Euclid's Elements
Book X
Incommensurable Magnitudes

Contents

Chapter 1

Definitions

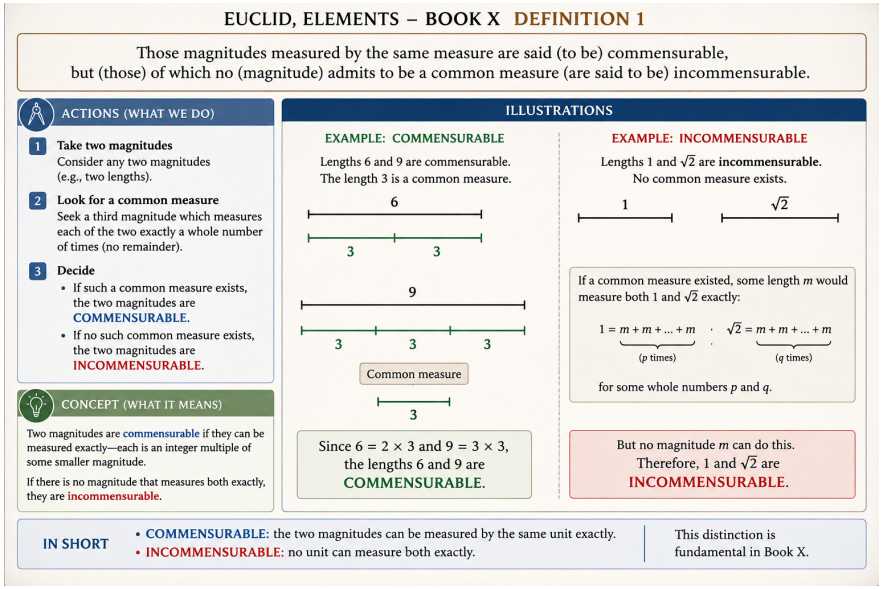
Definitions I

1.0.1 Definition 1

Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.

Symbolic Form

common measure exists $\iff$ commensurable	no common measure $\iff$ inc

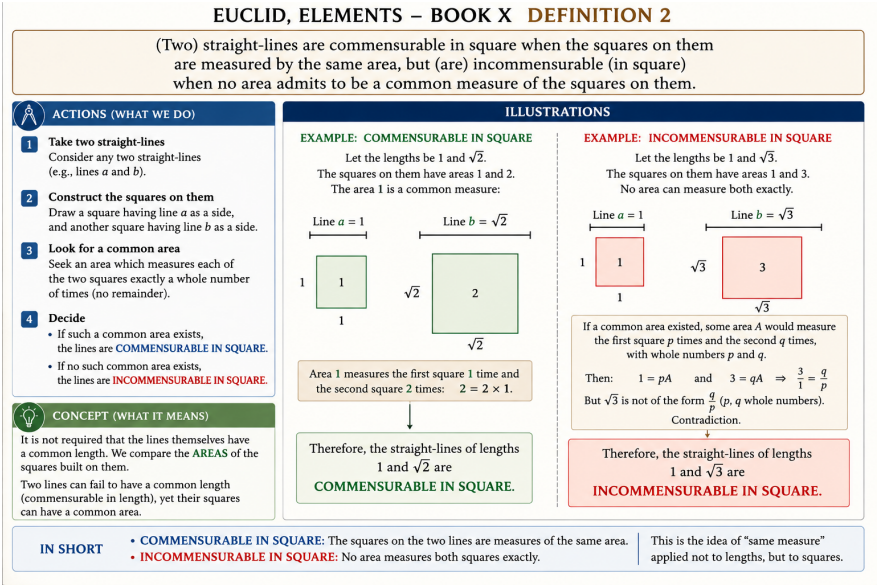


1.0.2 Definition 2

(Two) straight-lines are commensurable in square when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.

Symbolic Form

$$a^2 \sim b^2 \iff a, b \text{ are commensurable in square}$$

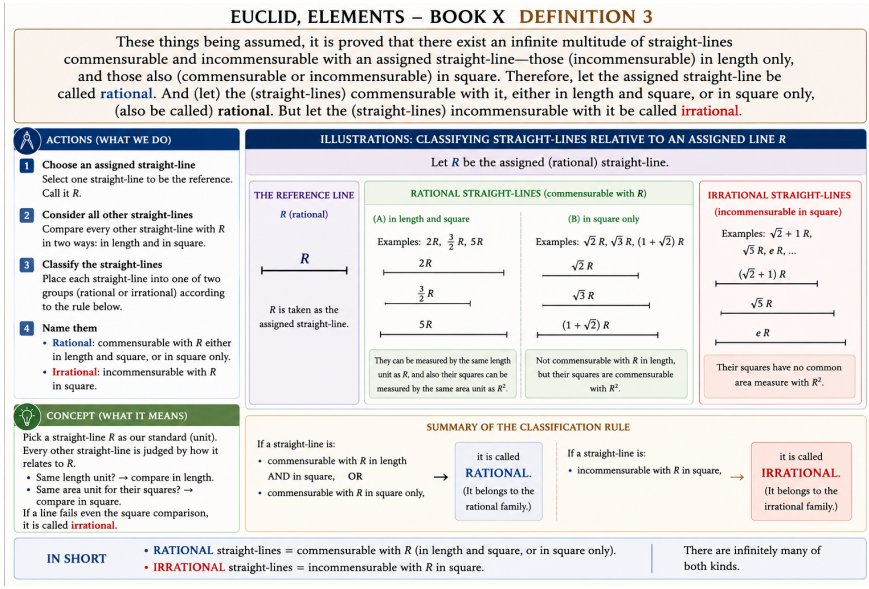


1.0.3 Definition 3

These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straightline—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square. Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.

Symbolic Form

assigned line r rational	$x \sim r$ in length or in square $\Rightarrow$ x rational	$x \nsim r$ =



1.0.4 Definition 4

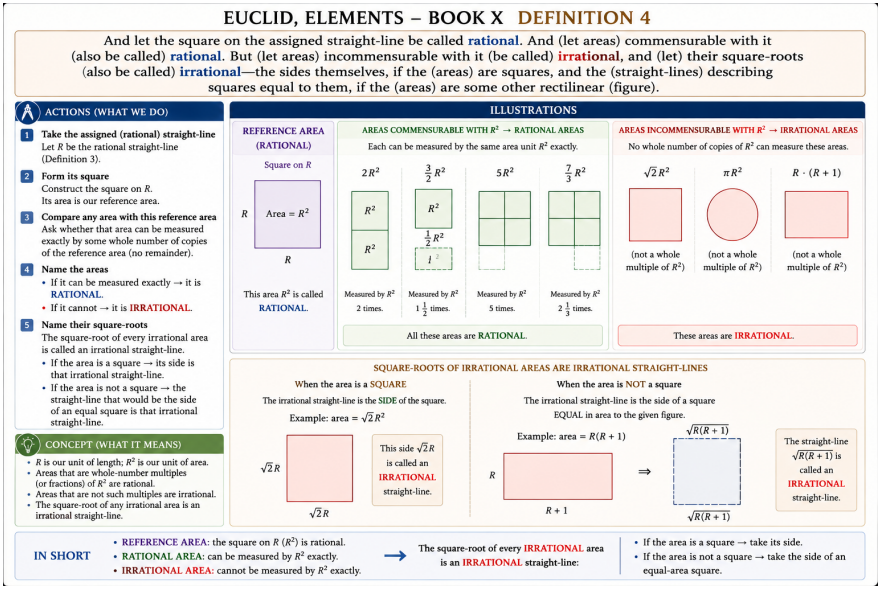
And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their squareroots (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).

Symbolic Form

r^2 rational area

$A \sim r^2 \Rightarrow A$ rational

$A \nsim r^2 \Rightarrow \sqrt{A}$ irrational



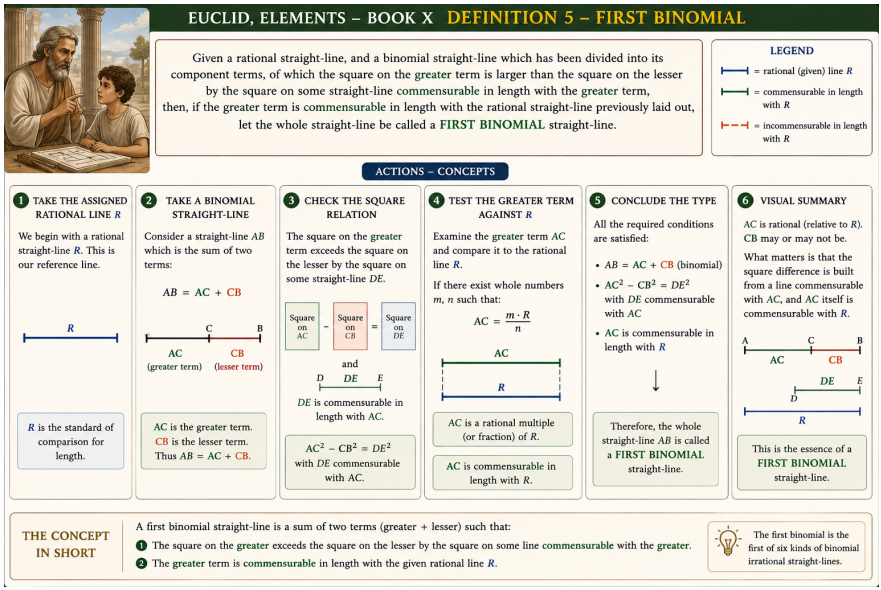
Definitions II

1.0.5 Definition 5

Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \sim u,\ u \sim r \Rightarrow x \text{ first binomial}$$

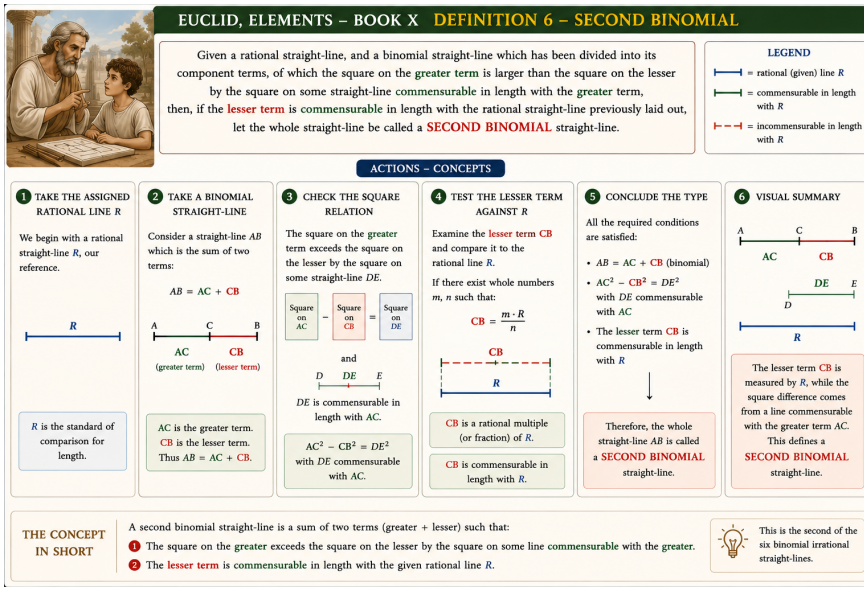


1.0.6 Definition 6

And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \sim u,\ v \sim r \Rightarrow x \text{ second binomial}$$

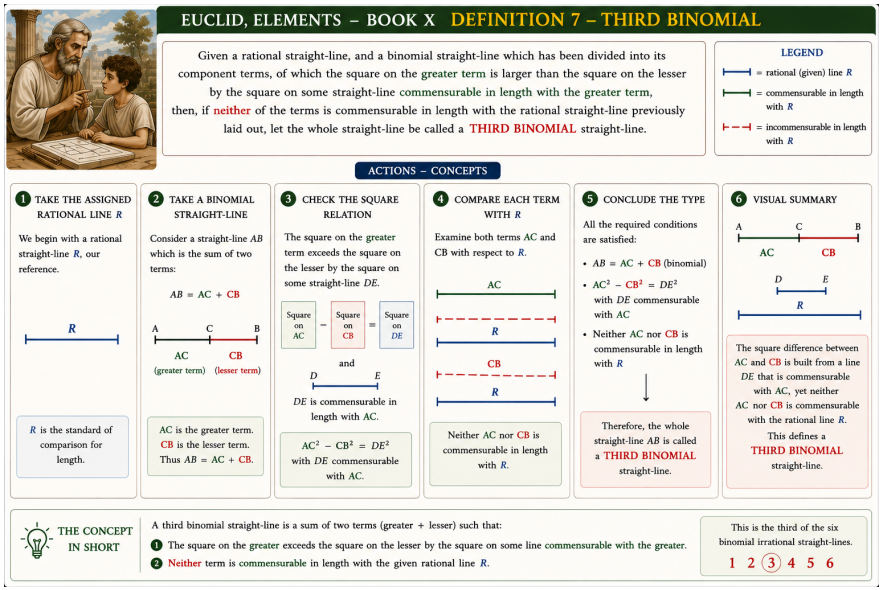


1.0.7 Definition 7

And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \sim u,\ u \nsim r,\ v \nsim r \Rightarrow x \text{ third binomial}$$

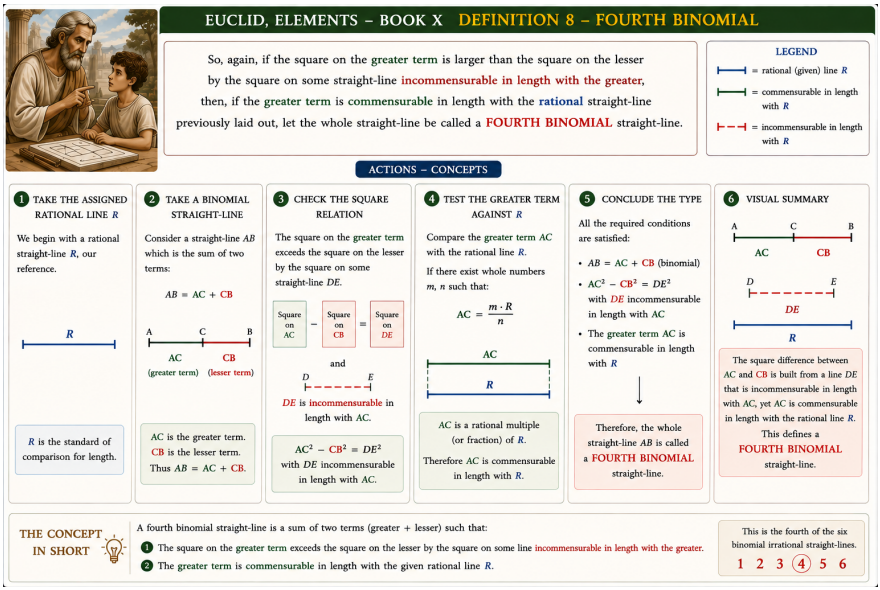


1.0.8 Definition 8

So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ u \sim r \Rightarrow x \text{ fourth binomial}$$

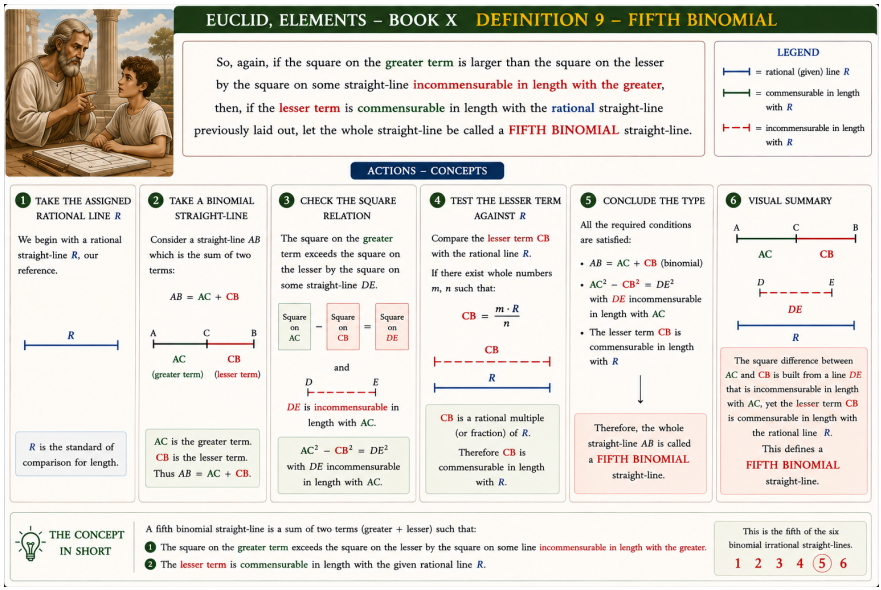


1.0.9 Definition 9

And if the lesser (term is commensurable), a fifth (binomial straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ v \sim r \Rightarrow x \text{ fifth binomial}$$

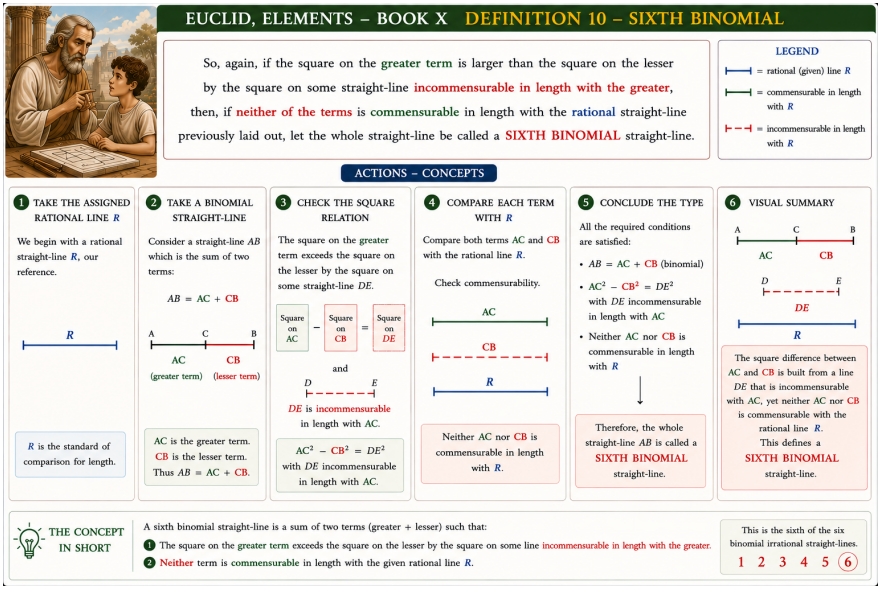


1.0.10 Definition 10

And if neither (term is commensurable), a sixth (binomial straight-line).

Symbolic Form

$$x = u + v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ u \nsim r,\ v \nsim r \Rightarrow x \text{ sixth binomial}$$



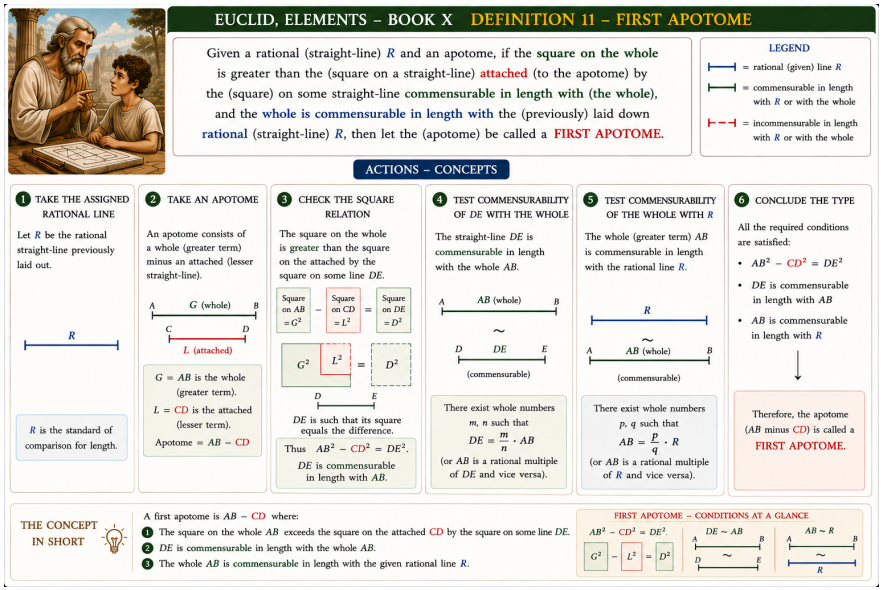
Definitions III

1.0.11 Definition 11

Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \sim u,\ u \sim r \Rightarrow x \text{ first apotome}$$

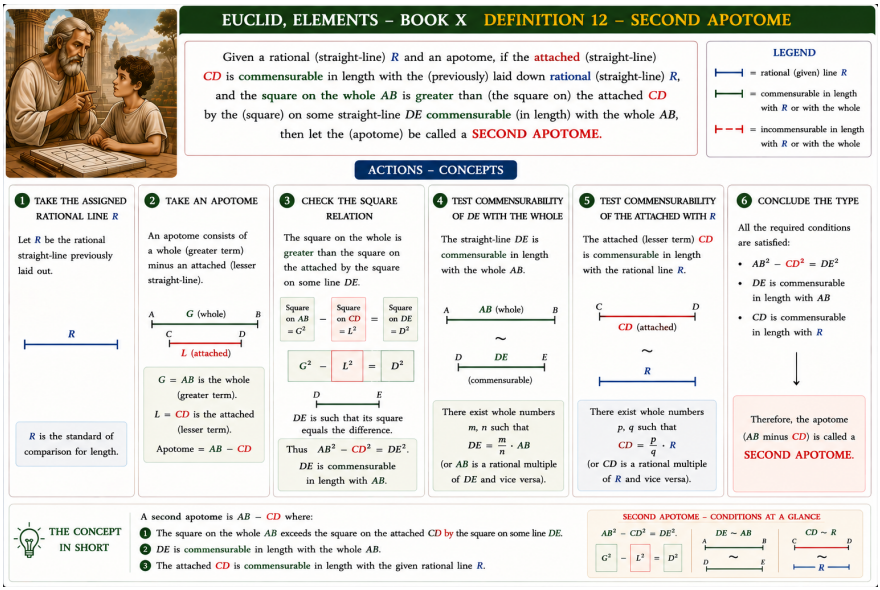


1.0.12 Definition 12

And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \sim u,\ v \sim r \Rightarrow x \text{ second apotome}$$

