

EUCLID'S ELEMENTS

BOOK XV

The Mutual Inscription and Comparison of the
Regular Solids

Illustrated edition

Contents

BOOK XV

Book XV

Proposition 1 – To inscribe a pyramid in a given cube

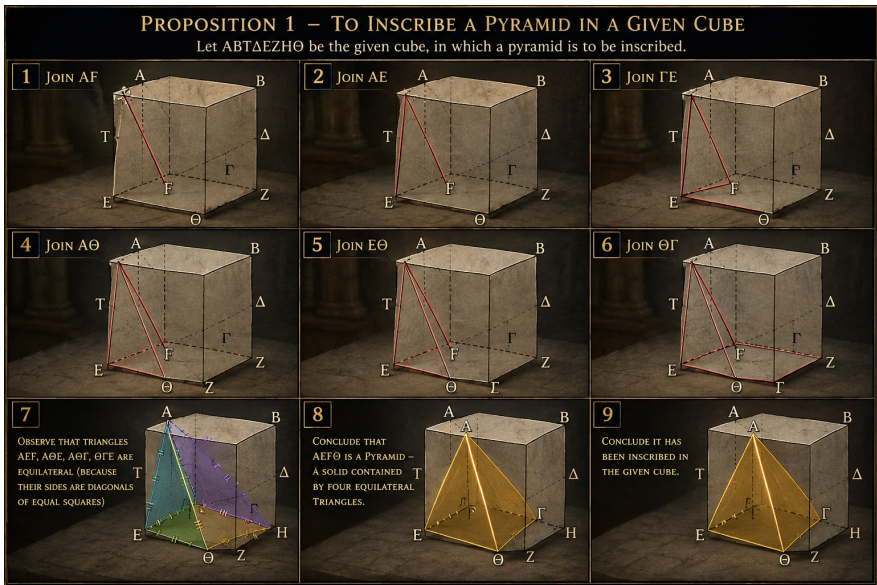
Statement

Let $AB\Gamma\Delta EZH\Theta$ be the given cube, in which a pyramid is to be inscribed.

Proof Actions

1. Join AF .
2. Join AE .
3. Join ΓE .
4. Join $A\Theta$.
5. Join $E\Theta$.
6. Join $\Theta\Gamma$.
7. Observe that triangles AEF , $A\Theta E$, $A\Theta\Gamma$, $\Theta\Gamma E$ are equilateral, because their sides are diagonals of equal squares.

8. Conclude that $AEF\Theta$ is a pyramid, a solid contained by four equilateral triangles.
9. Conclude that it has been inscribed in the given cube.



Proposition 1 – To inscribe a pyramid in a given cube

Proposition 2 – To inscribe an octahedron in a given pyramid

Statement

Let $ABT\Delta$ be the given pyramid, with vertex Δ , in which an octahedron is to be inscribed.

Proof Actions

1. Bisect AB at point E .
2. Bisect AF at point Z .
3. Bisect $A\Delta$ at point H .
4. Bisect $B\Delta$ at point Θ .
5. Bisect $B\Gamma$ at point K .
6. Bisect $\Gamma\Delta$ at point Λ .
7. Join ΘK .
8. Join $\Theta\Lambda$.
9. Join EZ .
10. Join ZH .
11. Join the remaining connecting lines.
12. Observe that $\Theta K = HZ$.
13. Observe that ΘK is both equal and parallel to HZ .
14. Observe that ΘH is both equal and parallel to ZK .
15. Conclude that ΘKZH is equilateral.
16. State that it is also right-angled.

17. Draw perpendiculars from the prescribed point to the planes $EZBH$, $ZTEH$, and $EZ\Theta K$.
18. Demonstrate that the triangles erected on square ΘKZH are equilateral.

PROPOSITION 2 – TO INSCRIBE AN OCTAHEDRON IN A GIVEN PYRAMID
Let $ABT\Delta$ be the given pyramid, with vertex Δ , in which an octahedron is to be inscribed.

1 BISECT AB AT POINT E	2 BISECT AF AT POINT Z	3 BISECT AD AT POINT H	4 BISECT BD AT POINT O	5 BISECT BT AT POINT K
6 BISECT $\Gamma\Delta$ AT POINT Λ	7 JOIN OK	8 JOIN $\Theta\Lambda$	9 JOIN EZ	10 JOIN ZH
11 JOIN THE REST OF THE CONNECTING LINES	12 OBSERVE THAT $OK = HZ$ (SINCE $AB = 2OK = 2HZ$)	13 OBSERVE THAT OK IS BOTH EQUAL AND PARALLEL TO HZ	14 OBSERVE THAT OH IS BOTH EQUAL AND PARALLEL TO ZK	15 CONCLUDE THAT $OKZH$ IS EQUILATERAL
16 STATE THAT IT IS ALSO RIGHT-ANGLED (PROOF INCOMPLETE IN MANUSCRIPT)	17 DRAW PERPENDICULARS FROM KA TO THE PLANES $EZBH$, $ZTEH$, $EZOK$	18 DEMONSTRATE THAT TRIANGLES ENJOINED ON SQUARE $OKZH$ ARE EQUILATERAL		

