawn through the bisection points, the intersection of those planes and the cube's diameter bisect each other.

1 Let AF be a cube. Its opposite faces CF and AH are bisected at the points shown. K = midpoint of CF O = midpoint of CD L = midpoint of BF P = midpoint of AD M = midpoint of EF Q = midpoint of AB N = midpoint of AF R = midpoint of EH  Each of K, L, M, N, O, P, Q, R is the midpoint of the corresponding edge.	2 Through the bisection points, draw the planes KN and OR. Let US be their common section. Let DG be the diameter of the cube.  US is the intersection of the planes KN and OR. DG is the (space) diagonal of the cube.	3 Join DU, UE, BS, and SG.  These segments will be used in the comparisons that follow.	4 Since $DO \parallel PE$ and DU is a transversal, the alternate interior angles $\angle DOU$ and $\angle UPE$ are equal.  Hence $\angle DOU = \angle UPE$.
5 Also $DO = PE$ (halves of equal edges) and $OU = UP$ (halves of equal edges). With the included angle equal (step 4), $\triangle DOU \cong \triangle UPE$ (side-angle-side). Therefore $DU = UE$ and $\angle OUD = \angle PUE$.  Thus $DU = UE$ and $\angle OUD = \angle PUE$.	6 Hence the adjacent angles $\angle OUD$ and $\angle PUE$ are supplementary and equal; thus each is a right angle, so D, U, E are in a straight line (Prop. 1.14). Similarly, R, S, G are in a straight line, and $BS = SG$.  UE is collinear with UD, and SG is collinear with SB.	7 Since $CA \parallel DB$ and $CA \parallel EG$ (opposite edges of a cube), we get $DB \parallel EG$. The lines DE and BG join these parallels; therefore $DE \parallel BG$.  Thus $DE \parallel BG$.	8 From $DE \parallel BG$, the alternate interior angles are equal: $\angle EDT = \angle BGT$ (with transversal DG), and $\angle DTU = \angle GTS$ (vertical angles).  Hence $T = DG \cap US$.
9 Also $DU = GS$ (each is half of the equal line DE and BG respectively). In $\triangle DTU$ and $\triangle GTS$: $\angle EDT = \angle BGT$ (step 8), $\angle DTU = \angle GTS$ (step 8), $DU = GS$ (above). Therefore $\triangle DTU \cong \triangle GTS$ (A-A-S). Hence $DT = TG$ and $UT = TS$.  Thus $DT = TG$ and $UT = TS$.	10 Therefore, the point T bisects both the diameter and the common section. That is, $DT = TG$ (so T is the midpoint of DG), $UT = TS$ (so T is the midpoint of US). The intersection of the planes drawn through the midpoints of opposite faces and the cube's diameter bisect each other.	 ★ Summary. In a cube, cut each pair of opposite faces by planes through the midpoints of their edges. The common section of these planes and the cube's space diagonal bisect each other. (The intersection point is the center of the cube.)	

2.39 Proposition 39

Statement

Two prisms of equal height, one with a parallelogram base and the other with a triangular base, are equal if the parallelogram is double the triangle.

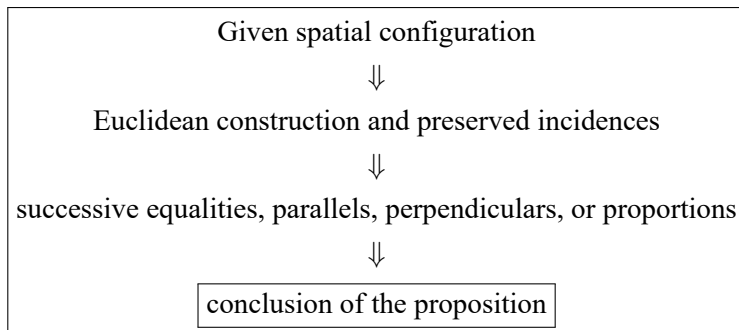
Given / Construction

Use the labelled configuration and construction specified in the proof actions below; all previously constructed points, lines, planes, incidences, and relative positions persist unchanged.

Proof Actions

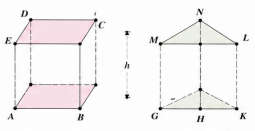
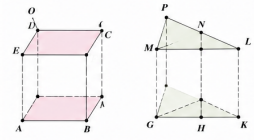
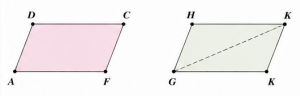
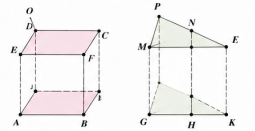
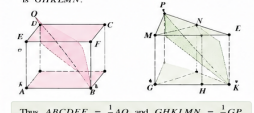
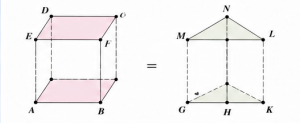
1. Let prism $ABCDEF$ have parallelogram base AF , prism $GHKLMN$ triangular base GHK , equal heights, and $\text{area}(AF) = 2 \text{ area}(GHK)$.
2. Complete parallelepipeds AO, GP on the corresponding bases and heights.
3. Complete the parallelogram on GHK ; its diagonal divides it into two equal triangles, so its area is twice GHK and hence equals AF .
4. By Proposition 31, parallelepipeds AO and GP are equal.
5. By Proposition 28, each given prism is half of its corresponding parallelepiped.
6. Halves of equal solids are equal.
7. Therefore the two prisms are equal.

Complete Cinematic Logic



PROPOSITION 39

Two prisms of equal height, one with a parallelogram base and the other with a triangular base, are equal if the parallelogram is double the triangle.

1 Let the prism $ABCDEF$ have the parallelogram base AF and the prism $GHLMN$ have the triangular base GHK. Let their heights be equal. Suppose also that $\text{area}(AF) = 2 \cdot \text{area}(GHK).$  Parallelogram base AF Triangular base GHK Given: equal heights and $\text{area}(AF) = 2 \cdot \text{area}(GHK)$.	2 Complete the parallelepiped AO on the same base AF and the same height h as the first prism. Complete the parallelepiped GP on the same base GHK and the same height h as the second prism.  Parallelepipeds AO and GP have the same heights as the prisms.	3 In the parallelogram HK, the diagonal HK divides it into two equal triangles. Hence $\text{area}(HK) = 2 \cdot \text{area}(GHK)$. Since $\text{area}(AF) = 2 \cdot \text{area}(GHK)$, we have $\text{area}(AF) = \text{area}(HK).$  Thus the bases AF and HK are equal in area.
4 The parallelepipeds AO and GP have equal bases ($AF = HK$) and equal heights. Therefore, by Proposition 31, $AO = GP.$  Hence $AO = GP$.	5 Each prism is half of its corresponding parallelepiped. For $ABCDEF$, the diagonal plane DBE cuts the parallelepiped AO into two equal prisms; one of them is $ABCDEF$. For $GHLMN$, the diagonal plane HPL cuts the parallelepiped GP into two equal prisms; one of them is $GHLMN$.  Thus $ABCDEF = \frac{1}{2}AO$ and $GHLMN = \frac{1}{2}GP$ (Proposition 28).	6 Since $AO = GP$ (step 4), taking half of equal solids gives equal solids. Therefore, $ABCDEF = GHLMN.$  Hence the two prisms are equal.

★ **Summary.** If two prisms of equal height—one with a parallelogram base and the other with a triangular base—have the parallelogram double the triangle, then the prisms are equal.