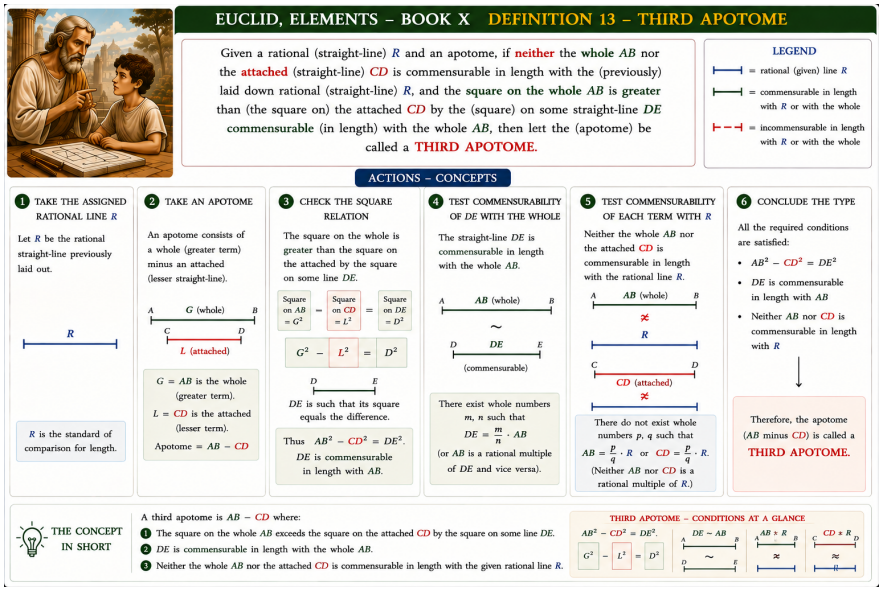


1.0.13 Definition 13

And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \sim u,\ u \nsim r,\ v \nsim r \Rightarrow x \text{ third apotome}$$

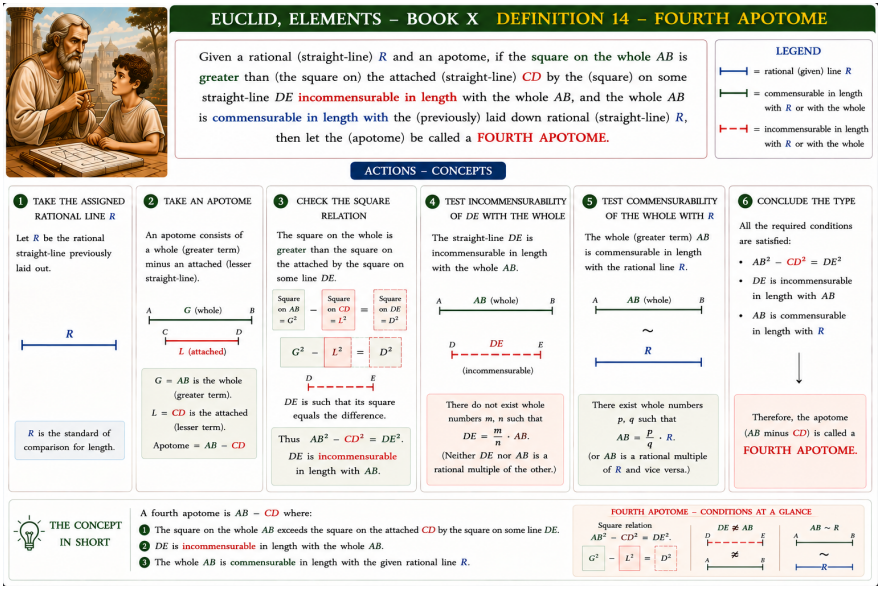


1.0.14 Definition 14

Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ u \sim r \Rightarrow x \text{ fourth apotome}$$

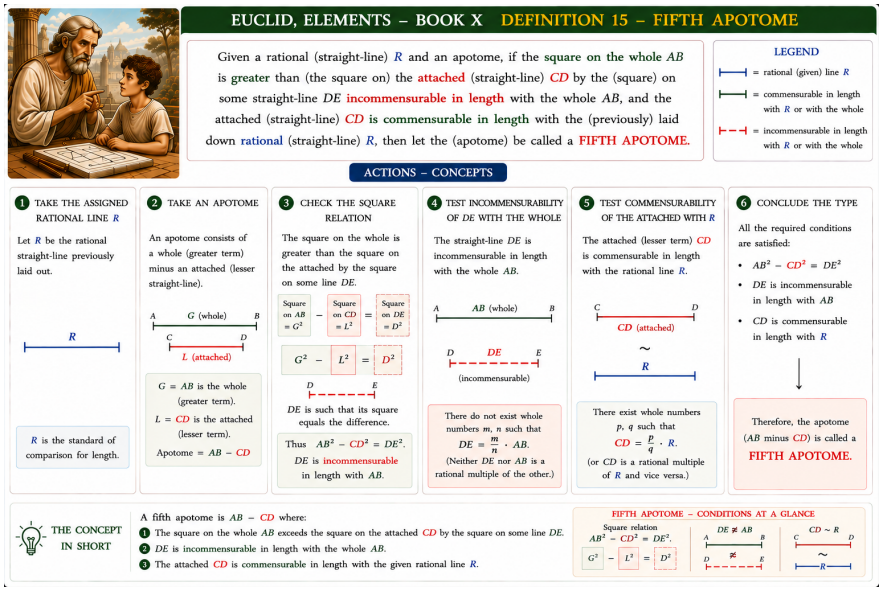


1.0.15 Definition 15

And if the attached (straight-line is commensurable), a fifth (apotome).

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ v \sim r \Rightarrow x \text{ fifth apotome}$$



1.0.16 Definition 16

And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

Symbolic Form

$$x = u - v,\ u^2 - v^2 = \square(w),\ w \nsim u,\ u \nsim r,\ v \nsim r \Rightarrow x \text{ sixth apotome}$$

EUCLID, ELEMENTS – BOOK X DEFINITION 16 – SIXTH APOTOME

Given a rational (straight-line) R and an apotome, if the square on the whole AB is greater than (the square on) the attached (straight-line) CD by the (square) on some straight-line DE incommensurable in length with the whole AB, and neither the whole AB nor the attached (straight-line) CD is commensurable in length with the (previously) laid down rational (straight-line) R, then let the (apotome) be called a SIXTH APOTOME.

LEGEND

- = rational (given) line R
- = commensurable in length with R or with the whole
- = incommensurable in length with R or with the whole

ACTIONS – CONCEPTS

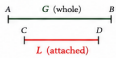
1 TAKE THE ASSIGNED RATIONAL LINE R

Let R be the rational straight-line previously laid out.

R is the standard of comparison for length.

2 TAKE AN APOTOME

An apotome consists of a whole (greater term) minus an attached (lesser straight-line).



$G = AB$ is the whole (greater term).
$L = CD$ is the attached (lesser term).
Apotome $= AB - CD$

3 CHECK THE SQUARE RELATION

The square on the whole is greater than the square on the attached by the square on some line DE.

Square on AB $= G^2$ – Square on CD $= L^2$ = Square on DE $= D^2$

$G^2 - L^2 = D^2$

DE is such that its square equals the difference.

Thus $AB^2 - CD^2 = DE^2$.
DE is incommensurable in length with AB.

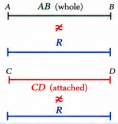
4 TEST INCOMMENSURABILITY OF DE WITH THE WHOLE

The straight-line DE is incommensurable in length with the whole AB.

There do not exist whole numbers m, n such that $DE = \frac{m}{n} \cdot AB$.
(Neither DE nor AB is a rational multiple of the other.)

5 TEST INCOMMENSURABILITY OF EACH TERM WITH R

Neither the whole AB nor the attached CD is commensurable in length with the rational line R.



There do not exist whole numbers p, q such that $AB = \frac{p}{q} \cdot R$ or $CD = \frac{p}{q} \cdot R$.
(Neither AB nor CD is a rational multiple of R.)

6 CONCLUDE THE TYPE

All the required conditions are satisfied:

- $AB^2 - CD^2 = DE^2$
- DE is incommensurable in length with AB
- Neither AB nor CD is commensurable in length with R

Therefore, the apotome (AB minus CD) is called a SIXTH APOTOME.

THE CONCEPT IN SHORT

A sixth apotome is $AB - CD$ where:

1. The square on the whole AB exceeds the square on the attached CD by the square on some line DE.
2. DE is incommensurable in length with the whole AB.
3. Neither the whole AB nor the attached CD is commensurable in length with the given rational line R.

SIXTH APOTOME – CONDITIONS AT A GLANCE

Square relation
$AB^2 - CD^2 = DE^2$.
$G^2 - L^2 = D^2$

$DE \sim AB$
(incommensurable)

$AB \not\sim R$; $CD \not\sim R$

Chapter 2

Propositions

2.1 Proposition 1

Statement

> *If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude.*

Proof Actions

1. Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater.

2. I say that if (a part) greater than half is subtracted from AB, and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude C.

3. For C, when multiplied (by some number), will sometimes be greater than AB [Def. 5.4].

4. Let it have been (so) multiplied.

5. And let DE be (both) a multiple of C, and greater than AB.

6. And let DE have been divided into the (divisions) DF, FG, GE, equal to C.

7. And let BH, (which is) greater than half, have been subtracted from AB.

8. And (let) HK, (which is) greater than half, (have been subtracted) from AH.

9. And let this happen continually, until the divisions in AB become equal in number to the divisions in DE.

10. Therefore, let the divisions (in AB) be AK, KH, HB, being equal in number to DF, FG, GE.

11. And since DE is greater than AB, and EG, (which is) less than half, has been subtracted from DE, and BH, (which is) greater than half, from AB, the remainder GD is thus greater than the remainder HA.

12. And since GD is greater than HA, and the half GF has been subtracted from GD, and HK, (which is) greater than half, from HA, the remainder DF is thus greater than the remainder AK. And DF (is) equal to C.

13. C is thus also greater than AK.

14. Thus, AK (is) less than C.

15. Thus, the magnitude AK, which is less than the lesser laid out magnitude C, is left over from the magnitude AB.

16. (Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves. generally attributed to Eudoxus of Cnidus.

Complete Cinematic Logic

Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater.

⇓

And let DE have been divided into the (divisions) DF, FG, GE, equal to C.

⇓

And since DE is greater than AB, and EG, (which is) less than half, has been subtracted from DE, and BH, (which is) greater than half, from AB, the remainder GD is thus greater than the remainder HA.

⇓

(Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves. generally attributed to Eudoxus of Cnidus.

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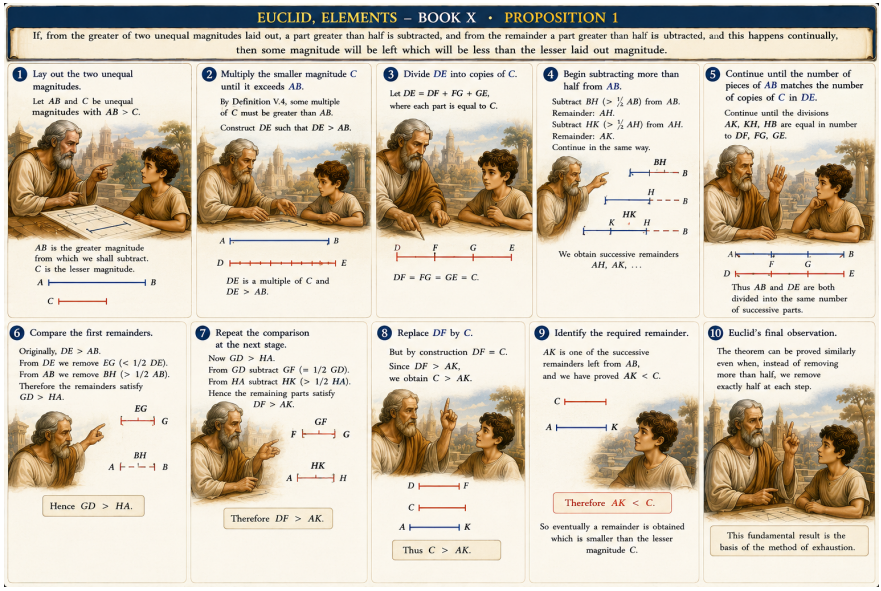
Conclusion: Proposition 1 established.



2.2 Proposition 2

Statement

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

Proof Actions

1. For, AB and CD being two unequal magnitudes, and AB (being) the lesser, let the remainder never measure the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater.
2. I say that the magnitudes AB and CD are incommensurable.
3. For if they are commensurable then some magnitude will measure them (both).
4. If possible, let it (so) measure (them), and let it be E.
5. And let AB leave CF less than itself (in) measuring FD, and let CF leave AG less than itself (in) measuring BG, and let this happen continually, until some magnitude which is less than E is left.
6. Let (this) have occurred, and let AG, (which is) less than E, have been left.
7. Therefore, since E measures AB, but AB measures DF, E will thus also measure FD.
8. And it also measures the whole (of) CD.
9. Thus, it will also measure the remainder CF. But, CF measures BG.
10. Thus, E also measures BG.
11. And it also measures the whole (of) AB.
12. Thus, it will also measure the remainder AG, the greater (measuring) the lesser.

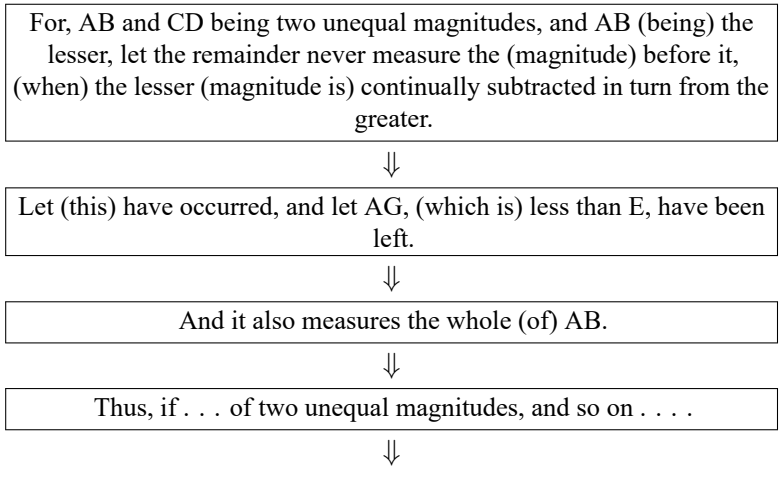
13. The very thing is impossible.

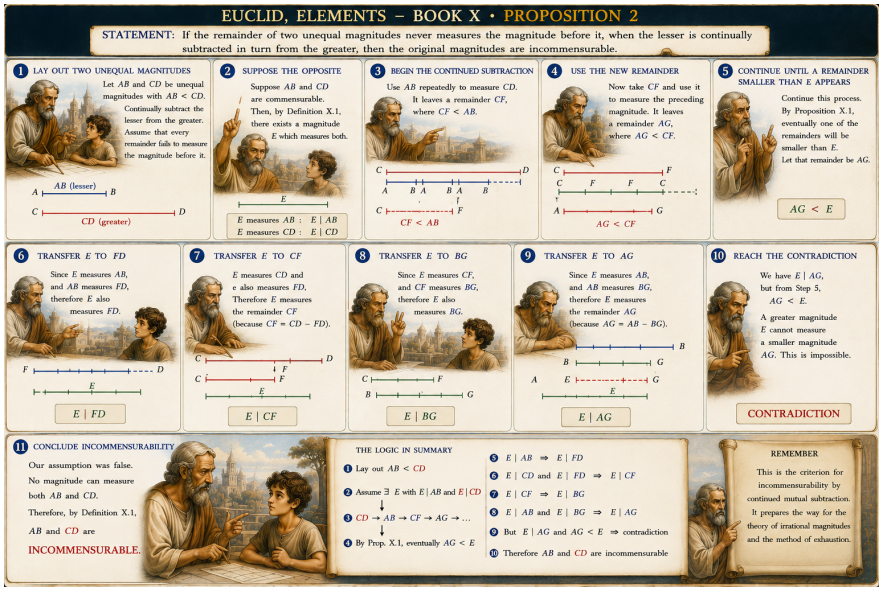
14. Thus, some magnitude cannot measure (both) the magnitudes AB and CD.

15. Thus, the magnitudes AB and CD are incommensurable [Def. 10.1].

16. Thus, if . . . of two unequal magnitudes, and so on . . . .

Complete Cinematic Logic





2.3 Proposition 3

Statement

> *To find the greatest common measure of two given commensurable magnitudes.*

Proof Actions

1. Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser.
2. So, it is required to find the greatest common measure of AB and CD.
3. For the magnitude AB either measures, or (does) not (measure), CD.
4. Therefore, if it measures (CD), and (since) it also measures itself, AB is thus a common measure of AB and CD.
5. And (it is) clear that (it is) also (the) greatest.
6. For a (magnitude) greater than magnitude AB cannot measure AB.

7. So let AB not measure CD.

8. And continually subtracting in turn the lesser (magnitude) from the greater, the remaining (magnitude) will (at) some time measure the (magnitude) before it, on account of AB and CD not being incommensurable [Prop. 10.2].

9. And let AB leave EC less than itself (in) measuring ED, and let EC leave AF less than itself (in) measuring FB, and let AF measure CE.

10. Therefore, since AF measures CE, but CE measures FB, AF will thus also measure FB.

11. And it also measures itself.

12. Thus, AF will also measure the whole (of) AB. But, AB measures DE.

13. Thus, AF will also measure ED. And it also measures CE.

14. Thus, it also measures the whole of CD.

15. Thus, AF is a common measure of AB and CD.

16. So I say that (it is) also (the) greatest (common measure).

17. For, if not, there will be some magnitude, greater than AF, which will measure (both) AB and CD. Let it be G.

18. Therefore, since G measures AB, but AB measures ED, G will thus also measure ED.

19. And it also measures the whole of CD.

20. Thus, G will also measure the remainder CE. But CE measures FB.

21. Thus, G will also measure FB.

22. And it also measures the whole (of) AB.

23. And (so) it will measure the remainder AF, the greater (measuring) the lesser.

24. The very thing is impossible.

25. Thus, some magnitude greater than AF cannot measure (both) AB and CD.

26. Thus, AF is the greatest common measure of AB and CD.

27. Thus, the greatest common measure of two given commensurable magnitudes, AB and CD, has been found.

28. (Which is) the very thing it was required to show.

29. Corollary So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

Complete Cinematic Logic

Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser.

⇓

Therefore, since AF measures CE, but CE measures FB, AF will thus also measure FB.

⇓

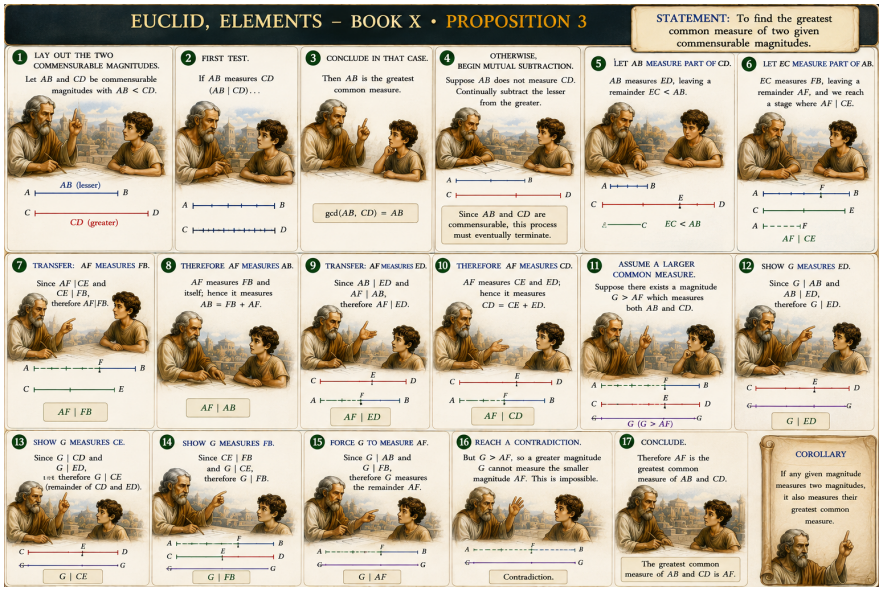
Thus, G will also measure the remainder CE. But CE measures FB.

⇓

Corollary So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

⇓

Conclusion: Proposition 3 established.



2.4 Proposition 4

Statement

> *To find the greatest common measure of three given commensurable magnitudes.*

Proof Actions

1. Let A, B, C be the three given commensurable magnitudes.
2. So it is required to find the greatest common measure of A, B, C.
3. For let the greatest common measure of the two (magnitudes) A and B have been taken [Prop. 10.3], and let it be D.
4. So D either measures, or [does] not [measure], C.
5. Let it, first of all, measure (C).
6. Therefore, since D measures C, and it also measures A and B, D thus measures A, B, C.

7. Thus, D is a common measure of A, B, C.

8. And (it is) clear that (it is) also (the) greatest (common measure).

9. For no magnitude larger than D measures (both) A and B. So let D not measure C.

10. I say, first, that C and D are commensurable.

11. For if A, B, C are commensurable then some magnitude will measure them which will clearly also measure A and B.

12. Hence, it will also measure D, the greatest common measure of A and B [Prop. 10.3 corr.]. And it also measures C.

13. Hence, the aforementioned magnitude will measure (both) C and D.

14. Thus, C and D are commensurable [Def. 10.1].

15. Therefore, let their greatest common measure have been taken [Prop. 10.3], and let it be E.

16. Therefore, since E measures D, but D measures (both) A and B, E will thus also measure A and B. And it also measures C.