Proposition 2 – To inscribe an octahedron in a given pyramid

Proposition 3 – To inscribe an octahedron in a given cube

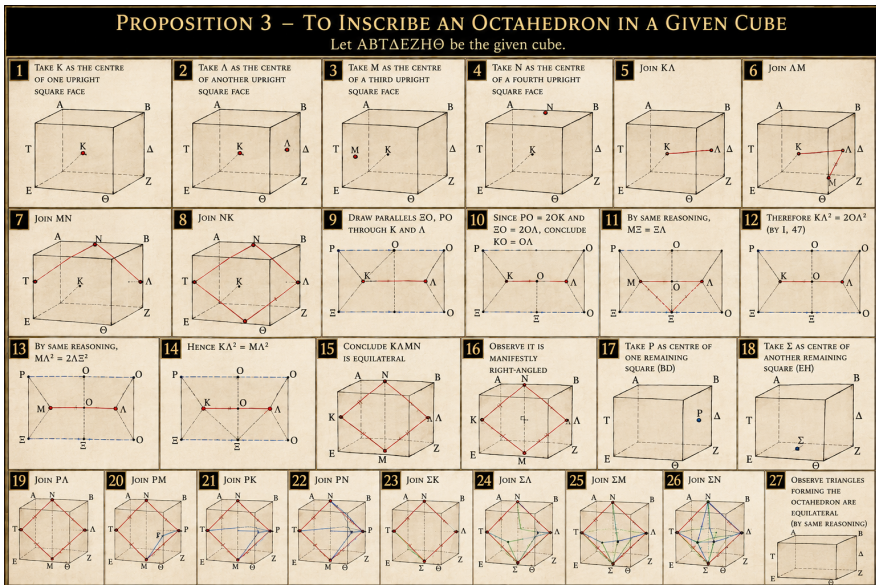
Statement

Let $ABT\Delta EZH\Theta$ be the given cube.

Proof Actions

1. Take K as the centre of one upright square face.
2. Take Λ as the centre of another upright square face.
3. Take M as the centre of a third upright square face.
4. Take N as the centre of a fourth upright square face.
5. Join $K\Lambda$.
6. Join ΛM .
7. Join MN .
8. Join NK .
9. Draw the prescribed parallels through K and Λ .
10. From the doubled segments, conclude that the relevant half-segments are equal.
11. By the same reasoning, obtain the corresponding equality on the opposite side.
12. Therefore $K\Lambda^2$ is twice the square on the corresponding half-segment, by I.47.
13. By the same reasoning, $M\Lambda^2$ is twice the square on its corresponding half-segment.
14. Hence $K\Lambda^2 = M\Lambda^2$.
15. Conclude that $K\Lambda MN$ is equilateral.

16. Observe that it is manifestly right-angled.
17. Take P as the centre of one remaining square face.
18. Take Σ as the centre of the other remaining square face.
19. Join $P\Lambda$.
20. Join PM .
21. Join PK .
22. Join PN .
23. Join ΣK .
24. Join $\Sigma\Lambda$.
25. Join ΣM .
26. Join ΣN .
27. Observe that the triangles forming the octahedron are equilateral by the same reasoning.



Proposition 3 – To inscribe an octahedron in a given cube

Proposition 4 – To inscribe a cube in a given octahedron

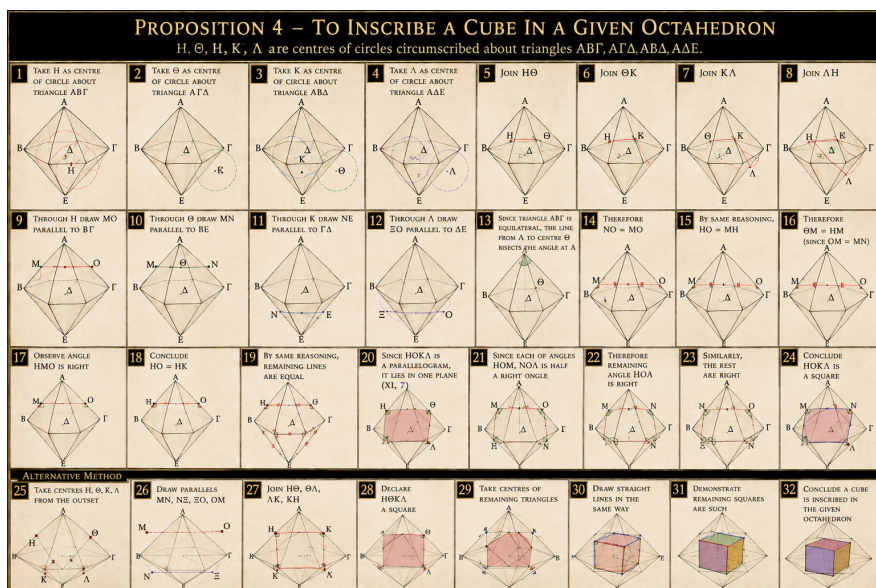
Statement

Let H, Θ, K, Λ be centres of circles circumscribed about the appropriate triangular faces of the octahedron.