17. Thus, E measures A, B, C.

18. Thus, E is a common measure of A, B, C.

19. So I say that (it is) also (the) greatest (common measure).

20. For, if possible, let F be some magnitude greater than E, and let it measure A, B, C.

21. And since F measures A, B, C, it will thus also measure A and B, and will (thus) measure the greatest common measure of A and B [Prop. 10.3 corr.].

22. And D is the greatest common measure of A and B. Thus, F measures D. And it also measures C.

23. Thus, F measures (both) C and D.

24. Thus, F will also measure the greatest common measure of C and D [Prop. 10.3 corr.]. And it is E.

25. Thus, F will measure E, the greater (measuring) the lesser.

26. The very thing is impossible.

27. Thus, some [magnitude] greater than the magnitude E cannot measure A, B, C.

28. Thus, if D does not measure C then E is the greatest common measure of A, B, C.

29. And if it does measure (C) then D itself (is the greatest common measure).

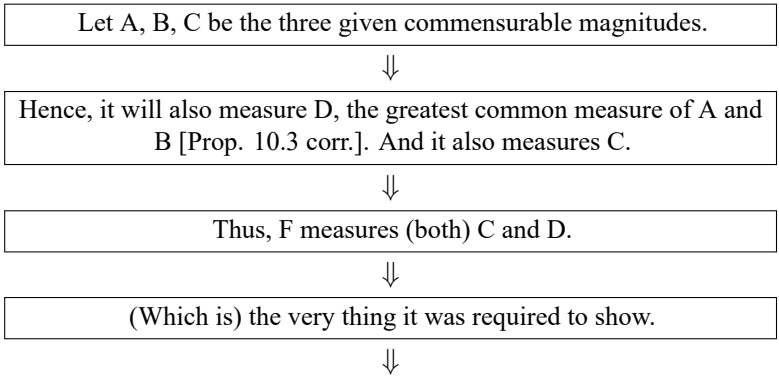
30. Thus, the greatest common measure of three given commensurable magnitudes has been found.

31. [(Which is) the very thing it was required to show.] Corollary So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

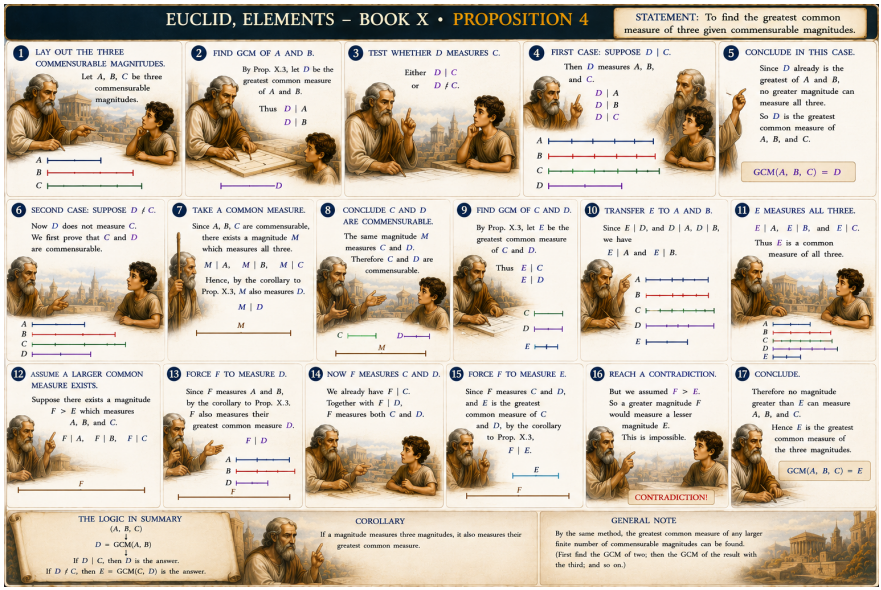
32. So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward.

33. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 4 established.



2.5 Proposition 5

Statement

> *Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.*

Proof Actions

1. Let A and B be commensurable magnitudes.
2. I say that A has to B the ratio which (some) number (has) to (some) number.
3. For if A and B are commensurable (magnitudes) then some magnitude will measure them.
4. Let it (so) measure (them), and let it be C.
5. And as many times as C measures A, so many units let there be in D.
6. And as many times as C measures B, so many units let there be in E.

7. Therefore, since C measures A according to the units in D, and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the magnitude C (measures) A.

8. Thus, as C is to A, so a unit (is) to D [Def. 7.20].

9. Thus, inversely, as A (is) to C, so D (is) to a unit [Prop. 5.7 corr.].

10. Again, since C measures B according to the units in E, and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B.

11. Thus, as C is to B, so a unit (is) to E [Def. 7.20].

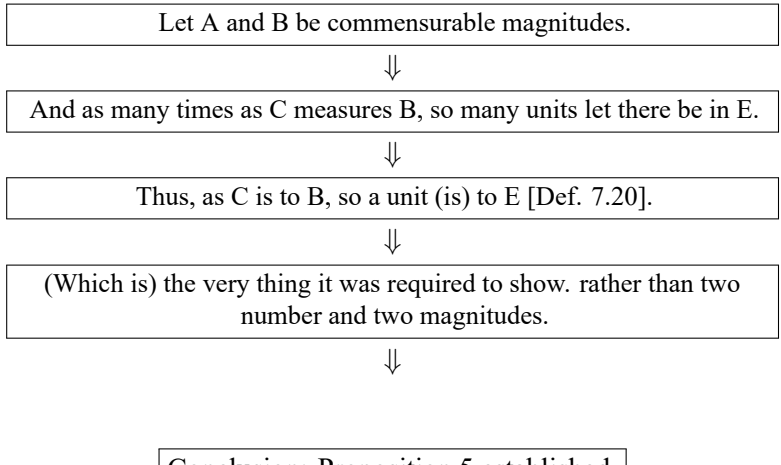
12. And it was also shown that as A (is) to C, so D (is) to a unit.

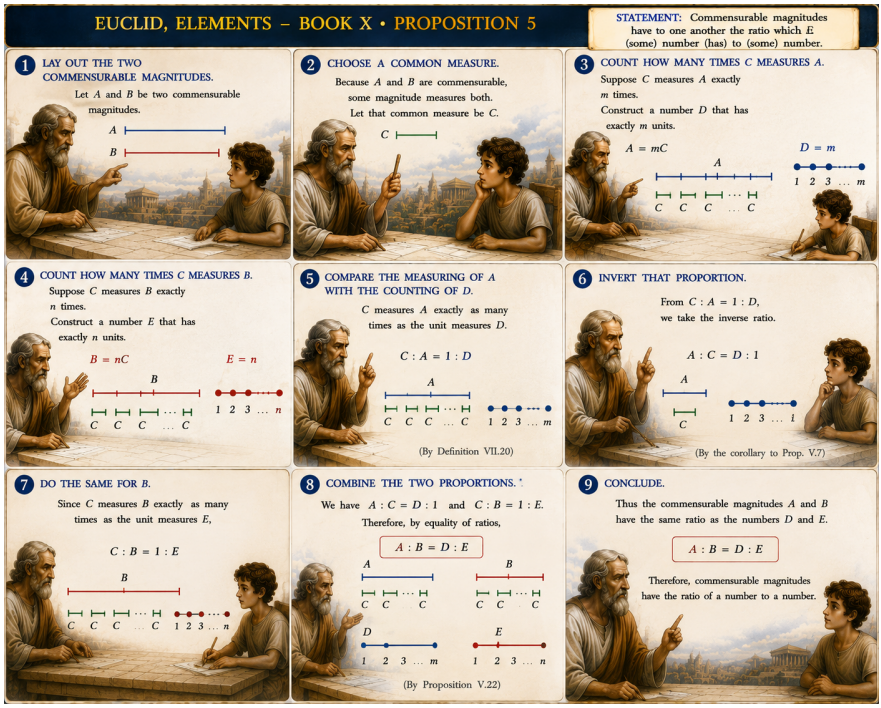
13. Thus, via equality, as A is to B, so the number D (is) to the (number) E [Prop. 5.22].

14. Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E.

15. (Which is) the very thing it was required to show. rather than two number and two magnitudes.

Complete Cinematic Logic





2.6 Proposition 6

Statement

> *If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.*

Proof Actions

1. For let the two magnitudes A and B have to one another the ratio which the number D (has) to the number E.

2. I say that the magnitudes A and B are commensurable.

3. For, as many units as there are in D, let A have been And let C be divided into so many equal (divisions). equal to one of them.

4. And as many units as there are in E, let F be the sum of so many magnitudes equal to C.

5. Therefore, since as many units as there are in D, so many magnitudes equal to C are also in A, therefore whichever part a unit is of D, C is also the same part of A.

6. Thus, as C is to A, so a unit (is) to D [Def. 7.20].

7. And a unit measures the number D. Thus, C also measures A.

8. And since as C is to A, so a unit (is) to the [number] D, thus, inversely, as A (is) to C, so the number D (is) to a unit [Prop. 5.7 corr.].

9. Again, since as many units as there are in E, so many (magnitudes) equal to C are also in F, thus as C is to F, so a unit (is) to the [number] E [Def. 7.20].

10. And it was also shown that as A (is) to C, so D (is) to a unit.

11. Thus, via equality, as A is to F, so D (is) to E [Prop. 5.22].

12. But, as D (is) to E, so A is to B.

13. And thus as A (is) to B, so (it) also is to F [Prop. 5.11].

14. Thus, A has the same ratio to each of B and F.

15. Thus, B is equal to F [Prop. 5.9]. And C measures F.

16. Thus, it also measures B.

17. But, in fact, (it) also (measures) A.

18. Thus, C measures (both) A and B.

19. Thus, A is commensurable with B [Def. 10.1].

20. Thus, if two magnitudes . . . to one another, and so on . . . .

21. Corollary So it is clear, from this, that if there are two numbers, like D and E, and a straight-line, like A, then it is possible to contrive that as the number D (is) to the number E, so the straight-line (is) to (another) straight-line (i.e., F).

22. And if the mean proportion, (say) B, is taken of A and F, then as A is to F, so the (square) on A (will be) to the (square) on B.

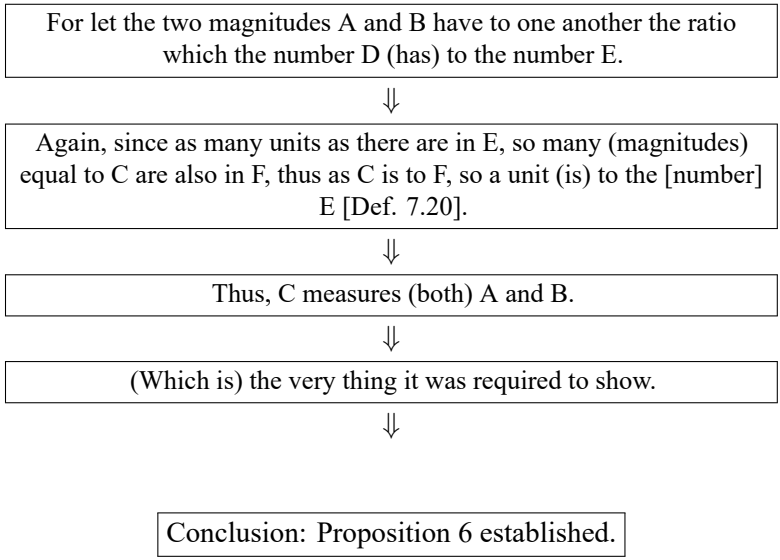
23. That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.].

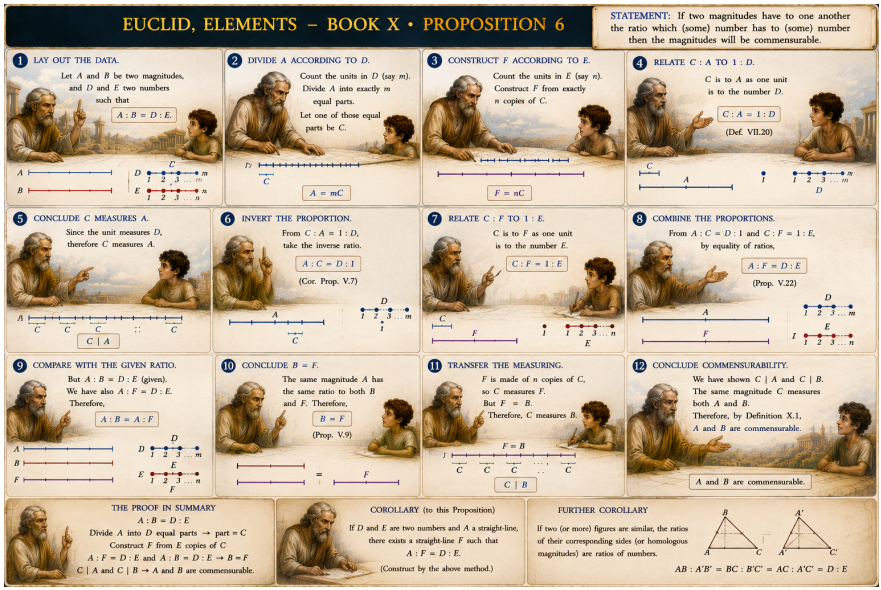
24. But, as A (is) to F, so the number D is to the number E.

25. Thus, it has also been contrived that as the number D (is) to the number E, so the (figure) on the straight-line A (is) to the (similar figure) on the straight-line B.

26. (Which is) the very thing it was required to show.

Complete Cinematic Logic





2.7 Proposition 7

Statement

> *Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.*

Proof Actions

1. Let A and B be incommensurable magnitudes.
2. I say that A does not have to B the ratio which (some) number (has) to (some) number.
3. For if A has to B the ratio which (some) number (has) to (some) number then A will be commensurable with B [Prop. 10.6]. But it is not.
4. Thus, A does not have to B the ratio which (some) number (has) to (some) number.
5. Thus, incommensurable numbers do not have to one another, and so on . . . .

Complete Cinematic Logic

Let A and B be incommensurable magnitudes.

⇓

I say that A does not have to B the ratio which (some) number (has) to (some) number.

⇓

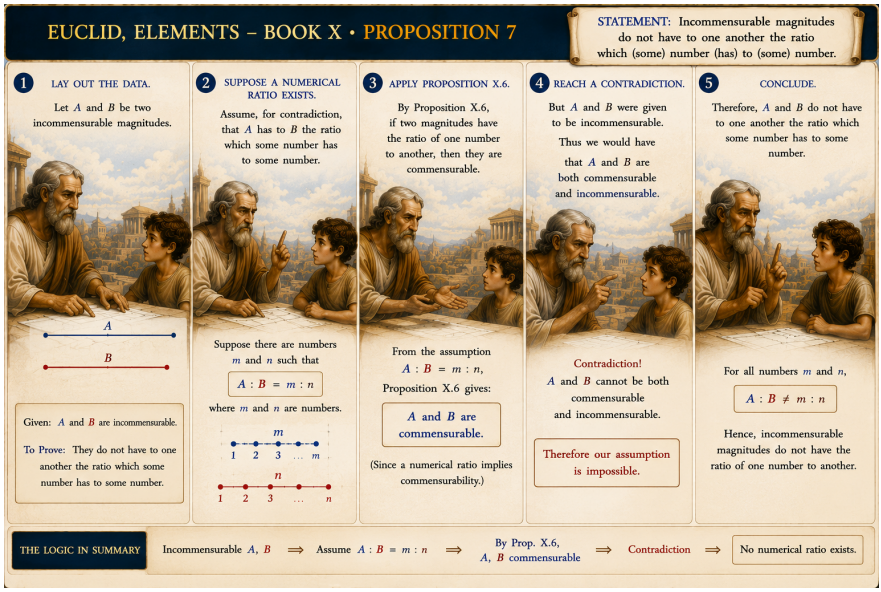
Thus, A does not have to B the ratio which (some) number (has) to (some) number.

⇓

Thus, incommensurable numbers do not have to one another, and so on . . . .

⇓

Conclusion: Proposition 7 established.



2.8 Proposition 8

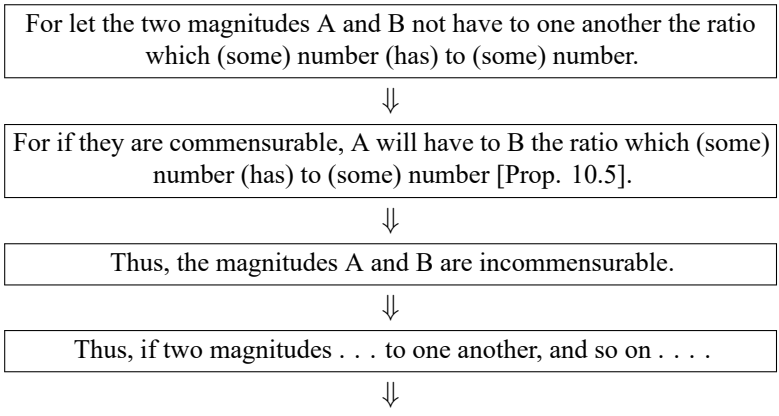
Statement

If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be incommensurable.

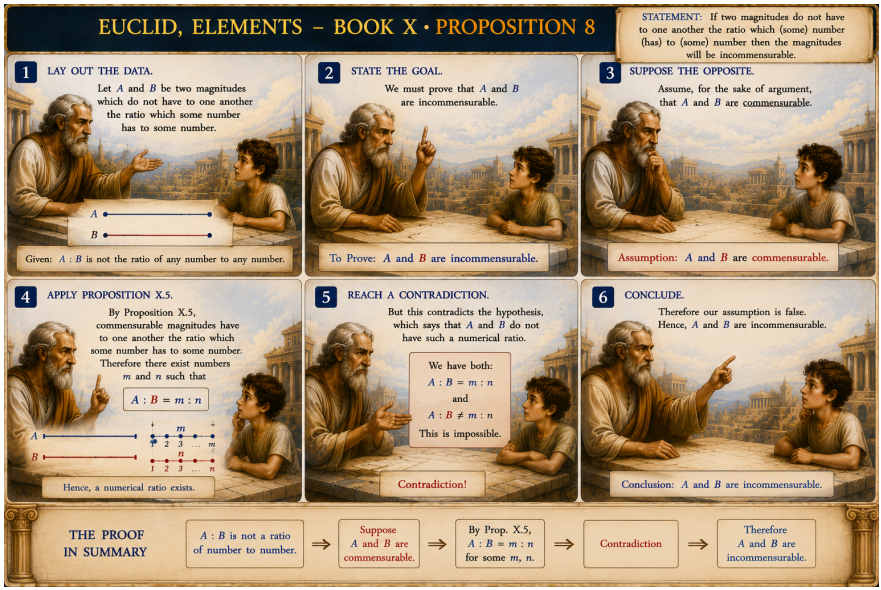
Proof Actions

1. For let the two magnitudes A and B not have to one another the ratio which (some) number (has) to (some) number.
2. I say that the magnitudes A and B are incommensurable.
3. For if they are commensurable, A will have to B the ratio which (some) number (has) to (some) number [Prop. 10.5].
4. But it does not have (such a ratio).
5. Thus, the magnitudes A and B are incommensurable.
6. Thus, if two magnitudes . . . to one another, and so on . . . .

Complete Cinematic Logic



Conclusion: Proposition 8 established.



2.9 Proposition 9

Statement

> *Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.*

Proof Actions

1. For let A and B be (straight-lines which are) commensurable in length.

2. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number.

3. For since A is commensurable in length with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5].

4. Let it have (that) which C (has) to D.

5. Therefore, since as A is to B, so C (is) to D.

6. But the (ratio) of the square on A to the square on B is the square of the ratio of A to B.

7. For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.].

8. And the (ratio) of the square on C to the square on D is the square of the ratio of the [number] C to the [number] D.

9. For there exits one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11].

10. And, thus, as the square on A is to the square on B, so the square [number] on the (number) C (is) to the square [number] on the [number] D.

11. And so let the square on A be to the (square) on B as the square (number) on C (is) to the [square] (number) on D.

12. I say that A is commensurable in length with B.

13. For since as the square on A is to the [square] on B, so the square (number) on C (is) to the [square] (number) on D.

14. But, the ratio of the square on A to the (square) on B is the square of the (ratio) of A to B [Prop. 6.20 corr.].

15. And the (ratio) of the square [number] on the [number] C to the square [number] on the [number] D is the square of the ratio of the [number] C to the [number] D [Prop. 8.11].

16. Thus, as A is to B, so the [number] C also (is) to the [number] D.

17. A, thus, has to B the ratio which the number C has to the number D.

18. Thus, A is commensurable in length with B [Prop. 10.6].

19. And so let A be incommensurable in length with B.

20. I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

21. For if the square on A has to the [square] on B the ratio which (some) square number (has) to (some) square number then A will be commensurable (in length) with B. But it is not.

22. Thus, the square on A does not have to the [square] on the B the ratio which (some) square number (has) to (some) square number.

23. So, again, let the square on A not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

24. I say that A is incommensurable in length with B.

25. For if A is commensurable (in length) with B then the (square) on A will have to the (square) on B the ratio which (some) square number (has) to (some) square number.

26. But it does not have (such a ratio).

27. Thus, A is not commensurable in length with B.

28. Thus, (squares) on (straight-lines which are) com290 mensurable in length, and so on . . . .

29. Corollary And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straightlines commensurable) in square (are) not always also (commensurable) in length. :: γ2 : δ2. : β :: γ : δ.

Complete Cinematic Logic

For let A and B be (straight-lines which are) commensurable in length.

⇓

And, thus, as the square on A is to the square on B, so the square [number] on the (number) C (is) to the square [number] on the [number] D.