Proof Actions

1. Take H as centre of the circle about triangle $AB\Gamma$.
2. Take Θ as centre of the circle about triangle $A\Gamma\Delta$.
3. Take K as centre of the circle about triangle $AB\Delta$.
4. Take Λ as centre of the circle about triangle $A\Delta E$.
5. Join $H\Theta$.
6. Join ΘK .
7. Join $K\Lambda$.
8. Join ΛH .
9. Through H draw MO parallel to $B\Gamma$.
10. Through Θ draw MN parallel to the corresponding side.
11. Through K draw the prescribed parallel.
12. Through Λ draw the prescribed parallel.
13. Since triangle $AB\Gamma$ is equilateral, the line from A to its centre bisects the angle at A .
14. Therefore the corresponding segments are equal.
15. By the same reasoning obtain the next equality.
16. Hence the next pair of segments are equal.

17. Observe that the relevant angle is right.
18. Conclude the corresponding sides are equal.
19. By the same reasoning, the remaining lines are equal.
20. Since the resulting quadrilateral is a parallelogram, it lies in one plane, by XI.7.
21. Each of the indicated angles is half a right angle.
22. Therefore the remaining angle is right.
23. Similarly the rest are right.
24. Conclude that the quadrilateral is a square.
25. Alternatively, take the centres from the outset.
26. Draw the required parallels.
27. Join the centres cyclically.
28. Declare the resulting quadrilateral a square.
29. Take the centres of the remaining triangles.
30. Draw straight lines in the same way.
31. Demonstrate that the remaining quadrilaterals are squares.
32. Conclude that a cube is inscribed in the given octahedron.



Proposition 4 – To inscribe a cube in a given octahedron

Proposition 5 – To inscribe a dodecahedron in a given icosahedron

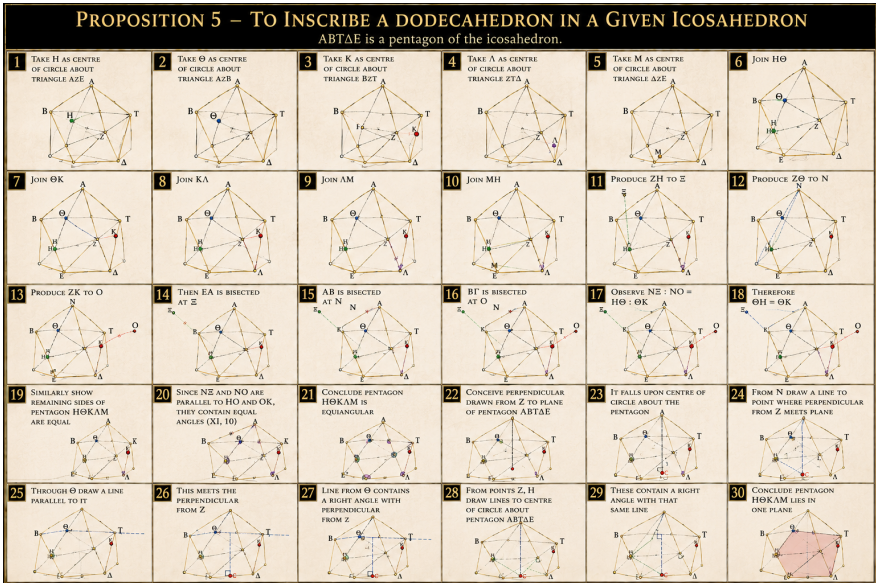
Statement

Let $ABT\Delta E$ be a pentagon of the icosahedron.

Proof Actions

1. Take H as centre of the circle about triangle AZE .
2. Take Θ as centre of the circle about triangle AZB .
3. Take K as centre of the circle about triangle BZT .
4. Take Λ as centre of the circle about triangle $ZT\Delta$.
5. Take M as centre of the circle about triangle ΔZE .
6. Join $H\Theta$.
7. Join ΘK .
8. Join $K\Lambda$.
9. Join ΛM .
10. Join MH .
11. Produce ZH to Ξ .
12. Produce $Z\Theta$ to N .
13. Produce ZK to O .
14. Then EA is bisected at Ξ .
15. AB is bisected at N .
16. $B\Gamma$ is bisected at O .
17. Observe the stated proportion between the corresponding segments.

18. Therefore $\Theta H = \Theta K$.
19. Similarly show that the remaining sides of pentagon $H\Theta K\Lambda M$ are equal.
20. Since the constructed lines are parallel to the corresponding sides, they contain equal angles, by XI.10.
21. Conclude that the pentagon is equiangular.
22. Conceive a perpendicular drawn from Z to the plane of pentagon $ABT\Delta E$.
23. It falls upon the centre of the circle about the pentagon.
24. From N draw a line to the point where the perpendicular from Z meets the plane.
25. Through Θ draw a line parallel to it.
26. This meets the perpendicular from Z .
27. The line from Θ contains a right angle with the perpendicular from Z .
28. From points Z, H draw lines to the centre of the circle about pentagon $ABT\Delta E$.
29. These contain a right angle with that same line.
30. Conclude that pentagon $H\Theta K\Lambda M$ lies in one plane.



Proposition 5 – To inscribe a dodecahedron in a given icosahedron

On the Number of Edges and Vertices

Edges of the icosahedron

1. The icosahedron is contained by 20 triangles.
2. Each triangle has 3 sides.
3. Thus $20 \times 3 = 60$ incidences of face-sides.
4. Since every edge is shared by two faces, $60/2 = 30$ edges.

Edges of the dodecahedron

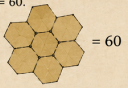
1. The dodecahedron is contained by 12 pentagons.
2. Each pentagon has 5 sides.
3. Thus $12 \times 5 = 60$ incidences of face-sides.
4. Since every edge is shared by two faces, $60/2 = 30$ edges.

Vertices of the icosahedron

1. Five triangles meet at each solid angle.
2. The 20 triangular faces contribute $20 \times 3 = 60$ face-vertices.
3. Therefore $60/5 = 12$ vertices.

Vertices of the dodecahedron

1. Three pentagons meet at each solid angle.
2. The 12 pentagonal faces contribute $12 \times 5 = 60$ face-vertices.
3. Therefore $60/3 = 20$ vertices.

ON THE NUMBER OF EDGES AND VERTICES			
EDGES OF THE ICOSAEDRON			
1 Note icosahedron is contained by 20 triangles. 	2 Each triangle has 3 sides. 	3 Multiply $20 \times 3 = 60$.  = 60	4 Take half = 30 edges.  30 edges
 REASON FOR TAKING HALF: Each single edge is counted twice (shared by two faces).			
EDGES OF THE DODECAHEDRON			
1 Note dodecahedron is contained by 12 pentagons. 	2 Each pentagon has 5 sides. 	3 Multiply $12 \times 5 = 60$.  = 60	4 Take half = 30 edges.  30 edges
 REASON FOR TAKING HALF: Each single edge is counted twice (shared by two faces).			
VERTICES OF THE ICOSAEDRON			
1 Note 5 triangles contain each solid angle. 	2 Take number of triangular faces (20) $\times$ 3 (sides per triangle) = 60.  = 60	3 Divide by 5 = 12 vertices.  12 vertices	
 Since 5 triangles meet at each vertex, each vertex is counted 5 times in the total 60. Therefore, $60 \div 5 = 12$ vertices.			
VERTICES OF THE DODECAHEDRON			
1 Note 3 pentagons contain each solid angle. 	2 Take number of pentagonal faces (12) $\times$ 5 (sides per pentagon) = 60.  = 60	3 Divide by 3 = 20 vertices.  20 vertices	
 Since 3 pentagons meet at each vertex, each vertex is counted 3 times in the total 60. Therefore, $60 \div 3 = 20$ vertices.			

On the number of edges and vertices

On the Inclinations of the Planes (after Isidore)

Cube

The planes cut one another at a right angle.

Pyramid

1. Set out one triangle.
2. Take the ends of one side as centres.
3. Use the perpendicular drawn from the vertex to the base as radius.
4. Describe arcs.
5. Let the arcs cut one another.
6. Draw straight lines from the intersection to the centres.
7. These contain the inclination of the planes.

Octahedron

1. Describe a square on the side of the triangle.
2. Take the ends of the diagonal as centres.
3. Use the perpendicular of the triangle as radius.
4. Describe arcs.
5. Draw straight lines from the common section to the centres.
6. These contain the angle by which the inclination falls short of two right angles.

Icosahedron

1. Describe a pentagon on the side of the triangle.
2. Join the straight line subtending two sides.
3. Take the ends of this line as centres.
4. Use the perpendicular of the triangle as radius.
5. Describe arcs.
6. Draw straight lines from the common section to the centres.
7. These contain the angle by which the inclination falls short of two right angles.

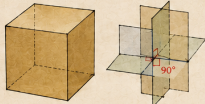
Dodecahedron

1. Set out one pentagon.
2. Join the straight line subtending two sides.
3. Take the ends of this line as centres.
4. Use the perpendicular drawn from the point of bisection to the side of the pentagon parallel to it as radius.
5. Describe arcs.
6. Draw straight lines from the intersection to the centres.
7. These contain the angle by which the inclination falls short of two right angles.