⇓

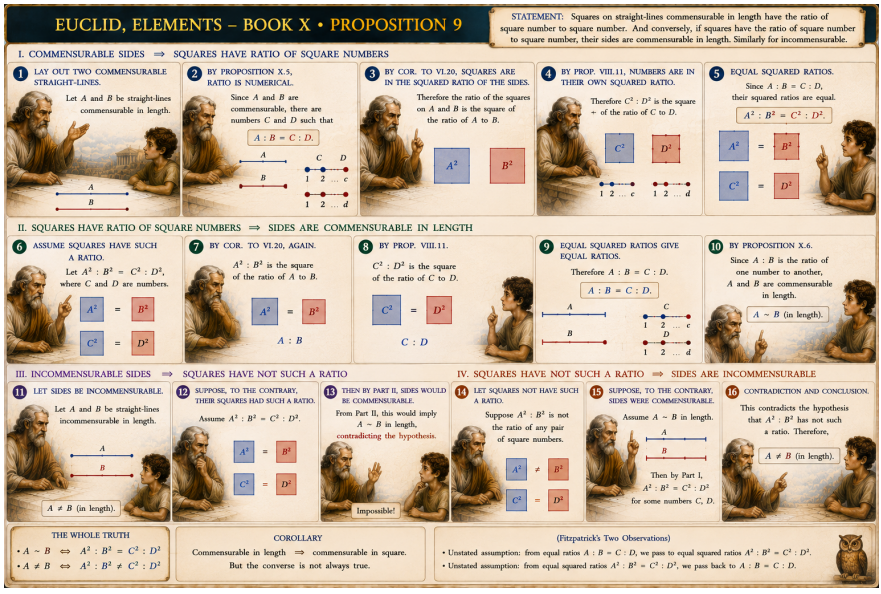
I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

⇓

Corollary And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straightlines commensurable) in square (are) not always also (commensurable) in length. :: γ2 : δ2. : β :: γ : δ.

⇓

Conclusion: Proposition 9 established.



2.10 Proposition 10

Statement

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length

Proof Actions

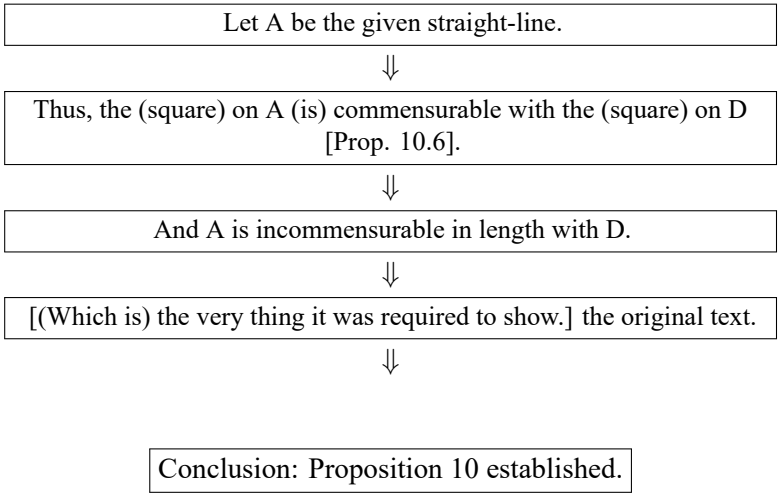
1. Let A be the given straight-line.
2. So it is required to find two straight-lines incommensurable with A, the one (incommensurable) in length only, the other also (incommensurable) in square.
3. For let two numbers, B and C, not having to one another the ratio which (some) square number (has) to (some) square number—that is to say, not (being) similar plane (numbers)—have been taken.
4. And let it be contrived that as B (is) to C, so the square on A (is) to the square on D.
5. For we learned (how to do this) [Prop. 10.6 corr.].
6. Thus, the (square) on A (is) commensurable with the (square) on D [Prop. 10.6].
7. And since B does not have to C the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on D the ratio which (some) square number (has) to (some) square number either.
8. Thus, A is incommensurable in length with D [Prop. 10.9].
9. Let the (straight-line) E (which is) in mean proportion to A and D have been taken [Prop. 6.13].
10. Thus, as A is to D, so the square on A (is) to the (square) on E [Def. 5.9].
11. And A is incommensurable in length with D.
12. Thus, the square on A is also incommensurble with the square on E [Prop. 10.11].

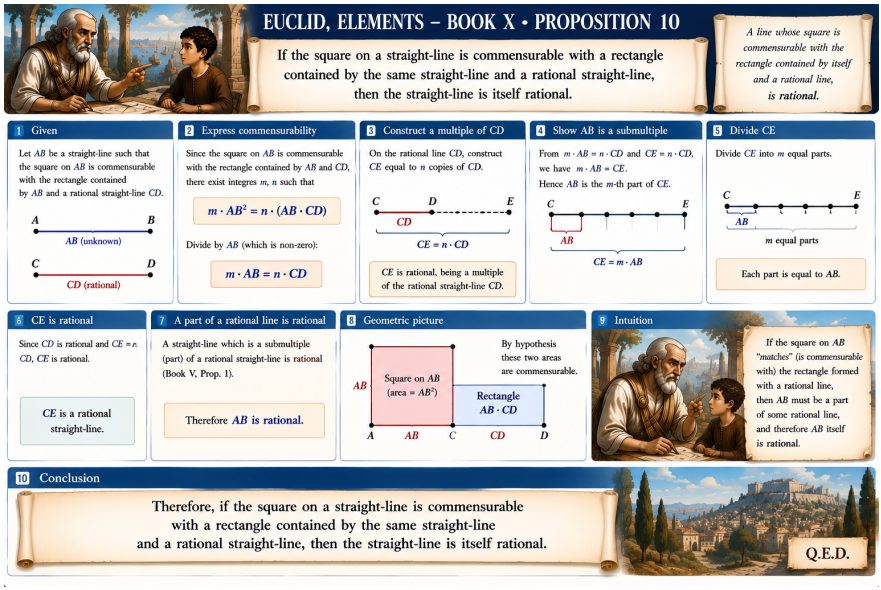
13. Thus, A is incommensurable in square with E.

14. Thus, two straight-lines, D and E, (which are) incommensurable with the given straight-line A, have been found, the one, D, (incommensurable) in length only, the other, E, (incommensurable) in square, and, clearly, also in length.

15. [(Which is) the very thing it was required to show.] the original text.

Complete Cinematic Logic





2.11 Proposition 11

Statement

> *If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.*

Proof Actions

1. Let A, B, C, D be four proportional magnitudes, (such that) as A (is) to B, so C (is) to D.
2. And let A be commensurable with B.
3. I say that C will also be commensurable with D.
4. For since A is commensurable with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5].

5. And as A is to B, so C (is) to D.

6. Thus, C also has to D the ratio which (some) number (has) to (some) number.

7. Thus, C is commensurable with D [Prop. 10.6].

8. And so let A be incommensurable with B.

9. I say that C will also be incommensurable with D.

10. For since A is incommensurable with B, A thus does not have to B the ratio which (some) number (has) to (some) number [Prop. 10.7].

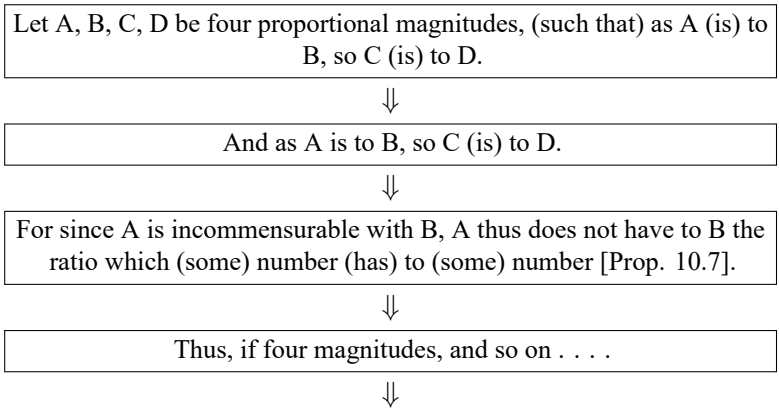
11. And as A is to B, so C (is) to D.

12. Thus, C does not have to D the ratio which (some) number (has) to (some) number either.

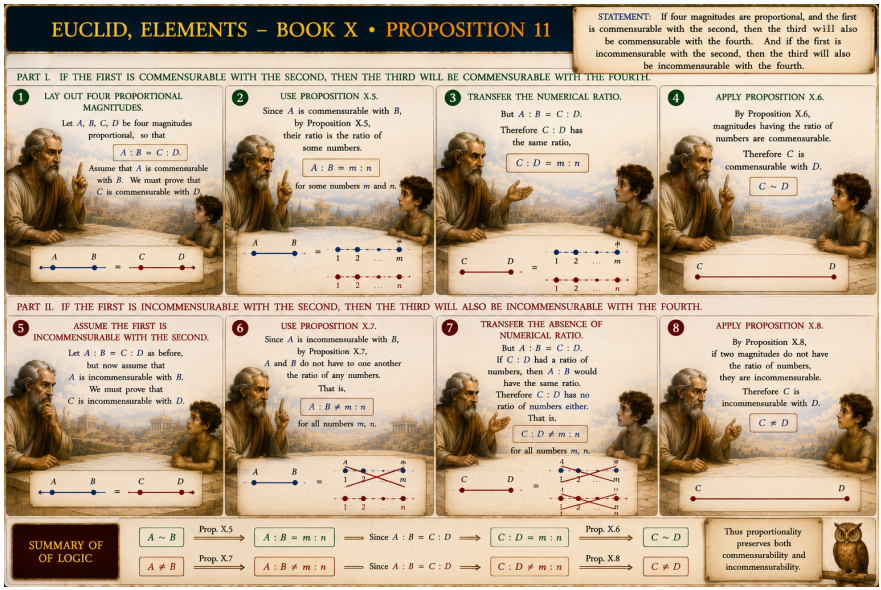
13. Thus, C is incommensurable with D [Prop. 10.8].

14. Thus, if four magnitudes, and so on . . . .

Complete Cinematic Logic



Conclusion: Proposition 11 established.



2.12 Proposition 12

Statement

> *(Magnitudes) commensurable with the same magnitude are also commensurable with one another.*

Proof Actions

1. For let A and B each be commensurable with C.
2. I say that A is also commensurable with B.
3. For since A is commensurable with C, A thus has to C the ratio which (some) number (has) to (some) Let it have (the ratio) which D number [Prop. 10.5]. (has) to E.
4. Again, since C is commensurable with B, C thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5].
5. Let it have (the ratio) which F (has) to G.

6. And for any multitude whatsoever of given ratios—(namely,) those which D has to E, and F to G—let the numbers H, K, L (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4].

7. Hence, as D is to E, so H (is) to K, and as F (is) to G, so K (is) to L.

8. Therefore, since as A is to C, so D (is) to E, but as D (is) to E, so H (is) to K, thus also as A is to C, so H (is) to K [Prop. 5.11].

9. Again, since as C is to B, so F (is) to G, but as F (is) to G, [so] K (is) to L, thus also as C (is) to B, so K (is) to L [Prop. 5.11].

10. And also as A is to C, so H (is) to K.

11. Thus, via equality, as A is to B, so H (is) to L [Prop. 5.22].

12. Thus, A has to B the ratio which the number H (has) to the number L.

13. Thus, A is commensurable with B [Prop. 10.6].

14. Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another.

15. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let A and B each be commensurable with C.

⇓

And for any multitude whatsoever of given ratios—(namely,) those which D has to E, and F to G—let the numbers H, K, L (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4].

⇓

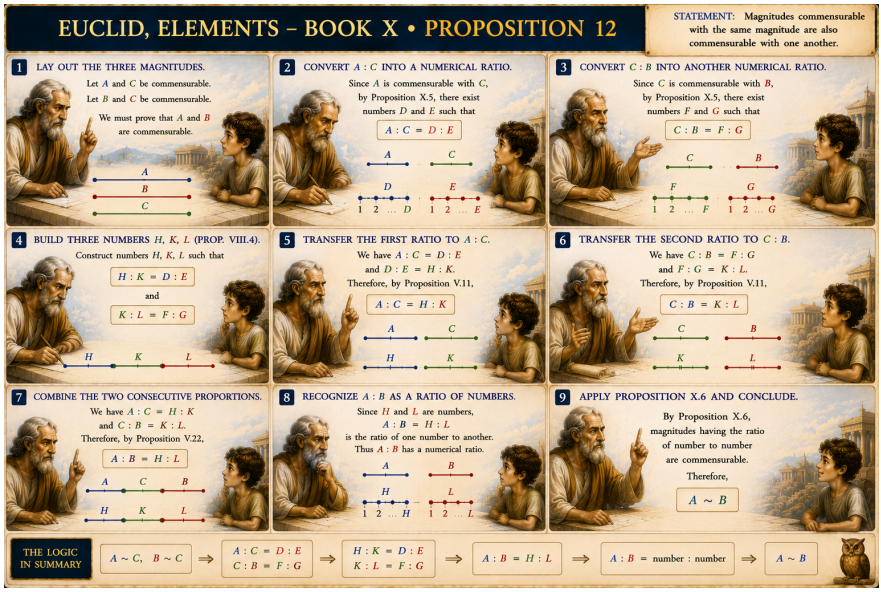
Thus, via equality, as A is to B, so H (is) to L [Prop. 5.22].

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 12 established.



2.13 Proposition 13

Statement

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.

Proof Actions

1. Let A and B be two commensurable magnitudes, and let one of them, A, be incommensurable with some other (magnitude), C.
2. I say that the remaining (magnitude), B, is also incommensurable with C.
3. For if B is commensurable with C, but A is also commensurable with B, A is thus also commensurable with C [Prop. 10.12].
4. But, (it is) also incommensurable (with C).
5. The very thing (is) impossible.
6. Thus, B is not commensurable with C.
7. Thus, (it is) incommensurable.
8. Thus, if two magnitudes are commensurable, and so on . . . .
9. Lemma For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater (straight-line is) larger than (the square on) the lesser.
10. Let AB and C be the two given unequal straight-lines, and let AB be the greater of them.
11. So it is required to find by (the square on) which (straight-line) the square on AB (is) greater than (the square on) C.
12. Let the semi-circle ADB have been described on AB.
13. And let AD, equal to C, have been inserted into it [Prop. 4.1].

14. And let DB have been joined.

15. So (it is) clear that the angle ADB is a right-angle [Prop. 3.31], and that the square on AB (is) greater than (the square on) AD—that is to say, (the square on) C—by (the square on) DB [Prop. 1.47].

16. And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likeso.

17. Let AD and DB be the two given straight-lines.

18. And let it be necessary to find the square-root of (the sum of the squares on) them.

19. For let them have been laid down such as to encompass a right-angle—(namely), that (angle encompassed) by AD and DB.

20. And let AB have been joined.

21. (It is) again clear that AB is the square-root of (the sum of the squares on) AD and DB [Prop. 1.47].

22. (Which is) the very thing it was required to show. α being greater than β, to find a straight-line of length γ such that

Complete Cinematic Logic

Let A and B be two commensurable magnitudes, and let one of them, A, be incommensurable with some other (magnitude), C.

⇓

Thus, if two magnitudes are commensurable, and so on . . . .

⇓

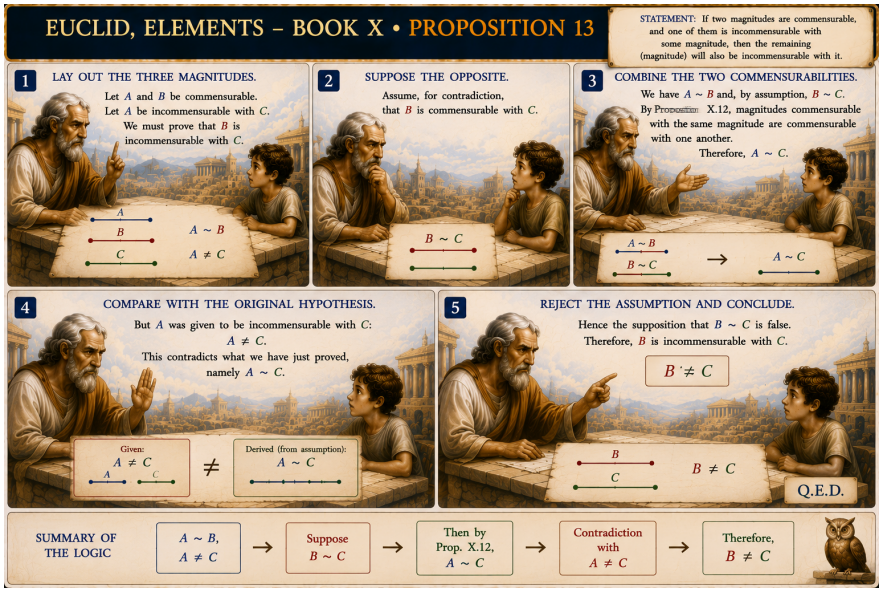
So (it is) clear that the angle ADB is a right-angle [Prop. 3.31], and that the square on AB (is) greater than (the square on) AD—that is to say, (the square on) C—by (the square on) DB [Prop. 1.47].

⇓

(Which is) the very thing it was required to show. α being greater than β, to find a straight-line of length γ such that

⇓

Conclusion: Proposition 13 established.



2.14 Proposition 14

Statement

> *If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.*

Proof Actions

1. Let A, B, C, D be four proportional straight-lines, (such that) as A (is) to B, so C (is) to D.

2. And let the square on A be greater than (the square on) B by the (square) on E, and let the square on C be greater than (the square on) D by the (square) on F.

3. I say that A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F.

4. For since as A is to B, so C (is) to D, thus as the (square) on A is to the (square) on B, so the (square) on C (is) to the (square) on D [Prop. 6.22].

5. But the (sum of the squares) on E and B is equal to the (square) on A, and the (sum of the squares) on D and F is equal to the (square) on C.

6. Thus, as the (sum of the squares) on E and B is to the (square) on B, so the (sum of the squares) on D and F (is) to the (square) on D.

7. Thus, via separation, as the (square) on E is to the (square) on B, so the (square) on F (is) to the (square) on D [Prop. 5.17].

8. Thus, also, as E is to B, so F (is) to D Thus, inversely, as B is to E, so D (is) [Prop. 6.22]. to F [Prop. 5.7 corr.].

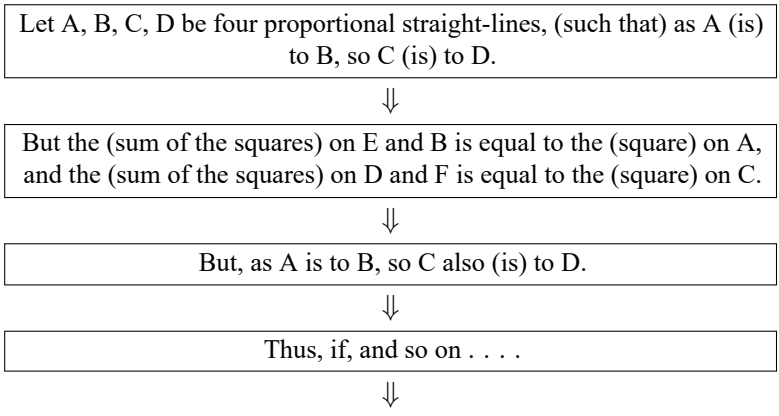
9. But, as A is to B, so C also (is) to D.

10. Thus, via equality, as A is to E, so C (is) to F [Prop. 5.22].

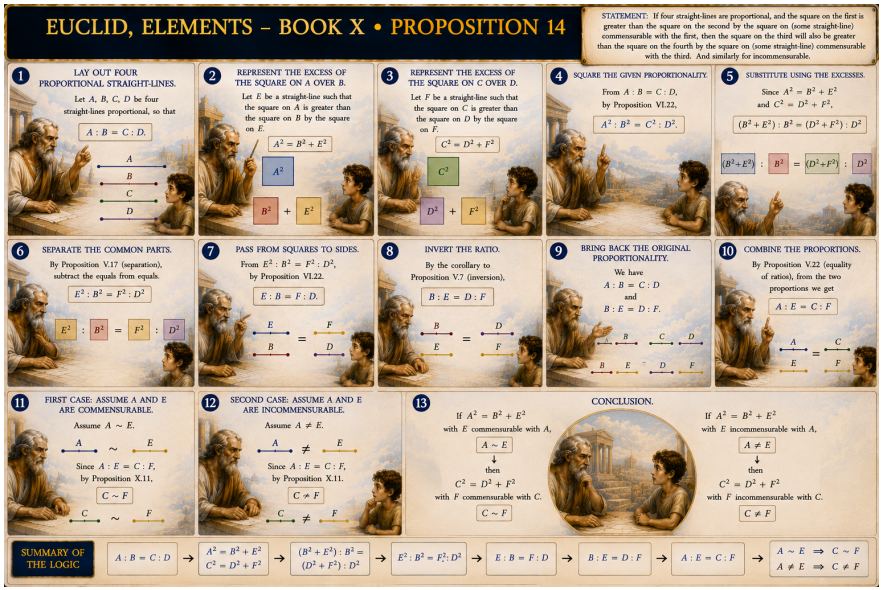
11. Therefore, A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F [Prop. 10.11].

12. Thus, if, and so on . . . .

Complete Cinematic Logic



Conclusion: Proposition 14 established.



2.15 Proposition 15

Statement

> *If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).*

Proof Actions

1. For let the two commensurable magnitudes AB and BC be laid down together.

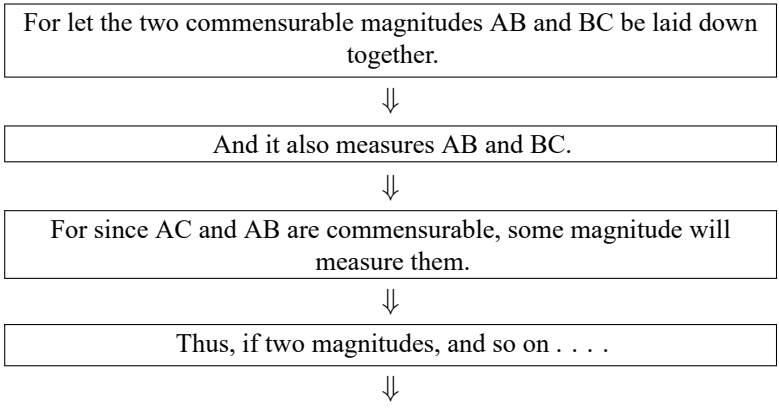
2. I say that the whole AC is also commensurable with each of AB and BC.