ON THE INCLINATIONS OF THE PLANES (AFTER ISIDORE)

CUBE

• Observe planes cut one another at a right angle



1 Set out one triangle



2 Take ends of one side as centres



3 Use perpendicular drawn from vertex to base as radius



4 Describe arcs



5 Let arcs cut one another



6 Draw straight lines from intersection to centres



7 These contain the inclination of the planes



OCTAHEDRON

1 Describe a square on the side of the triangle



2 Take ends of the diagonal as centres



3 Use perpendicular of the triangle as radius



4 Describe arcs



5 Draw straight lines from common section to centres



6 These contain the angle by which inclination falls short of two right angles



ICOSAHEDRON

1 Describe a pentagon on the side of the triangle



2 Join the straight line subtending two sides



3 Take ends of this line as centres



4 Use perpendicular of the triangle as radius



5 Describe arcs



6 Draw straight lines from common section to centres



7 These contain the angle by which inclination falls short of two right angles



DODECAHEDRON

1 Set out one pentagon



2 Join the straight line subtending two sides



3 Take ends of this line as centres



4 Use perpendicular drawn from point of bisection to the side of the pentagon parallel to it as radius



5 Describe arcs



6 Draw straight lines from intersection to centres



7 These contain the angle by which inclination falls short of two right angles



Right angle (one right angle) = 90°

Angle contained in the inclination of two planes

 (pyramid) $\alpha < 90^\circ$

Angle by which inclination falls short of two right angles

 $\beta, \gamma, \delta < 180^\circ$

In all cases the drawn angle is the dihedral angle between the two planes forming the solid angle.

On the inclinations of the planes (after Isidore)

20

The Pyramid – Detailed Demonstration

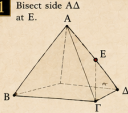
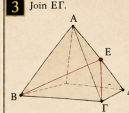
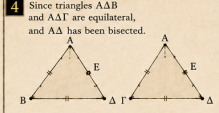
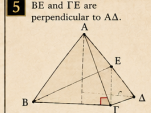
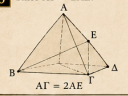
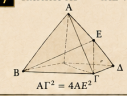
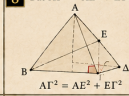
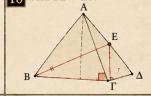
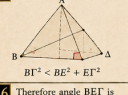
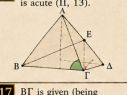
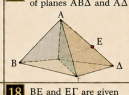
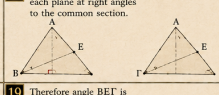
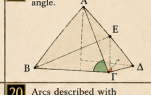
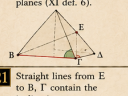
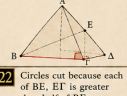
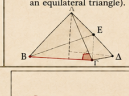
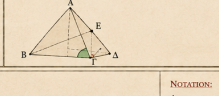
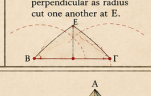
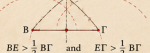
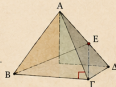
Statement

Given pyramid $AB\Gamma\Delta$, contained by four equilateral triangles, with $AB\Gamma$ as base and Δ as vertex.

Proof Actions

1. Bisect the relevant side at E .
2. Join BE .
3. Join $E\Gamma$.
4. Since the adjacent triangles are equilateral and the common side has been bisected, the resulting lines are perpendicular to that side.
5. Thus BE and $E\Gamma$ are perpendicular to the common section.
6. Since $A\Gamma = 2AE$.
7. Therefore $A\Gamma^2 = 4AE^2$.
8. But $A\Gamma^2 = AE^2 + E\Gamma^2$.
9. Therefore $A\Gamma^2 : E\Gamma^2 = 4 : 3$.
10. And $BE = E\Gamma$.
11. Therefore $B\Gamma^2 < BE^2 + E\Gamma^2$.
12. Hence angle BEG is acute, by II.13.
13. The common side is the common section of the two planes.
14. BE and $E\Gamma$ are drawn in the respective planes at right angles to the common section.
15. They contain an acute angle.
16. Therefore angle BEG is the inclination of the planes, by XI def. 6.

17. $B\Gamma$ is given.
18. BE and $E\Gamma$ are given.
19. Therefore angle $B E \Gamma$ is given, by Data 38.
20. Arcs described with centres B, Γ and with the perpendicular as radius cut one another at E .
21. Straight lines from E to B, Γ contain the inclination.
22. The circles cut because each of $BE, E\Gamma$ is greater than half of $B\Gamma$.