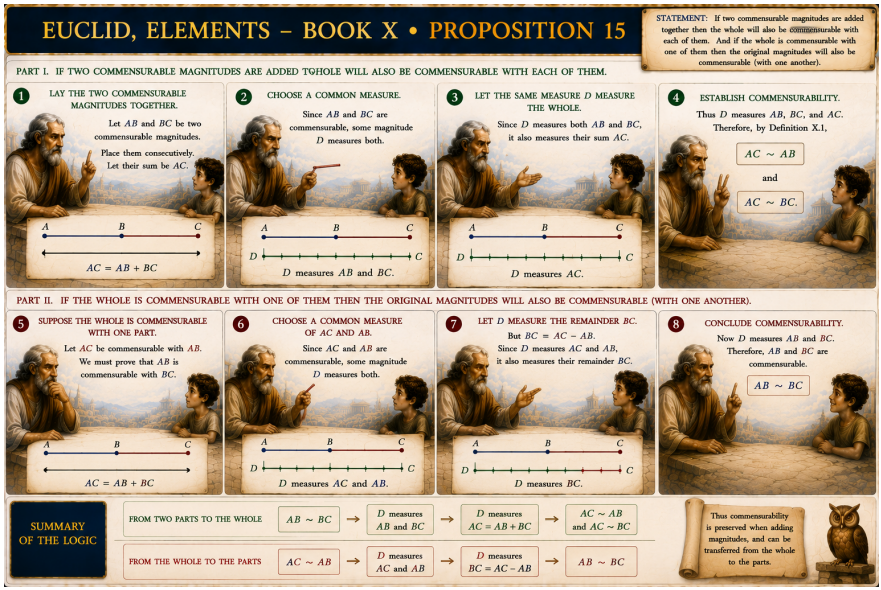
3. For since AB and BC are commensurable, some magnitude will measure them.

4. Let it (so) measure (them), and let it be D.

5. Therefore, since D measures (both) AB and BC, it will also measure the whole AC.
6. And it also measures AB and BC.
7. Thus, D measures AB, BC, and AC.
8. Thus, AC is commensurable with each of AB and BC [Def. 10.1].
9. And so let AC be commensurable with AB.
10. I say that AB and BC are also commensurable.
11. For since AC and AB are commensurable, some magnitude will measure them.
12. Let it (so) measure (them), and let it be D.
13. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB.
14. Thus, D will measure (both) AB and BC.
15. Thus, AB and BC are commensurable [Def. 10.1].
16. Thus, if two magnitudes, and so on . . . .

Complete Cinematic Logic





2.16 Proposition 16

Statement

> *If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).*

Proof Actions

1. For let the two incommensurable magnitudes AB and BC be laid down together.
2. I say that that the whole AC is also incommensurable with each of AB and BC.
3. For if CA and AB are not incommensurable then some magnitude will measure [them].
4. If possible, let it (so) measure (them), and let it be D.

5. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB.
6. Thus, D measures (both) AB and BC.
7. Thus, AB and BC are commensurable [Def. 10.1].
8. But they were also assumed (to be) incommensurable.
9. The very thing is impossible.
10. Thus, some magnitude cannot measure (both) CA and AB.
11. Thus, CA and AB are incommensurable [Def. 10.1].
12. So, similarly, we can show that AC and CB are also incommensurable.
13. Thus, AC is incommensurable with each of AB and BC.
14. And so let AC be incommensurable with one of AB and BC.
15. So let it, first of all, be incommensurable with AB.
16. I say that AB and BC are also incommensurable.
17. For if they are commensurable then some magnitude will measure them.
18. Let it (so) measure (them), and let it be D.
19. Therefore, since D measures (both) AB and BC, it will thus also measure the whole AC. And it also measures AB.
20. Thus, D measures (both) CA and AB.
21. Thus, CA and AB are commensurable [Def. 10.1].
22. But they were also assumed (to be) incommensurable.
23. The very thing is impossible.
24. Thus, some magnitude cannot measure (both) AB and BC.
25. Thus, AB and BC are incommensurable [Def. 10.1].
26. Thus, if two. . . magnitudes, and so on . . . .

27. Lemma If a parallelogram, falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).

28. For let the parallelogram AD, falling short by the square figure DB, have been applied to the straight-line AB.

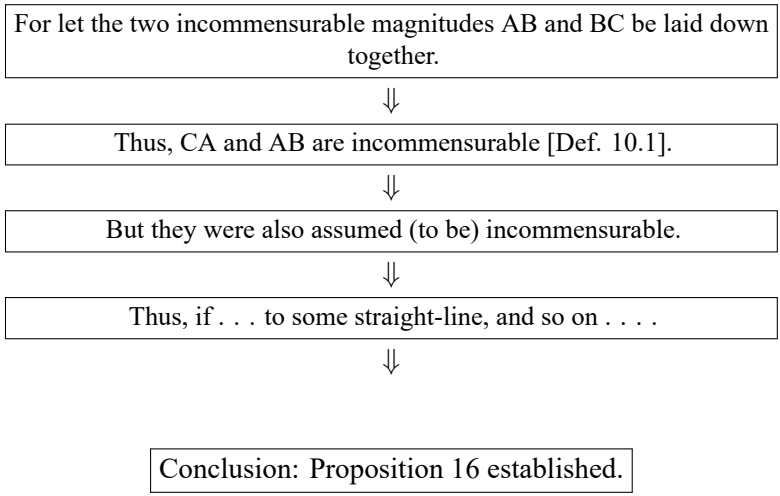
29. I say that AD is equal to the (rectangle contained) by AC and CB.

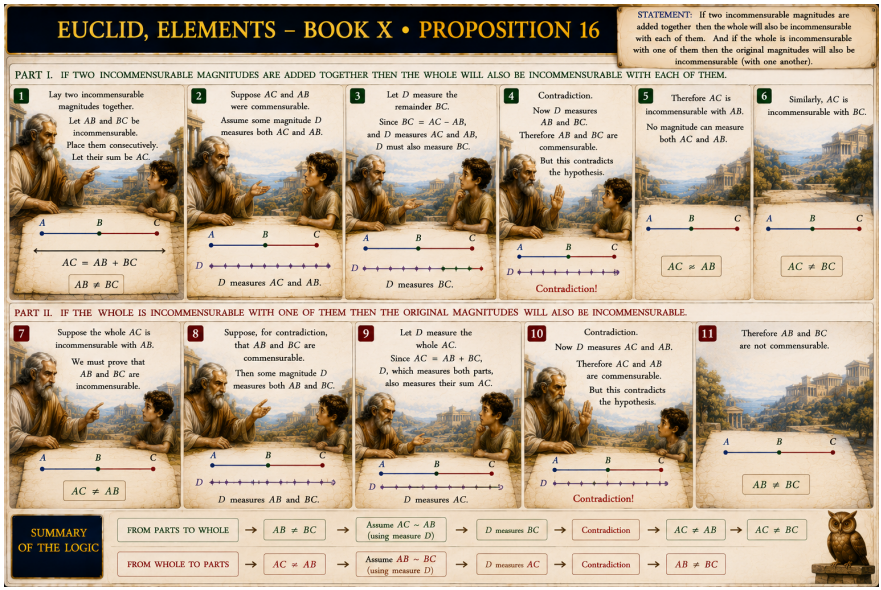
30. For since DB is a And it is immediately obvious. square, DC is equal to CB.

31. And AD is the (rectangle contained) by AC and CD—that is to say, by AC and CB.

32. Thus, if . . . to some straight-line, and so on . . . .

Complete Cinematic Logic





2.17 Proposition 17

Statement

> *If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.*

Proof Actions

1. Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater.

2. And let a (rectangle) equal to the fourth part of the (square) on the lesser, A—that is, (equal) to the (square) on half of A—falling short by a square figure, have been applied to BC.

3. And let it be the (rectangle contained) by BD and DC [see previous lemma].

4. And let BD be commensurable in length with DC.

5. I say that that the square on BC is greater than the (square on) A by (the square on some straight-line) commensurable (in length) with (BC).

6. For let BC have been cut in half at the point E [Prop. 1.10].

7. And let EF be made equal to DE [Prop. 1.3].

8. Thus, the remainder DC is equal to BF.

9. And since the straight-line BC has been cut into equal (pieces) at E, and into unequal (pieces) at D, the rectangle contained by BD and DC, plus the square on ED, is thus equal to the square on EC [Prop. 2.5].

10. (The same) also (for) the quadruples.

11. Thus, four times the (rectangle contained) by BD and DC, plus the quadruple of the (square) on DE, is equal to four times the square on EC.

12. But, the square on A is equal to the quadruple of the (rectangle contained) by BD and DC, and the square on DF is equal to the quadruple of the (square) on DE. For DF is double DE.

13. And the square on BC is equal to the quadruple of the (square) on EC.

14. For, again, BC is double CE.

15. Thus, the (sum of the) squares on A and DF is equal to the square on BC.

16. Hence, the (square) on BC is greater than the (square) on A by the (square) on DF.

17. Thus, BC is greater in square than A by DF.

18. It must also be shown that BC is commensurable (in length) with DF.

19. For since BD is commensurable in length with DC, BC is thus also commensurable in length with CD [Prop. 10.15].

20. But, CD is commensurable in length with CD plus BF.

21. For CD is equal to BF [Prop. 10.6].

22. Thus, BC is also commensurable in length with BF plus CD [Prop. 10.12].

23. Hence, BC is also commensurable in length with the remainder FD [Prop. 10.15].

24. Thus, the square on BC is greater than (the square on) A by the (square) on (some straight-line) commensurable (in length) with (BC).

25. And so let the square on BC be greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC).

26. And let a (rectangle) equal to the fourth (part) of the (square) on A, falling short by a square figure, have been applied to BC.

27. And let it be the (rectangle contained) by BD and DC.

28. It must be shown that BD is commensurable in length with DC.

29. For, similarly, by the same construction, we can show that the square on BC is greater than the (square on) A by the (square) on FD.

30. And the square on BC is greater than the (square on) A by the (square) on (some straightline) commensurable (in length) with (BC).

31. Thus, BC is commensurable in length with FD.

32. Hence, BC is also commensurable in length with the remaining sum of BF and DC [Prop. 10.15].

33. But, the sum of BF and DC is commensurable [in length] with DC [Prop. 10.6].

34. Hence, BC is also commensurable in length with CD [Prop. 10.12].

35. Thus, via separation, BD is also commensurable in length with DC [Prop. 10.15].

36. Thus, if there are two unequal straight-lines, and so on . . . . p α2 -β2 are commensurable when α -x DC, and β = A) then α and

Complete Cinematic Logic

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater.

⇓

And the square on BC is equal to the quadruple of the (square) on EC.

⇓

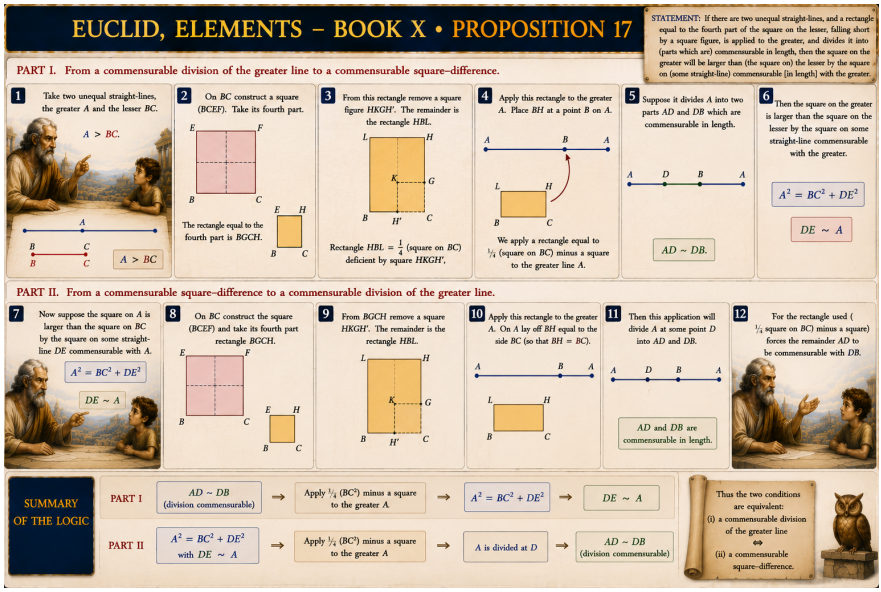
And so let the square on BC be greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC).

⇓

Thus, if there are two unequal straight-lines, and so on . . . . p α2 -β2 are commensurable when α -x DC, and β = A) then α and

⇓

Conclusion: Proposition 17 established.



2.18 Proposition 18

Statement

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Proof Actions

1. Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater.

2. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, A, falling short by a square figure, have been applied to BC.

3. And let it be the (rectangle contained) by BDC.

4. And let BD be incommensurable in length with DC.

5. I say that that the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).

6. For, similarly, by the same construction as before, we can show that the square on BC is greater than the (square on) A by the (square) on FD.

7. [Therefore] it must be shown that BC is incommensurable in length with DF.

8. For since BD is incommensurable in length with DC, BC is thus also incommensurable in length with CD [Prop. 10.16].

9. But, DC is commensurable (in length) with the sum of BF and DC [Prop. 10.6].

10. And, thus, BC is incommensurable (in length) with the sum of BF and DC [Prop. 10.13].

11. Hence, BC is also incommensurable in length with the remainder FD [Prop. 10.16].

12. And the square on BC is greater than the (square on) A by the (square) on FD.

13. Thus, the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).

14. So, again, let the square on BC be greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).

15. And let a (rectangle) equal to the fourth [part] of the (square) on A, falling short by a square figure, have been applied to BC.

16. And let it be the (rectangle contained) by BD and DC.

17. It must be shown that BD is incommensurable in length with DC.

18. For, similarly, by the same construction, we can show that the square on BC is greater than the (square) on A by the (square) on FD.

19. But, the square on BC is greater than the (square) on A by the (square) on (some straight-line) incommensurable (in length) with (BC).

20. Thus, BC is incommensurable in length with FD.

21. Hence, BC is also incommensurable (in length) with the remaining sum of BF and DC [Prop. 10.16].

22. But, the sum of BF and DC is commensurable in length with DC [Prop. 10.6].

23. Thus, BC is also incommensurable in length with DC [Prop. 10.13].

24. Hence, via separation, BD is also incommensurable in length with DC [Prop. 10.16].

25. Thus, if there are two . . . straight-lines, and so on . . . . p α2 -β2 are incommensurable when = DC, and β = A) then α and

Complete Cinematic Logic

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater.

⇓

But, DC is commensurable (in length) with the sum of BF and DC [Prop. 10.6].

⇓

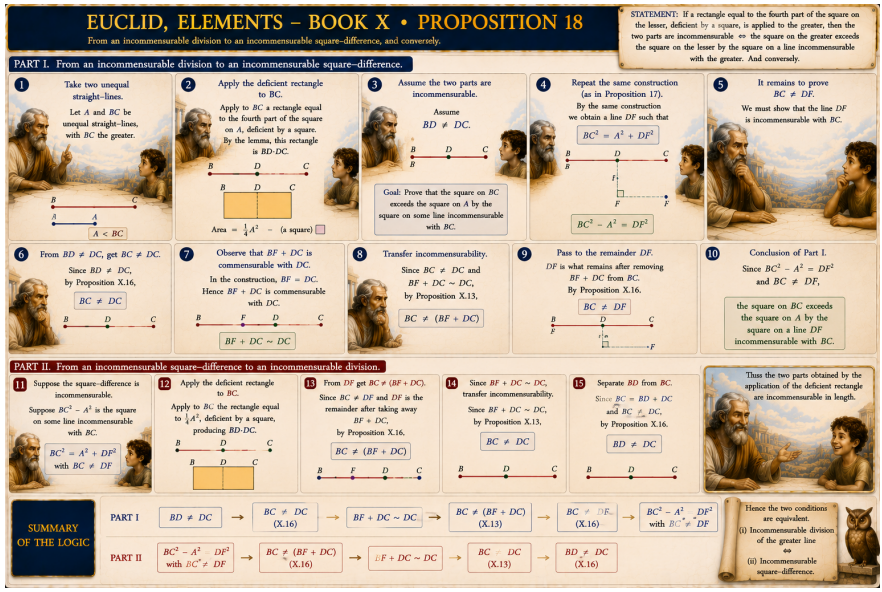
It must be shown that BD is incommensurable in length with DC.

⇓

Thus, if there are two . . . straight-lines, and so on . . . . p α2 -β2 are incommensurable when = DC, and β = A) then α and

⇓

Conclusion: Proposition 18 established.



2.19 Proposition 19

Statement

The rectangle contained by rational straight-lines (which are) commensurable in length is rational.

Proof Actions

1. For let the rectangle AC have been enclosed by the rational straight-lines AB and BC (which are) commensurable in length.

2. I say that AC is rational.

3. For let the square AD have been described on AB.

4. AD is thus rational [Def. 10.4].

5. And since AB is commensurable in length with BC, and AB is equal to BD, BD is thus commensurable in length with BC.

6. And as BD is to BC, so DA (is) to AC [Prop. 6.1].

7. Thus, DA is commensurable with AC [Prop. 10.11]. And DA (is) rational.

8. Thus, AC is also rational [Def. 10.4].

9. Thus, the . . . by rational straight-lines . . . commensurable, and so on . . . .

Complete Cinematic Logic

For let the rectangle AC have been enclosed by the rational straight-lines AB and BC (which are) commensurable in length.

⇓

AD is thus rational [Def. 10.4].

⇓

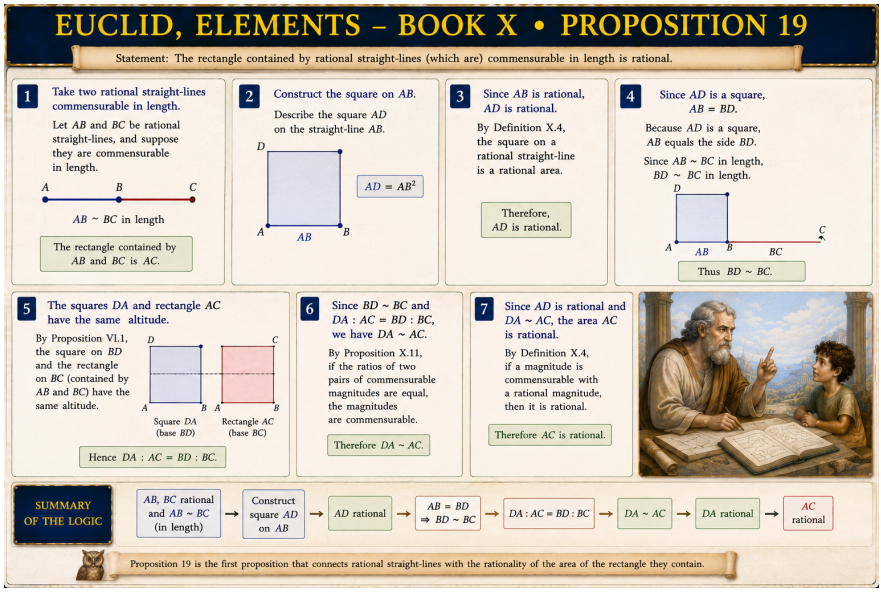
Thus, DA is commensurable with AC [Prop. 10.11]. And DA (is) rational.

⇓

Thus, the . . . by rational straight-lines . . . commensurable, and so on . . . .

⇓

Conclusion: Proposition 19 established.



2.20 Proposition 20

Statement

If a rational (area) is applied to a rational (straightline) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straightline) to which it is applied.

Proof Actions

1. For let the rational (area) AC have been applied to the rational (straight-line) AB, producing the (straight-line) BC as breadth.

2. I say that BC is rational, and commensurable in length with BA.

3. For let the square AD have been described on AB.

4. AD is thus rational [Def. 10.4].

5. And AC (is) also rational.

6. DA is thus commensurable with AC.

7. And as DA is to AC, so DB (is) to BC [Prop. 6.1].

8. Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA.

9. Thus, AB (is) also commensurable (in length) with BC. And AB is rational.

10. Thus, BC is also rational, and commensurable in length with AB [Def. 10.3].

11. Thus, if a rational (area) is applied to a rational (straight-line), and so on . . . .

Complete Cinematic Logic

For let the rational (area) AC have been applied to the rational (straight-line) AB, producing the (straight-line) BC as breadth.

⇓

AD is thus rational [Def. 10.4].

⇓

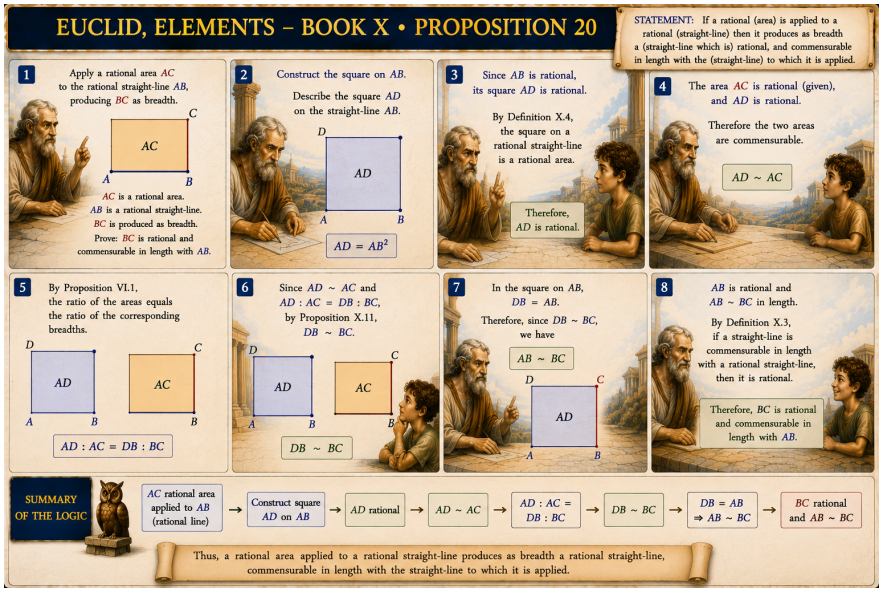
Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA.