THE PYRAMID – DETAILED DEMONSTRATION Given: Pyramid $AB\Gamma\Delta$ contained by four equilateral triangles, $AB\Gamma$ as base, Δ as vertex.					
1 Bisect side $A\Delta$ at E. 	2 Join BE. 	3 Join $E\Gamma$. 	4 Since triangles $A\Delta B$ and $A\Delta\Gamma$ are equilateral, and $A\Delta$ has been bisected. 	5 BE and $E\Gamma$ are perpendicular to $A\Delta$. 	
6 Since $A\Gamma = 2AE$.  $A\Gamma = 2AE$	7 Therefore $A\Gamma^2 = 4AE^2$.  $A\Gamma^2 = 4AE^2$	8 But $A\Gamma^2 = AE^2 + E\Gamma^2$.  $A\Gamma^2 = AE^2 + E\Gamma^2$	9 Therefore $A\Gamma^2 : E\Gamma^2 = 4 : 3$. $\frac{A\Gamma^2}{E\Gamma^2} = \frac{4}{3}$	10 And $BE = E\Gamma$. 	
11 Therefore $BE^2 < BE^2 + E\Gamma^2$.  $BE^2 < BE^2 + E\Gamma^2$	12 Hence angle BEG is acute (II. 13). 	13 Since $A\Delta$ is common section of planes $AB\Delta$ and $A\Delta\Gamma$. 	14 BE and $E\Gamma$ are drawn in each plane at right angles to the common section. 	15 They contain an acute angle. 	
16 Therefore angle BEG is the inclination of the planes (XI def. 6). 	17 $B\Gamma$ is given (being a side of the triangle). 	18 BE and $E\Gamma$ are given (being perpendiculars of an equilateral triangle). 	19 Therefore angle BEG is given (Data 38). 	20 Arcs described with centres B, Γ and with perpendicular as radius cut one another at E. 	
21 Straight lines from E to B, Γ contain the inclination A. 	22 Circles cut because each of BE, $E\Gamma$ is greater than half of $B\Gamma$. $BE > \frac{1}{2} B\Gamma$ and $E\Gamma > \frac{1}{2} B\Gamma$ 	SUMMARY: The angle BEG, constructed as above, is the inclination of the planes $AB\Delta$ and $A\Delta\Gamma$ of the pyramid.			
NOTATION: A = vertex B, Γ, Δ = base vertices E = midpoint of $A\Delta$ All faces are equilateral triangles.					

The pyramid – detailed demonstration

The Octahedron – Detailed Demonstration

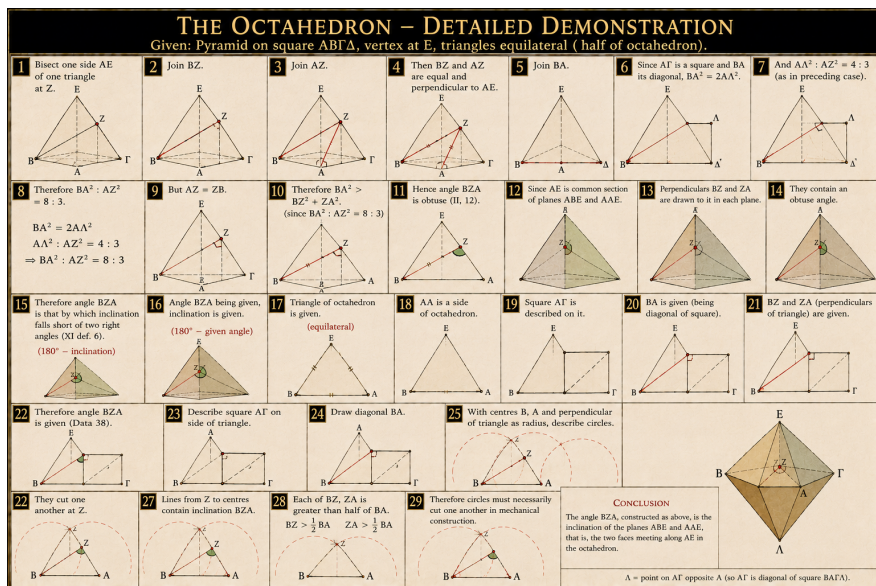
Statement

Given a pyramid on square $AB\Gamma\Delta$, with vertex E , whose triangular faces are equilateral; this is half of an octahedron.

Proof Actions

1. Bisect one side AE of one triangle at Z .
2. Join BZ .
3. Join AZ .
4. Then BZ and AZ are equal and perpendicular to AE .
5. Join BA .
6. Since the prescribed figure is a square and BA its diagonal, BA^2 is twice the square on a side.
7. The square on the side and AZ^2 are in the ratio 4 : 3, as in the preceding case.
8. Therefore $BA^2 : AZ^2 = 8 : 3$.
9. But $AZ = ZB$.
10. Therefore $BA^2 > BZ^2 + ZA^2$.
11. Hence angle BZA is obtuse, by II.12.
12. Since AE is the common section of the two planes.
13. Perpendiculars BZ and ZA are drawn to it in each plane.
14. They contain an obtuse angle.
15. Therefore angle BZA is that by which the inclination falls short of two right angles, by XI def. 6.
16. Angle BZA being given, the inclination is given.

17. The triangle of the octahedron is given.
18. Its side is given.
19. Describe a square on it.
20. The diagonal of the square is given.
21. BZ and ZA are given.
22. Therefore angle BZA is given, by Data 38.
23. Describe the square on the side of the triangle.
24. Draw its diagonal.
25. With the ends of the diagonal as centres and the perpendicular of the triangle as radius, describe circles.
26. They cut one another at Z .
27. Lines from Z to the centres contain angle BZA .
28. Each radius is greater than half of the diagonal.
29. Therefore the circles necessarily cut one another in the mechanical construction.