⇓

Thus, if a rational (area) is applied to a rational (straight-line), and so on . . . .

⇓

Conclusion: Proposition 20 established.



2.21 Proposition 21

Statement

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational—let it be called medial.

Proof Actions

1. For let the rectangle AC be contained by the rational straight-lines AB and BC (which are) commensurable in square only.
2. I say that AC is irrational, and its squareroot is irrational—let it be called medial.
3. For let the square AD have been described on AB.
4. AD is thus rational [Def. 10.4].
5. And since AB is incommensurable in length with BC.
6. For they were assumed to be commensurable in square only. And AB (is) equal to BD.
7. DB is thus also incommensurable in length with BC.
8. And as DB is to BC, so AD (is) to AC [Prop. 6.1].
9. Thus, DA [is] incommensurable with AC [Prop. 10.11]. And DA (is) rational.
10. Thus, AC is irrational [Def. 10.4].
11. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]—let it be called medial.
12. (Which is) the very thing it was required to show.
13. Lemma If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.

14. Let FE and EG be two straight-lines.

15. I say that as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG.

16. For let the square DF have been described on FE.

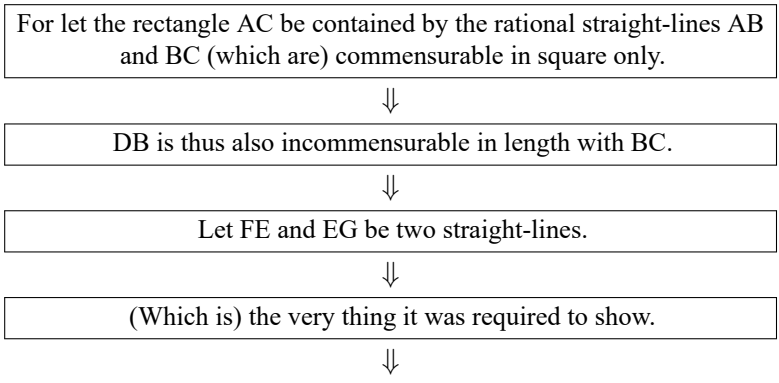
17. And let GD have been completed.

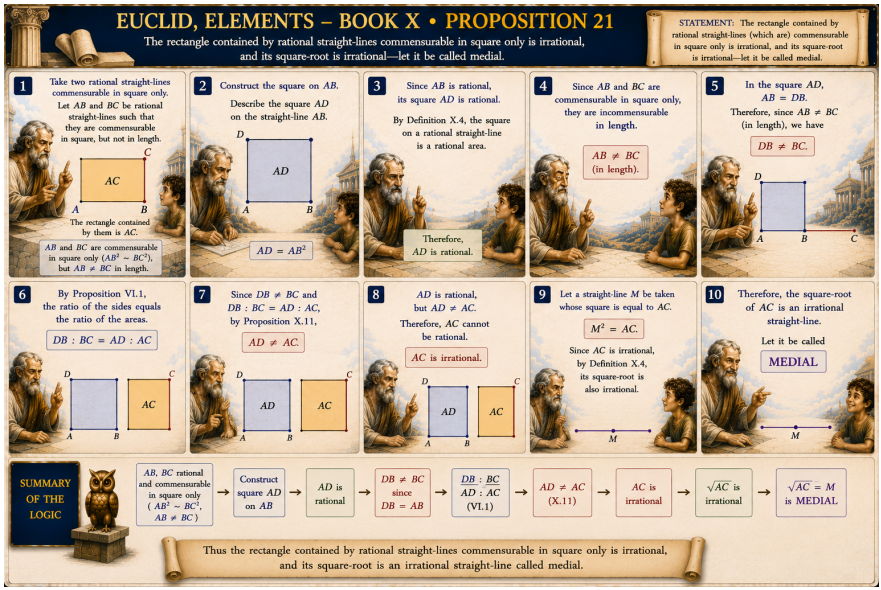
18. Therefore, since as FE is to EG, so FD (is) to DG [Prop. 6.1], and FD is the (square) on FE, and DG the (rectangle contained) by DE and EG—that is to say, the (rectangle contained) by FE and EG—thus as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG.

19. And also, similarly, as the (rectangle contained) by GE and EF is to the (square on) EF—that is to say, as GD (is) to FD—so GE (is) to EF.

20. (Which is) the very thing it was required to show.

Complete Cinematic Logic





2.22 Proposition 22

Statement

> *The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.*

Proof Actions

1. Let A be a medial (straight-line), and CB a rational (straight-line), and let the rectangular area BD, equal to the (square) on A, have been applied to BC, producing CD as breadth.

2. I say that CD is rational, and incommensurable in length with CB.

3. For since A is medial, the square on it is equal to a (rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21].

4. Let the square on (A) be equal to GF.

5. And the square on (A) is also equal to BD. Thus, BD is equal to GF.

6. And (BD) is also equiangular with (GF).

7. And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14].

8. Thus, proportionally, as BC is to EG, so EF (is) to CD.

9. And, also, as the (square) on BC is to the (square) on EG, so the (square) on EF (is) to the (square) on CD [Prop. 6.22].

10. And the (square) on CB is commensurable with the (square) on EG.

11. For they are each rational.

12. Thus, the (square) on EF is also commensurable with the (square) on CD [Prop. 10.11].

13. And the (square) on EF is rational.

14. Thus, the (square) on CD is also rational [Def. 10.4]. Thus, CD is rational.

15. And since EF is incommensurable in length with EG.

16. For they are commensurable in square only.

17. And as EF (is) to EG, so the (square) on EF (is) to the (rectangle contained) by FE and EG [see previous lemma].

18. The (square) on EF [is] thus incommensurable with the (rectangle contained) by FE and EG [Prop. 10.11].

19. But, the (square) on CD is commensurable with the (square) on EF.

20. For they are rational in square.

21. And the (rectangle contained) by DC and CB is commensurable with the (rectangle contained) by FE and EG.

22. For they are (both) equal to the (square) on A.

23. Thus, the (square) on CD is also incommensurable with the (rectangle contained) by DC and CB [Prop. 10.13].

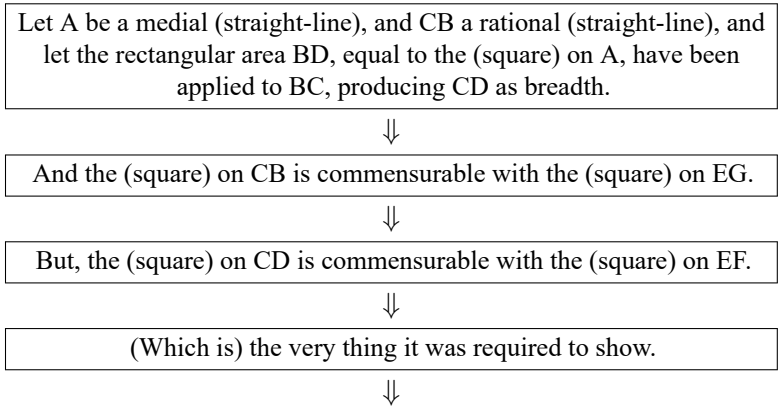
24. And as the (square) on CD (is) to the (rectangle contained) by DC and CB, so DC is to CB [see previous lemma].

25. Thus, DC is incommensurable in length with CB [Prop. 10.11].

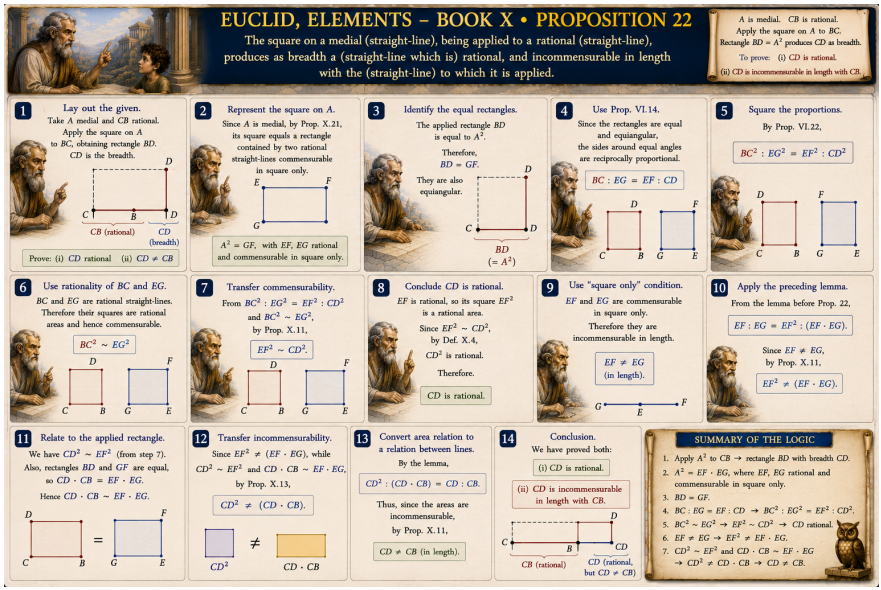
26. Thus, CD is rational, and incommensurable in length with CB.

27. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 22 established.



2.23 Proposition 23

Statement

> *A (straight-line) commensurable with a medial (straightline) is medial.*

Proof Actions

1. Let A be a medial (straight-line), and let B be commensurable with A.
2. I say that B is also a medial (staightline).
3. Let the rational (straight-line) CD be set out, and let the rectangular area CE, equal to the (square) on A, have been applied to CD, producing ED as width.
4. ED is thus rational, and incommensurable in length with CD [Prop. 10.22].
5. And let the rectangular area CF, equal to the (square) on B, have been applied to CD, producing DF as width.

6. Therefore, since A is commensurable with B, the (square) on A is also commensurable with the (square) on B.

7. But, EC is equal to the (square) on A, and CF is equal to the (square) on B.

8. Thus, EC is commensurable with CF.

9. And as EC is to CF, so ED (is) to DF [Prop. 6.1].

10. Thus, ED is commensurable in length with DF [Prop. 10.11].

11. And ED is rational, and incommensurable in length with CD.

12. DF is thus also rational [Def. 10.3], and incommensurable in length with DC [Prop. 10.13].

13. Thus, CD and DF are rational, and commensurable in square only.

14. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21].

15. Thus, the square-root of the (rectangle contained) by CD and DF is medial.

16. And the square on B is equal to the (rectangle contained) by CD and DF.

17. Thus, B is a medial (straight-line).

18. Corollary And (it is) clear, from this, that an (area) commensurable with a medial area is medial. a medial area is expressible as k1/2.

Complete Cinematic Logic

Let A be a medial (straight-line), and let B be commensurable with A.

⇓

But, EC is equal to the (square) on A, and CF is equal to the (square) on B.

⇓

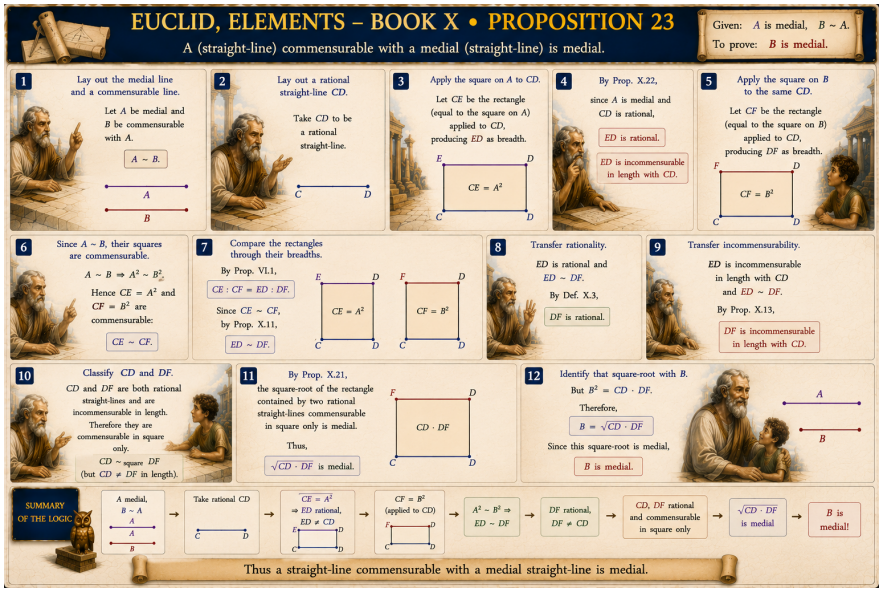
Thus, CD and DF are rational, and commensurable in square only.

⇓

Corollary And (it is) clear, from this, that an (area) commensurable with a medial area is medial. a medial area is expressible as k1/2.

⇓

Conclusion: Proposition 23 established.



2.24 Proposition 24

Statement

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

Proof Actions

1. For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in length. I say that AC is medial.

2. For let the square AD have been described on AB.

3. AD is thus medial [see previous footnote].

4. And since AB is commensurable in length with BC, and AB (is) equal to BD, DB is thus also commensurable in length with BC.

5. Hence, DA is also commensurable with AC [Props. 6.1, 10.11]. And DA (is) medial.

6. Thus, AC (is) also medial [Prop. 10.23 corr.].

7. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in length. I say that AC is medial.

⇓

AD is thus medial [see previous footnote].

⇓

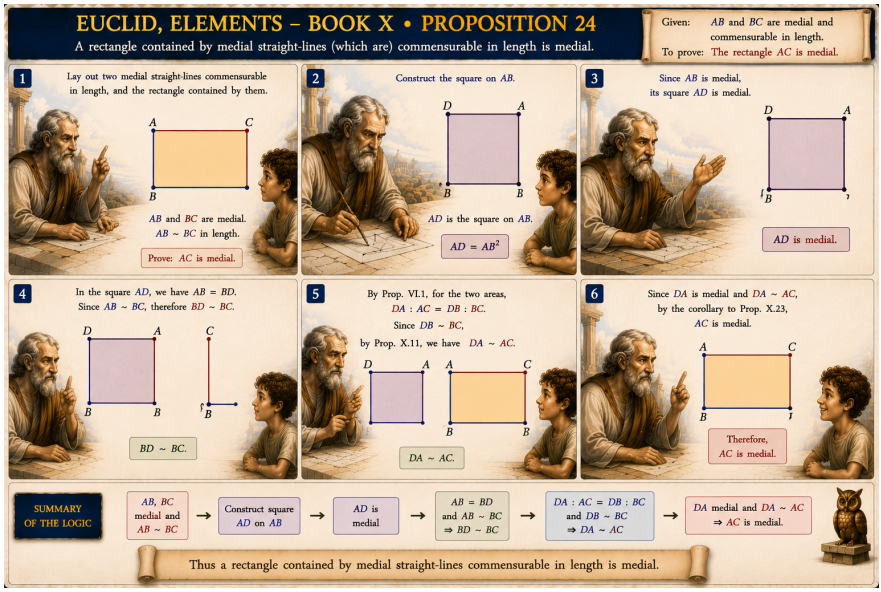
Hence, DA is also commensurable with AC [Props. 6.1, 10.11]. And DA (is) medial.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 24 established.



2.25 Proposition 25

Statement

The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.

Proof Actions

1. For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in square only.
2. I say that AC is either rational or medial.
3. For let the squares AD and BE have been described on (the straight-lines) AB and BC (respectively).
4. AD and BE are thus each medial.
5. And let the rational (straight-line) FG be laid out.
6. And let the rectangular parallelogram GH, equal to AD, have been applied to FG, producing FH as breadth.
7. And let the rectangular parallelogram MK, equal to AC, have been applied to HM, producing HK as breadth.
8. And, finally, let NL, equal to BE, have similarly been applied to KN, producing KL as breadth.
9. Thus, FH, HK, and KL are in a straight-line.
10. Therefore, since AD and BE are each medial, and AD is equal to GH, and BE to NL, GH and NL (are) thus each also medial.
11. And they are applied to the rational (straight-line) FG.
12. FH and KL are thus each rational, and incommensurable in length with FG [Prop. 10.22].
13. And since AD is commensurable with BE, GH is thus also commensurable with NL.

14. And as GH is to NL, so FH (is) to KL [Prop. 6.1].

15. Thus, FH is commensurable in length with KL [Prop. 10.11].

16. Thus, FH and KL are rational (straight-lines which are) commensurable in length.

17. Thus, the (rectangle contained) by FH and KL is rational [Prop. 10.19].

18. And since DB is equal to BA, and OB to BC, thus as DB is to BC, so AB (is) to BO.

19. But, as DB (is) to BC, so DA (is) to AC [Props. 6.1].

20. And as AB (is) to BO, so AC (is) to CO [Prop. 6.1].

21. Thus, as DA is to AC, so AC (is) to CO.

22. And AD is equal to GH, and AC to MK, and CO to NL.

23. Thus, as GH is to MK, so MK (is) to NL.

24. Thus, also, as FH is to HK, so HK (is) to KL [Props. 6.1, 5.11].

25. Thus, the (rectangle contained) by FH and KL is equal to the (square) on HK [Prop. 6.17].

26. And the (rectangle contained) by FH and KL (is) rational.

27. Thus, the (square) on HK is also rational. Thus, HK is rational.

28. And if it is commensurable in length with FG then HN is rational [Prop. 10.19].

29. And if it is incommensurable in length with FG then KH and HM are rational (straight-lines which are) commensurable in square only: thus, HN is medial [Prop. 10.21].

30. Thus, HN is either rational or medial. And HN (is) equal to AC.

31. Thus, AC is either rational or medial.

32. Thus, the . . . by medial straight-lines (which are) commensurable in square only, and so on . . . .

Complete Cinematic Logic

For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in square only.

⇓

And they are applied to the rational (straight-line) FG.

⇓

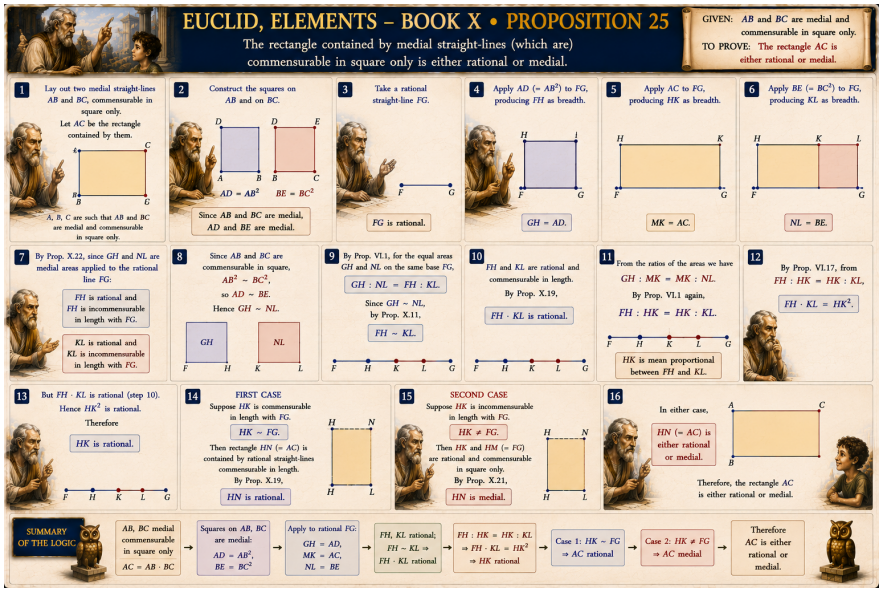
And AD is equal to GH, and AC to MK, and CO to NL.

⇓

Thus, the . . . by medial straight-lines (which are) commensurable in square only, and so on . . . .

⇓

Conclusion: Proposition 25 established.



2.26 Proposition 26

Statement

A medial (area) does not exceed a medial (area) by a rational (area).

Proof Actions

1. For, if possible, let the medial (area) AB exceed the medial (area) AC by the rational (area) DB.
2. And let the rational (straight-line) EF be laid down.
3. And let the rectangular parallelogram FH, equal to AB, have been applied to to EF, producing EH as breadth.
4. And let FG, equal to AC, have been cut off (from FH).
5. Thus, the remainder BD is equal to the remainder KH. And DB is rational.
6. Thus, KH is also rational.
7. Therefore, since AB and AC are each medial, and AB is equal to FH, and AC to FG, FH and FG are thus each also medial.
8. And they are applied to the rational (straight-line) EF.
9. Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22].
10. And since DB is rational, and is equal to KH, KH is thus also rational.
11. And (KH) is applied to the rational (straight-line) EF.
12. GH is thus rational, and commensurable in length with EF [Prop. 10.20].
13. But, EG is also rational, and incommensurable in length with EF.
14. Thus, EG is incommensurable in length with GH [Prop. 10.13].
15. And as EG is to GH, so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.].

16. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11].

17. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG.

18. For (EG and GH are) both rational.

19. And twice the (rectangle contained) by EG and GH is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6].

20. For (the former) is double the latter.

21. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13].

22. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH, that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16].

23. And the (sum of the squares) on EG and GH (is) rational.

24. Thus, the (square) on EH is irrational [Def. 10.4].

25. Thus, EH is irrational [Def. 10.4].

26. But, (it is) also rational.

27. The very thing is impossible.

28. Thus, a medial (area) does not exceed a medial (area) by a rational (area).

29. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For, if possible, let the medial (area) AB exceed the medial (area) AC by the rational (area) DB.

⇓

And since DB is rational, and is equal to KH, KH is thus also rational.