The octahedron – detailed demonstration

The Icosahedron – Detailed Demonstration

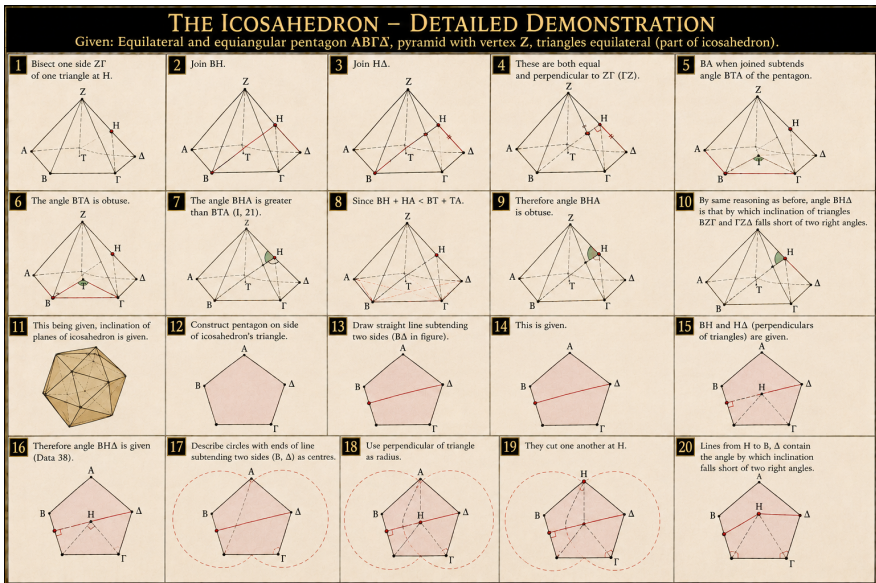
Statement

Given an equilateral and equiangular pentagon $AB\Gamma\Delta$, with a pyramid having vertex Z and equilateral triangular faces, forming part of an icosahedron.

Proof Actions

1. Bisect one side $Z\Gamma$ of one triangle at H .
2. Join BH .
3. Join $H\Delta$.
4. These are equal and perpendicular to ΓZ .
5. Join the straight line subtending the relevant angle of the pentagon.
6. The corresponding pentagonal angle is obtuse.
7. The constructed angle at H is greater than that pentagonal angle, by I.21.
8. Since the sum of the two constructed sides is less than the comparison sum.
9. Therefore the angle at H is obtuse.
10. By the same reasoning as before, angle $BH\Delta$ is that by which the inclination of the two triangular planes falls short of two right angles.
11. This being given, the inclination of the planes of the icosahedron is given.
12. Construct a pentagon on the side of the icosahedron's triangle.
13. Draw the straight line subtending two sides.
14. This line is given.

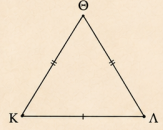
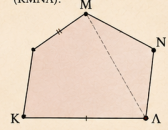
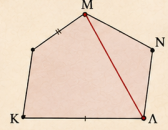
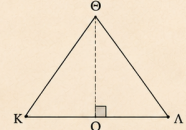
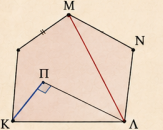
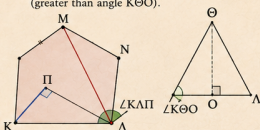
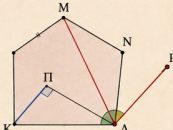
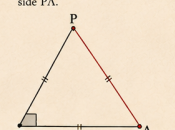
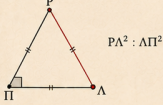
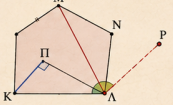
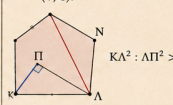
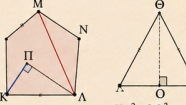
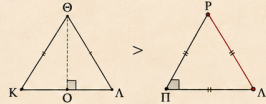
15. BH and $H\Delta$, being perpendiculars of equilateral triangles, are given.
16. Therefore angle $BH\Delta$ is given, by Data 38.
17. Describe circles with the ends of the line subtending two sides as centres.
18. Use the perpendicular of the triangle as radius.
19. They cut one another at H .
20. Lines from H to the two centres contain the angle by which the inclination falls short of two right angles.



The icosahedron – detailed demonstration

Supplementary Demonstration

1. Conceive separately equilateral triangle $\Theta K\Lambda$.
2. On $K\Lambda$ construct pentagon $KMN\Lambda$.
3. Join $M\Lambda$.
4. Draw ΘO , the perpendicular of triangle $\Theta K\Lambda$.
5. Draw perpendicular $K\Pi$ from K to $M\Lambda$.
6. Since angle $K\Lambda\Pi$ is greater than a third of a right angle, and greater than angle $K\Theta O$.
7. Construct angle $\Pi\Lambda P$ equal to $K\Theta O$.
8. Then $\Pi\Lambda$ is the perpendicular of an equilateral triangle with side $P\Lambda$.
9. Hence $P\Lambda^2 : \Lambda\Pi^2 = 4 : 3$.
10. But $K\Lambda > \Lambda P$.
11. Therefore $K\Lambda^2 : \Lambda\Pi^2 > 4 : 3$, by V.8.
12. And $K\Lambda^2 : \Theta O^2 = 4 : 3$.
13. Therefore $K\Lambda : \Lambda\Pi > K\Lambda : \Theta O$.
14. Hence $\Theta O > \Lambda\Pi$, by V.10.