⇓

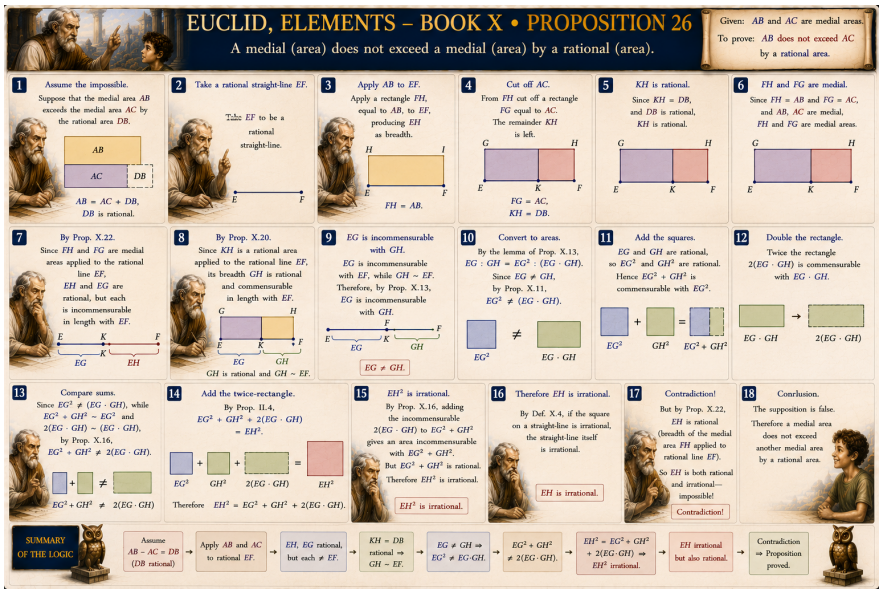
For (the former) is double the latter.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 26 established.



2.27 Proposition 27

Statement

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.

Proof Actions

1. Let the two rational (straight-lines) A and B, (which are) commensurable in square only, be laid down.
2. And let C—the mean proportional (straight-line) to A and B— have been taken [Prop. 6.13].
3. And let it be contrived that as A (is) to B, so C (is) to D [Prop. 6.12].
4. And since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on C [Prop. 6.17]—is thus medial [Prop 10.21].
5. Thus, C is medial [Prop. 10.21].
6. And since as A is to B, [so] C (is) to D, and A and B [are] commensurable in square only, C and D are thus also commensurable in square only [Prop. 10.11]. And C is medial.
7. Thus, D is also medial [Prop. 10.23].
8. Thus, C and D are medial (straight-lines which are) commensurable in square only.
9. I say that they also contain a rational (area).
10. For since as A is to B, so C (is) to D, thus, alternately, as A is to C, so B (is) to D [Prop. 5.16].
11. But, as A (is) to C, (so) C (is) to B.
12. And thus as C (is) to B, so B (is) to D [Prop. 5.11].
13. Thus, the (rectangle contained) by C and D is equal to the (square) on B [Prop. 6.17].

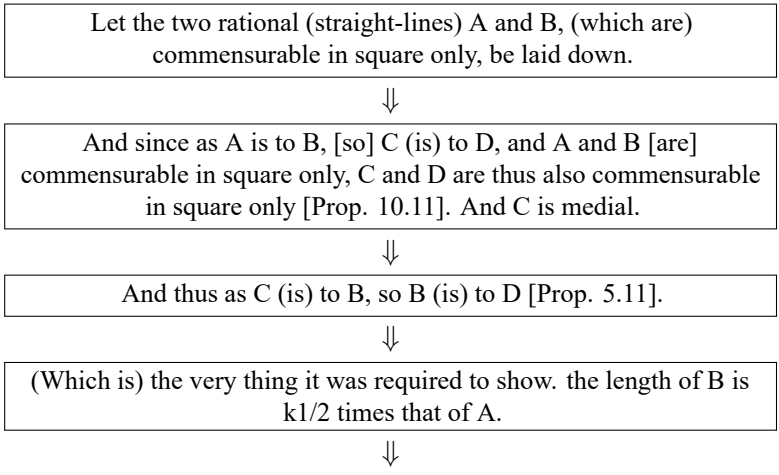
14. And the (square) on B (is) rational.

15. Thus, the (rectangle contained) by C and D [is] also rational.

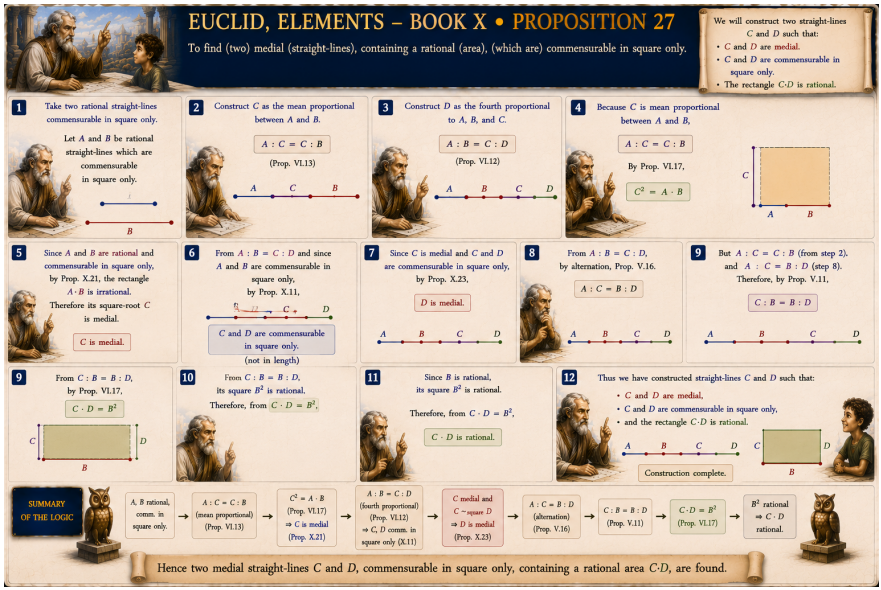
16. Thus, (two) medial (straight-lines, C and D), containing a rational (area), (which are) commensurable in square only, have been found.

17. (Which is) the very thing it was required to show. the length of B is k1/2 times that of A.

Complete Cinematic Logic



Conclusion: Proposition 27 established.



2.28 Proposition 28

Statement

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.

Proof Actions

1. Let the [three] rational (straight-lines) A, B, and C, (which are) commensurable in square only, be laid down.

2. And let, D, the mean proportional (straight-line) to A and B, have been taken [Prop. 6.13].

3. And let it be contrived that as B (is) to C, (so) D (is) to E [Prop. 6.12].

4. Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on D [Prop. 6.17]— is medial [Prop. 10.21].

5. Thus, D (is) medial [Prop. 10.21].

6. And since B and C are commensurable in square only, and as B is to C, (so) D (is) to E, D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial.

7. E (is) thus also medial [Prop. 10.23].

8. Thus, D and E are medial (straight-lines which are) commensurable in square only.

9. So, I say that they also enclose a medial (area).

10. For since as B is to C, (so) D (is) to E, thus, alternately, as B (is) to D, (so) C (is) to E [Prop. 5.16].

11. And as B (is) to D, (so) D (is) to A.

12. And thus as D (is) to A, (so) C (is) to E.

13. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16].

14. And the (rectangle contained) by A and C is medial [Prop. 10.21].

15. Thus, the (rectangle contained) by D and E (is) also medial.

16. Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found.

17. (Which is) the very thing it was required to show. where the lengths of B and C are k1/2 and k′1/2 times that of A, Lemma I To find two square numbers such that the sum of them is also square.

18. Let the two numbers AB and BC be laid down.

19. And let them be either (both) even or (both) odd.

20. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even.

21. Let AC have been cut in half at D.

22. And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers).

23. Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6].

24. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1].

25. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on BD.

26. And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD, and the (square) on CD—such that their difference—(namely,) the (rectangle) contained by AB and BC—is square whenever AB and BC are similar plane (numbers).

27. But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD, and the (square) on DC—between which the difference—(namely,) the (rectangle) contained by AB and BC—is not square.

28. (Which is) the very thing it was required to show.

29. Lemma II To find two square numbers such that the sum of them is not square.

30. For let the (number created) from (multiplying) AB and BC, as we said, be square. And (let) CA (be) even.

31. And let CA have been cut in half at D.

32. So it is clear that the square (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [see previous lemma].

33. Let the unit DE have been subtracted (from BD).

34. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is less than the square on BD.

35. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is not square.

36. For if it is square, it is either equal to the (square) on BE, or less than the (square) on BE, but cannot any more be greater (than the square on BE), lest the unit be divided.

37. First of all, if possible, let the (number created) from (multiplying) AB and BC, plus the (square) on CE, be equal to the (square) on BE.

38. And let GA be double the unit DE.

39. Therefore, since the whole of AC is double the whole of CD, of which AG is double DE, the remainder GC is thus double the remainder EC.

40. Thus, GC has been cut in half at E.

41. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the square on BE [Prop. 2.6].

42. But, the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the square on BE.

43. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE.

44. And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB.

45. The very thing is absurd.

46. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to the (square) on BE.

47. So I say that (it is) not less than the (square) on BE either.

48. For, if possible, let it be equal to the (square) on BF.

49. And (let) HA (be) double DF.

50. And it can again be inferred that HC (is) double CF.

51. Hence, CH has also been cut in half at F.

52. And, on account of this, the (number created) from (multiplying) HB and BC, plus the (square) on FC, becomes equal to the (square) on BF [Prop. 2.6].

53. And the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the (square) on BF.

54. Hence, the (number created) from (multiplying) HB and BC, plus the (square) on CF, will also be equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE.

55. The very thing is absurd.

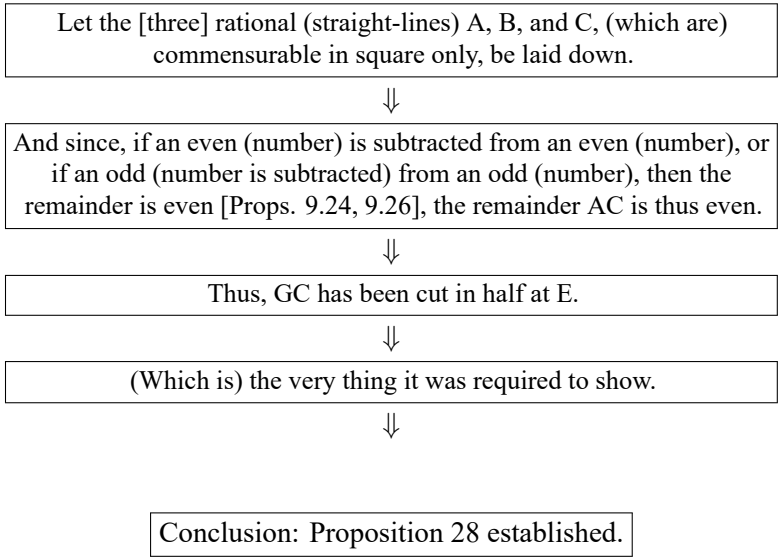
56. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to less than the (square) on BE.

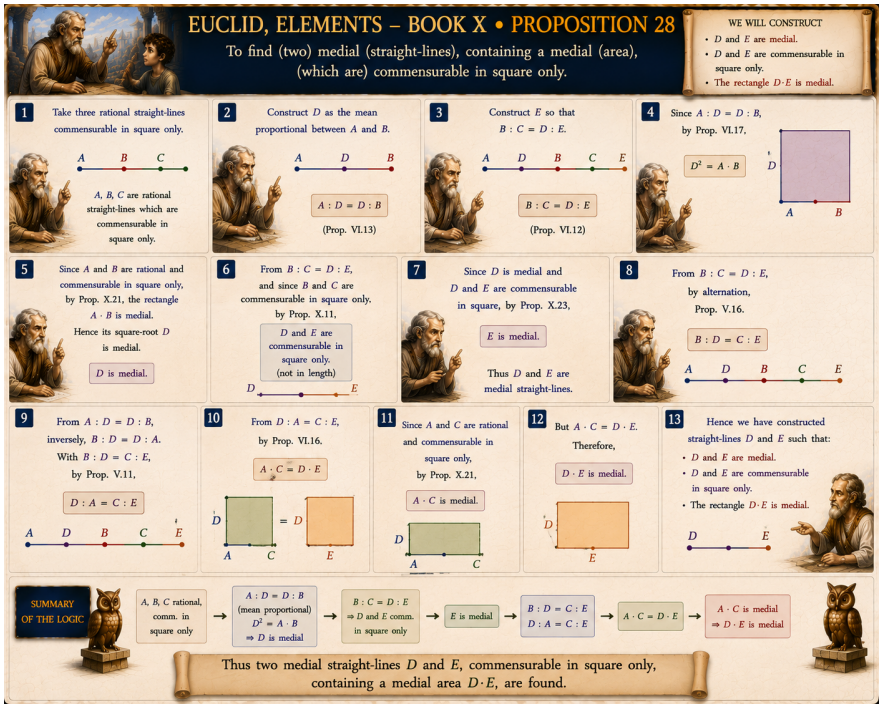
57. And it was shown that (is it) not equal to the (square) on BE either.

58. Thus, the (number created) from (multiplying) AB and BC, plus the square on CE, is not square.

59. (Which is) the very thing it was required to show.

Complete Cinematic Logic





2.29 Proposition 29

Statement

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.

Proof Actions

1. For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE, such that the difference between them, CE, is not square [Prop. 10.28 lem. I].

2. And let the semi-circle AFB have been drawn on AB.

3. And let it be contrived that as DC (is) to CE, so the square on BA (is) to the square on AF [Prop. 10.6 corr.].

4. And let FB have been joined.

5. [Therefore,] since as the (square) on BA is to the (square) on AF, so DC (is) to CE, the (square) on BA thus has to the (square) on AF the ratio which the number DC (has) to the number CE.

6. Thus, the (square) on BA is commensurable with the (square) on AF [Prop. 10.6].

7. And the (square) on AB (is) rational [Def. 10.4].

8. Thus, the (square) on AF (is) also ratioThus, AF (is) also rational.

9. And since DC does nal. not have to CE the ratio which (some) square number (has) to (some) square number, the (square) on BA thus does not have to the (square) on AF the ratio which (some) square number has to (some) square number either.

10. Thus, AB is incommensurable in length with AF [Prop. 10.9].

11. Thus, the rational (straight-lines) BA and AF are commensurable in square only.

12. And since as DC [is] to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47].

13. And CD has to DE the ratio which (some) square number (has) to (some) square number.

14. Thus, the (square) on AB also has to the (square) on BF the ratio which (some) square number has to (some) square number.

15. AB is thus commensurable in length with BF [Prop. 10.9].

16. And the (square) on AB is equal to the (sum of the squares) on AF and FB [Prop. 1.47].

17. Thus, the square on AB is greater than (the square on) AF by (the square on) BF, (which is) commensurable (in length) with (AB).

18. Thus, two rational (straight-lines), BA and AF, commensurable in square only, have been found such that the square on the greater, AB, is larger than (the square on) the lesser, AF, by the (square) on BF, (which is) commensurable in length with (AB).

19. (Which is) the very thing it was required to show. p where k = DE/CD.

Complete Cinematic Logic

For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE, such that the difference between them, CE, is not square [Prop. 10.28 lem. I].

⇓

And the (square) on AB (is) rational [Def. 10.4].

⇓

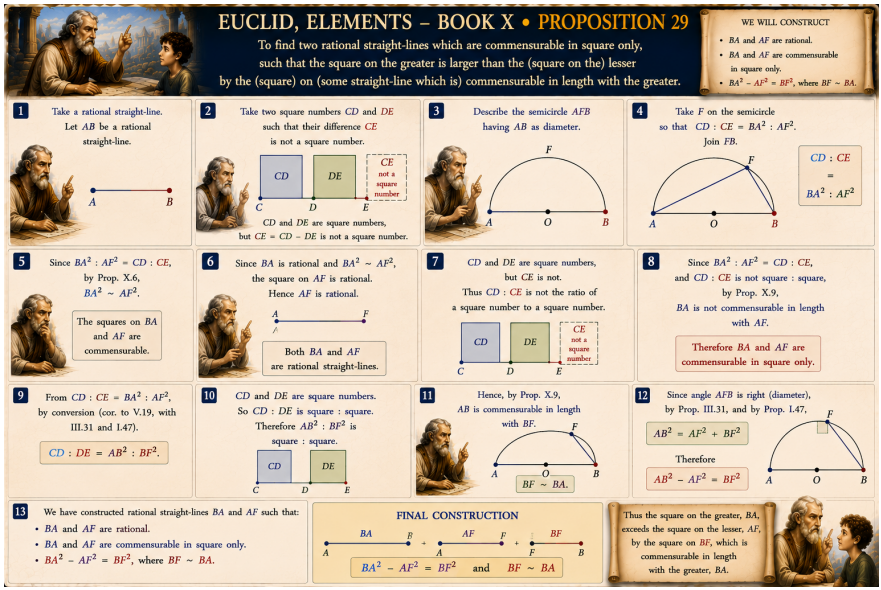
And CD has to DE the ratio which (some) square number (has) to (some) square number.

⇓

(Which is) the very thing it was required to show. p where k = DE/CD.

⇓

Conclusion: Proposition 29 established.



2.30 Proposition 30

Statement

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.

Proof Actions

1. Let the rational (straight-line) AB be laid out, and the two square numbers, CE and ED, such that the sum of them, CD, is not square [Prop. 10.28 lem. II].

2. And let the semi-circle AFB have been drawn on AB.

3. And let it be contrived that as DC (is) to CE, so the (square) on BA (is) to the (square) on AF [Prop. 10.6 corr].

4. And let FB have been joined.

5. So, similarly to the (proposition) before this, we can show that BA and AF are rational (straight-lines which are) commensurable in square only.

6. And since as DC is to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, And CD does not have to DE the ratio which 1.47].

7. (some) square number (has) to (some) square number.

8. Thus, the (square) on AB does not have to the (square) on BF the ratio which (some) square number has to (some) square number either.

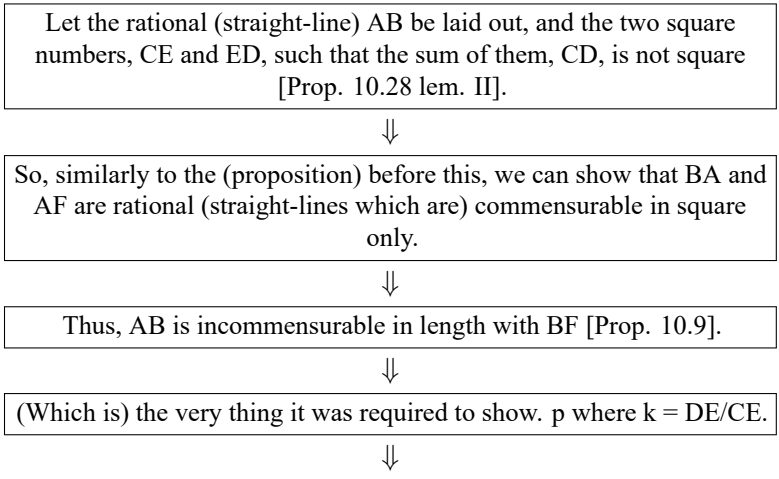
9. Thus, AB is incommensurable in length with BF [Prop. 10.9].

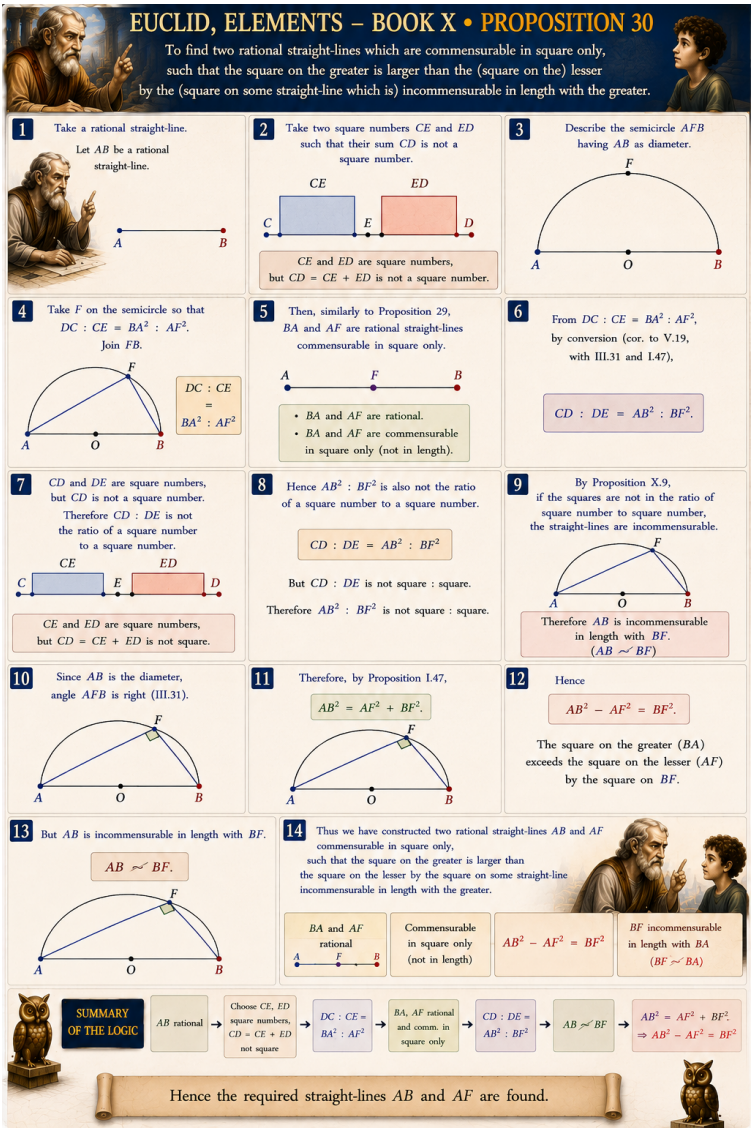
10. And the square on AB is greater than the (square on) AF by the (square) on FB [Prop. 1.47], (which is) incommensurable (in length) with (AB).

11. Thus, AB and AF are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than (the square on) AF by the (square) on FB, (which is) incommensurable (in length) with (AB).

12. (Which is) the very thing it was required to show. p where k = DE/CE.

Complete Cinematic Logic





2.31 Proposition 31

Statement

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square

on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.

Proof Actions

1. Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29].