SUPPLEMENTARY DEMONSTRATION			
1 Conceive separately equilateral triangle ΘKA. 	2 On KA construct pentagon $KMN\Lambda$ ($KMNA$). 	3 Join MA. 	4 Draw ΘO, the perpendicular of triangle ΘKA. 
5 Draw perpendicular $K\Pi$ from K to MA. 	6 Since angle $K\Lambda\Pi$ is greater than a third of a right angle (greater than angle $K\Theta O$). 	7 Construct angle $\Pi\Lambda P$ equal to $K\Theta O$. 	8 Then $\Pi\Lambda$ is perpendicular of equilateral triangle with side PA. 
9 Hence $PA^2 : \Lambda\Pi^2 = 4 : 3$.  $PA^2 : \Lambda\Pi^2 = 4 : 3$	10 But $K\Lambda > \Lambda P$. 	11 Therefore $K\Lambda^2 : \Lambda\Pi^2 > 4 : 3$ (V, 8).  $K\Lambda^2 : \Lambda\Pi^2 > 4 : 3$	12 And $K\Lambda^2 : \Theta O^2 = 4 : 3$.  $K\Lambda^2 : \Theta O^2 = 4 : 3$
13 Therefore $K\Lambda : \Lambda\Pi > K\Lambda : \Theta O$. $\frac{K\Lambda}{\Lambda\Pi} > \frac{K\Lambda}{\Theta O}$	14 Hence $\Theta O > \Lambda\Pi$ (V, 10). 		

Supplementary demonstration

The Dodecahedron – Detailed Demonstration

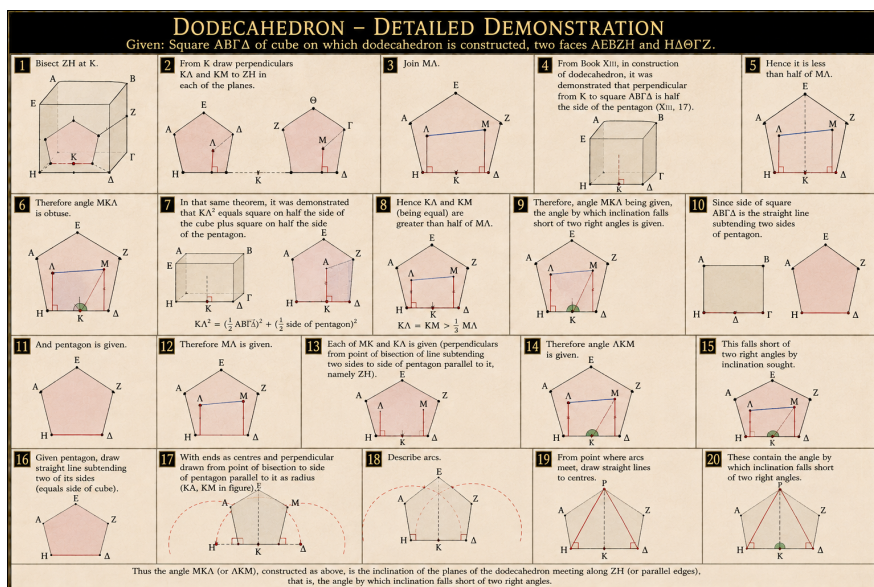
Statement

Given square $AB\Gamma\Delta$ of the cube on which the dodecahedron is constructed, with two adjacent pentagonal faces.

Proof Actions

1. Bisect ZH at K .
2. From K draw perpendiculars $K\Lambda$ and KM to ZH in the two planes.
3. Join $M\Lambda$.
4. From XIII.17, the perpendicular from K to square $AB\Gamma\Delta$ is half the side of the pentagon.
5. Hence it is less than half of $M\Lambda$.
6. Therefore angle $MK\Lambda$ is obtuse.
7. In the same theorem, it was demonstrated that $K\Lambda^2$ equals the square on half the side of the cube plus the square on half the side of the pentagon.
8. Hence $K\Lambda$ and KM , being equal, are greater than half of $M\Lambda$.
9. Therefore, angle $MK\Lambda$ being given, the angle by which the inclination falls short of two right angles is given.
10. Since the side of square $AB\Gamma\Delta$ is the straight line subtending two sides of the pentagon.
11. And the pentagon is given.
12. Therefore $M\Lambda$ is given.
13. Each of MK and $K\Lambda$ is given.
14. Therefore angle ΛKM is given.

15. This falls short of two right angles by the inclination sought.
16. Given the pentagon, draw the straight line subtending two of its sides; this equals the side of the cube.
17. With its ends as centres and with the prescribed perpendicular as radius, prepare the construction.
18. Describe arcs.
19. From the point where the arcs meet, draw straight lines to the centres.
20. These contain the angle by which the inclination falls short of two right angles.



The dodecahedron – detailed demonstration

Conclusion

Book XV has established constructions relating the regular solids to one another, together with supplementary demonstrations concerning their edges, vertices, and the inclinations of their planes.