2. And let the (square) on C be equal to the (rectangle contained) by A and B.

3. And the (rectangle contained by) A and B (is) medial [Prop. 10.21].

4. Thus, the (square) on C (is) also medial.

5. Thus, C (is) also medial [Prop. 10.21].

6. And let the (rectangle contained) by C and D be equal to the (square) on B.

7. And the (square) on B (is) rational.

8. Thus, the (rectangle contained) by C and D (is) also rational.

9. And since as A is to B, so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B, and the (rectangle contained) by C and D to the (square) on B, thus as A (is) to B, so the (square) on C (is) to the (rectangle contained) by C and D.

10. And as the (square) on C (is) to the (rectangle contained) by C and D, so C (is) to D [Prop. 10.21 lem.].

11. And thus as A (is) to B, so C (is) to D.

12. And A is commensurable in square only with B.

13. Thus, C (is) also commensurable in square only with D [Prop. 10.11]. And C is medial.

14. Thus, D (is) also medial [Prop. 10.23].

15. And since as A is to B, (so) C (is) to D, and the square on A is greater than (the square on) B by the (square) on (some straight-line) commensurable (in length) with (A), the square on C is thus also greater than (the square on) D by the (square) on (some straight-line) commensurable (in length) with (C) [Prop. 10.14].

16. Thus, two medial (straight-lines), C and D, commensurable in square only, (and) containing a rational (area), have been found.

17. And the square on C is greater than (the square on) D by the (square) on (some straight-line) commensurable in length with (C).

18. So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with C), provided that the square on A is greater than (the square on B) by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30]. respectively, where k is defined in the footnote to Prop. 10.29. that of A, respectively, where k is defined in the footnote to Prop. 10.30.

Complete Cinematic Logic

Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29].

⇓

And the (square) on B (is) rational.

⇓

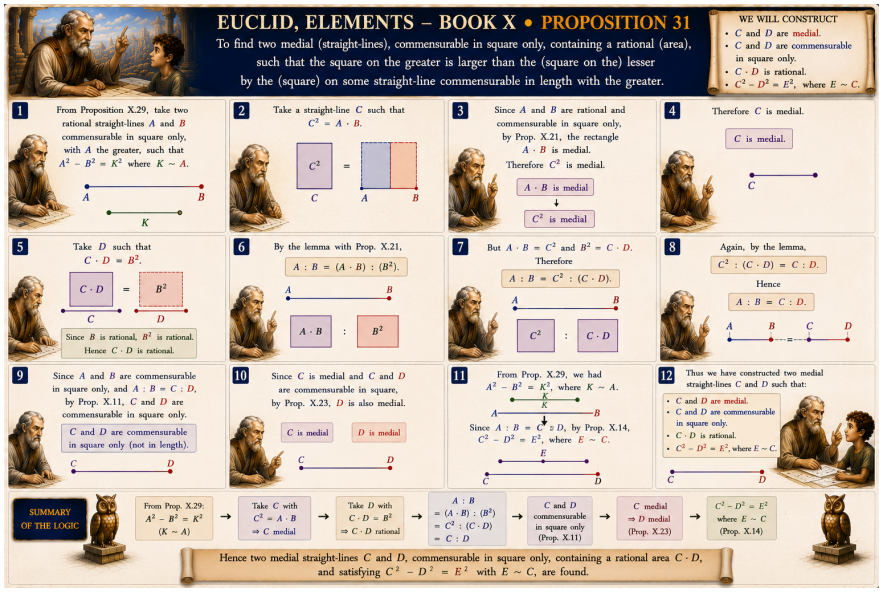
Thus, C (is) also commensurable in square only with D [Prop. 10.11]. And C is medial.

⇓

So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with C), provided that the square on A is greater than (the square on B) by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30]. respectively, where k is defined in the fo...

⇓

Conclusion: Proposition 31 established.



2.32 Proposition 32

Statement

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.

Proof Actions

1. Let three rational (straight-lines), A, B and C, commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29].

2. And let the (square) on D be equal to the (rectangle contained) by A and B.

3. Thus, the (square) on D (is) medial.

4. Thus, D is also medial [Prop. 10.21].

5. And let the (rectangle contained) by D and E be equal to the (rectangle contained) by B and C.

6. And since as the (rectangle contained) by A and B is to the (rectangle contained) by B and C, so A (is) to C [Prop. 10.21 lem.], but the (square) on D is equal to the (rectangle contained) by A and B, and the (rectangle contained) by D and E to the (rectangle contained) by B and C, thus as A is to C, so the (square) on D (is) to the (rectangle contained) by D and E.

7. And as the (square) on D (is) to the (rectangle contained) by D and E, so D (is) to E [Prop. 10.21 lem.].

8. And thus as A (is) to C, so D (is) to E.

9. And A (is) commensurable in square [only] with C.

10. Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial.

11. Thus, E (is) also medial [Prop. 10.23].

12. And since as A is to C, (so) D (is) to E, and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14].

13. So, I also say that the (rectangle contained) by D and E is medial.

14. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E, and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

15. Thus, two medial (straight-lines), D and E, commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater..

16. So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30]. where the length of B is k′1/2 times that of A, and k is defined in A, respectively, where the length of B is k′1/2 times that of A, and k is Lemma Let ABC be a right-angled triangle having the (angle) A a right-angle.

17. And let the perpendicular AD have been drawn.

18. I say that the (rectangle contained) by CBD is equal to the (square) on BA, and the (rectangle contained) by BCD (is) equal to the (square) on CA, and the (rectangle contained) by BD and DC (is) equal to the (square) on AD, and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC.

19. And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA.

20. For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABD and ADC are thus triangles (which

are) similar to the whole, ABC, and to one another [Prop. 6.8].

21. And since triangle ABC is similar to triangle ABD, thus as CB is to BA, so BA (is) to BD [Prop. 6.4].

22. Thus, the (rectangle contained) by CBD is equal to the (square) on AB [Prop. 6.17].

23. So, for the same (reasons), the (rectangle contained) by BCD is also equal to the (square) on AC.

24. And since if a (straight-line) is drawn from the rightangle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as BD is to DA, so AD (is) to DC.

25. Thus, the (rectangle contained) by BD and DC is equal to the (square) on DA [Prop. 6.17].

26. I also say that the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC.

27. For since, as we said, ABC is similar to ABD, thus as BC is to CA, so BA (is) to AD [Prop. 6.4].

28. Thus, the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC [Prop. 6.16].

29. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let three rational (straight-lines), A, B and C, commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29].

⇓

Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial.

⇓

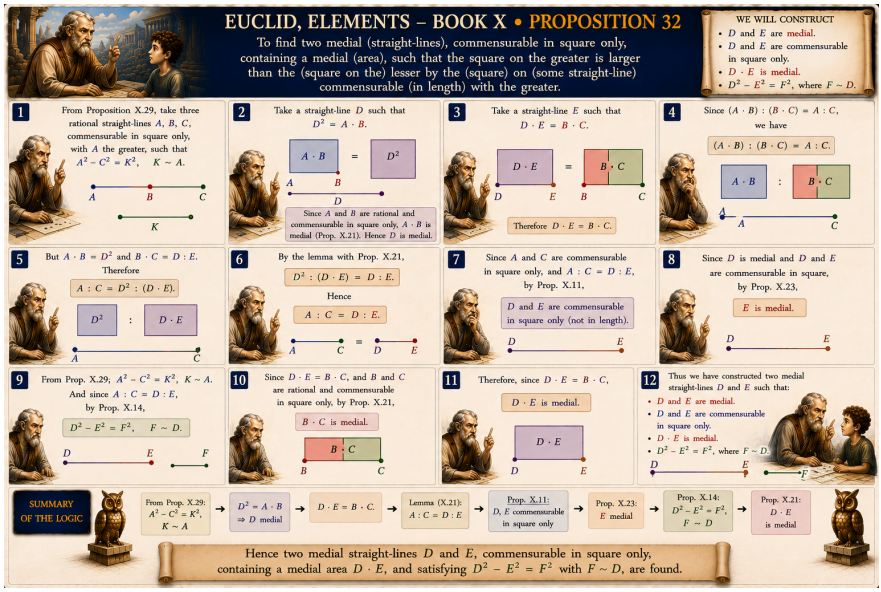
For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABD and ADC are thus triangles (which are) similar to the whole, ABC, and to one another [Prop. 6.8].

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 32 established.



2.33 Proposition 33

Statement

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Proof Actions

1. Let the two rational (straight-lines) AB and BC, (which are) commensurable in square only, be laid out such that the square on the greater, AB, is larger than (the square on) the lesser, BC, by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30].

2. And let BC have been cut in half at D.

3. And let a parallelogram equal to the (square) on ei317 ther of BD or DC, (and) falling short by a square figure, have been applied to AB [Prop. 6.28], and let it be the (rectangle contained) by AEB.

4. And let the semi-circle AFB have been drawn on AB.

5. And let EF have been drawn at right-angles to AB.

6. And let AF and FB have been joined.

7. And since AB and BC are [two] unequal straightlines, and the square on AB is greater than (the square on) BC by the (square) on (some straight-line which is) incommensurable (in length) with (AB).

8. And a parallelogram, equal to one quarter of the (square) on BC— that is to say, (equal) to the (square) on half of it—(and) falling short by a square figure, has been applied to AB, and makes the (rectangle contained) by AEB.

9. AE is thus incommensurable (in length) with EB [Prop. 10.18].

10. And as AE is to EB, so the (rectangle contained) by BA and AE (is) to the (rectangle contained) by AB and BE.

11. And the (rectangle contained) by BA and AE (is) equal to the (square) on AF, and the (rectangle contained) by AB and BE to the (square) on BF [Prop. 10.32 lem.].

12. The (square) on AF is thus incommensurable with the (square) on FB [Prop. 10.11].

13. Thus, AF and FB are incommensurable in square.

14. And since AB is rational, the (square) on AB is also rational.

15. Hence, the sum of the (squares) on AF and FB is also rational [Prop. 1.47].

16. And, again, since the (rectangle contained) by AE and EB is equal to the (square) on EF, and the (rectangle contained) by AE and EB was assumed (to be) equal to the (square) on BD, FE is thus equal to BD. Thus, BC is double FE.

17. And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6].

18. And the (rectangle contained) by AB and BC (is) medial [Prop. 10.21].

19. Thus, the (rectangle contained) by AB and EF (is) also medial [Prop. 10.23 corr.].

20. And the (rectangle contained) by AB and EF (is) equal to the (rectangle contained) by AF and FB [Prop. 10.32 lem.].

21. Thus, the (rectangle contained) by AF and FB (is) also medial.

22. And the sum of the squares on them was also shown (to be) rational.

23. Thus, the two straight-lines, AF and FB, (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

24. (Which is) the very thing it was required to show. + k2)1/2]/2 times that of AB, respectively, where k is defined in the

Complete Cinematic Logic

Let the two rational (straight-lines) AB and BC, (which are) commensurable in square only, be laid out such that the square on the greater, AB, is larger than (the square on) the lesser, BC, by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30].

⇓

AE is thus incommensurable (in length) with EB [Prop. 10.18].

⇓

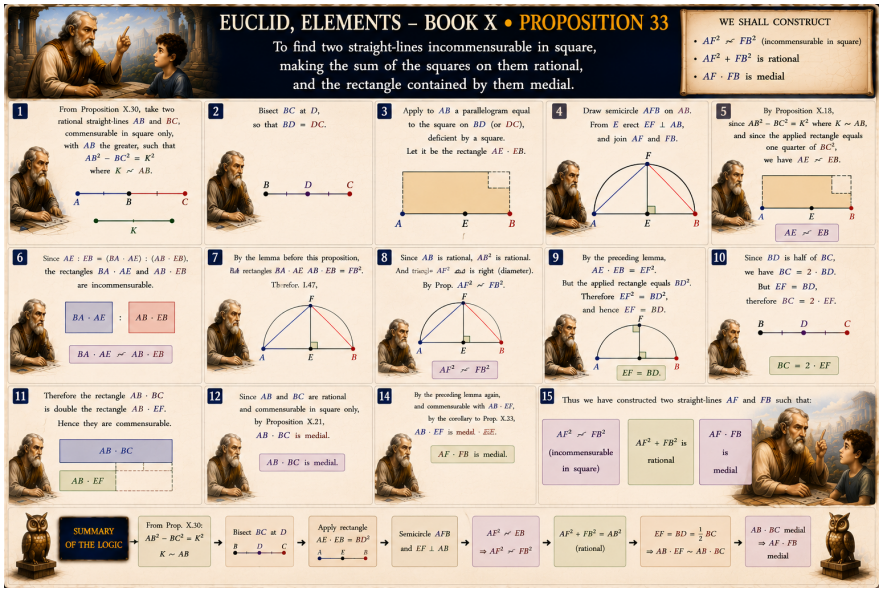
And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6].

⇓

(Which is) the very thing it was required to show. + k2)1/2]/2 times that of AB, respectively, where k is defined in the

⇓

Conclusion: Proposition 33 established.



2.34 Proposition 34

Statement

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Proof Actions

1. Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.31].

2. And let the semi-circle ADB have been drawn on AB.

3. And let BC have been cut in half at E.

4. And let a (rectangular) parallelogram equal to the (square) on BE, (and) falling short by a square figure, have been applied to AB, (and let it be) the (rectangle contained by) AFB [Prop. 6.28].

5. Thus, AF [is] incommensurable in length with FB [Prop. 10.18].

6. And let FD have been drawn from F at right-angles to AB.

7. And let AD and DB have been joined.

8. Since AF is incommensurable (in length) with FB, the (rectangle contained) by BA and AF is thus also incommensurable with the (rectangle contained) by AB and BF [Prop. 10.11].

9. And the (rectangle contained) by BA and AF (is) equal to the (square) on AD, and the (rectangle contained) by AB and BF to the (square) on DB [Prop. 10.32 lem.].

10. Thus, the (square) on AD is also incommensurable with the (square) on DB.

11. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47].

12. And since BC is double DF [see previous proposition], the (rectangle contained) by AB and BC (is) thus also double the (rectangle contained) by AB and FD.

13. And the (rectangle contained) by AB and BC (is) rational.

14. Thus, the (rectangle contained) by AB and FD (is) also rational [Prop. 10.6, Def. 10.4].

15. And the (rectangle contained) by AB and FD (is) equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.].

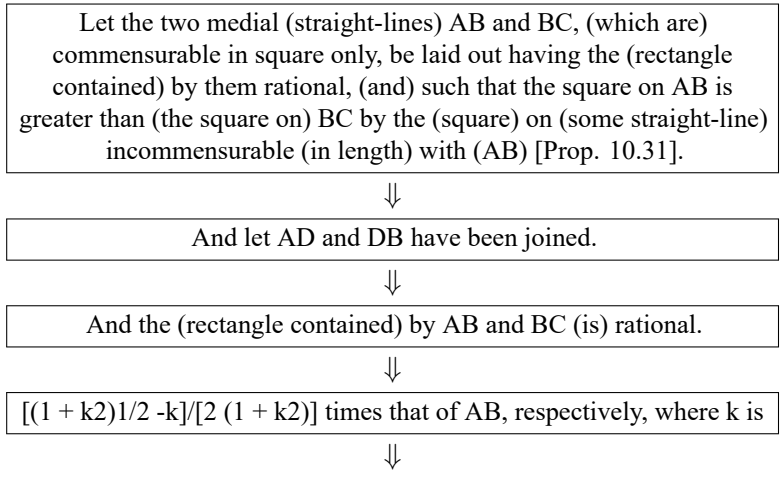
16. And hence the (rectangle contained) by AD and DB is rational.

17. Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

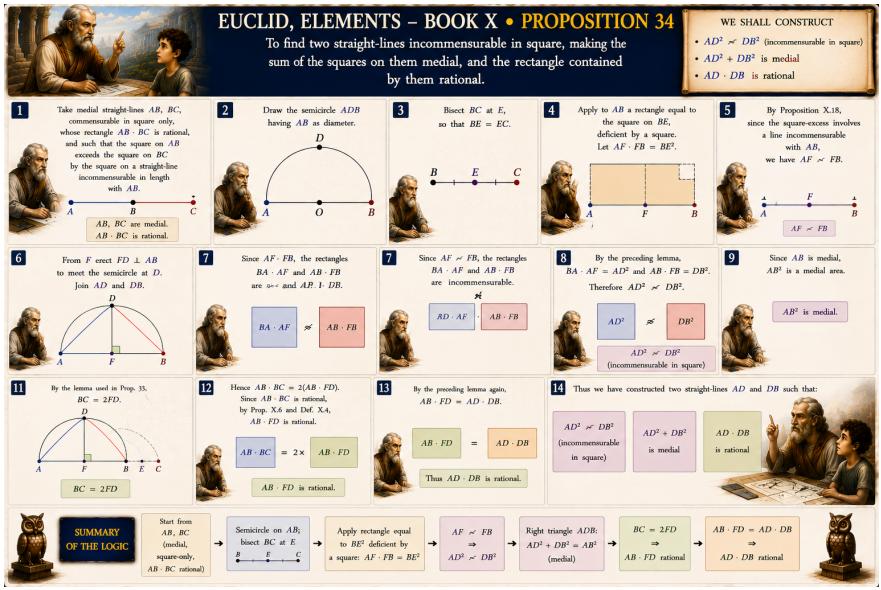
18. (Which is) the very thing it was required to show.

19. [(1 + k2)1/2 -k]/[2 (1 + k2)] times that of AB, respectively, where k is

Complete Cinematic Logic



Conclusion: Proposition 34 established.



2.35 Proposition 35

Statement

> *To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.*

Proof Actions

1. Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32].

2. And let the semi-circle ADB have been drawn on AB.

3. And let the remainder (of the figure) be generated similarly to the above (proposition).

4. And since AF is incommensurable in length with FB [Prop. 10.18], AD is also incommensurable in square with DB [Prop. 10.11].

5. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47].

6. And since the (rectangle contained) by AF and FB is equal to the (square) on each of BE and DF, BE is thus equal to DF. Thus, BC (is) double FD.

7. And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD.

8. And the (rectangle contained) by AB and BC (is) medial.

9. Thus, the (rectangle contained) by AB and FD (is) also medial.

10. And it is equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.].

11. Thus, the (rectangle contained) by AD and DB (is) also medial.

12. And since AB is incommensurable in length with BC, and CB (is) commensurable (in length) with BE, AB (is) thus also incommensurable in length with BE And hence the (square) on AB is also [Prop. 10.13]. incommensurable with the (rectangle contained) by AB and BE [Prop. 10.11].

13. But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47].

14. And the (rectangle contained) by AB and FD—that is to say, the (rectangle contained) by AD and DB—is equal to the (rectangle contained) by AB and BE.

15. Thus, the sum of the (squares) on AD and DB is incommensurable with the (rectangle contained) by AD and DB.

16. Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.

17. (Which is) the very thing it was required to show.

18. [1 -k/(1 + k2)1/2]/2 times that of AB, respectively, where k and k′ are

Complete Cinematic Logic

Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32].

⇓

And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD.

⇓

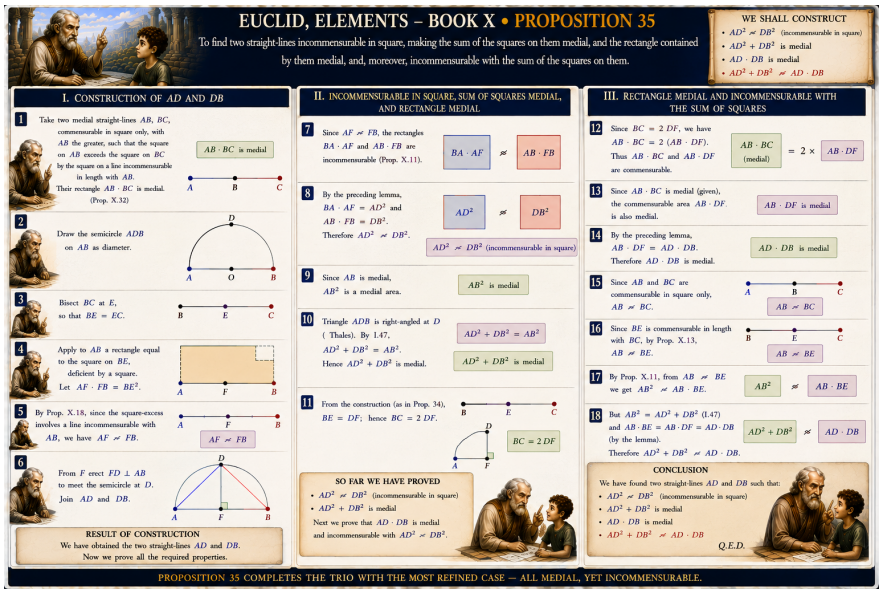
But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47].

⇓

[1 -k/(1 + k2)1/2]/2 times that of AB, respectively, where k and k′ are

⇓

Conclusion: Proposition 35 established.



2.36 Proposition 36

Statement

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational—let it be called a binomial (straight-line).

Proof Actions

1. For let the two rational (straight-lines), AB and BC, (which are) commensurable in square only, be laid down together.

2. I say that the whole (straight-line), AC, is irrational.

3. For since AB is incommensurable in length with BC—for they are commensurable in square only—and as AB (is) to BC, so the (rectangle contained) by ABC (is) to the (square) on BC, the (rectangle contained) by AB and BC is thus incommensurable with the (square) on BC [Prop. 10.11].

4. But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6].

5. And (the sum of) the (squares) on AB and BC is commensurable with the (square) on BC—for the rational (straight-lines) AB and BC are commensurable in square only [Prop. 10.15].

6. Thus, twice the (rectangle contained) by AB and BC is incommensurable with (the sum of) the (squares) on AB and BC [Prop. 10.13].

7. And, via composition, twice the (rectangle contained) by AB and BC, plus (the sum of) the (squares) on AB and BC—that is to say, the (square) on AC [Prop. 2.4]—is incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16].

8. And the sum of the (squares) on AB and BC (is) rational.

9. Thus, the (square) on AC [is] irrational [Def. 10.4].

10. Hence, AC is also irrational [Def. 10.4]—let it be called a binomial (straight-line).

11. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the two rational (straight-lines), AB and BC, (which are) commensurable in square only, be laid down together.

⇓

But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6].

⇓

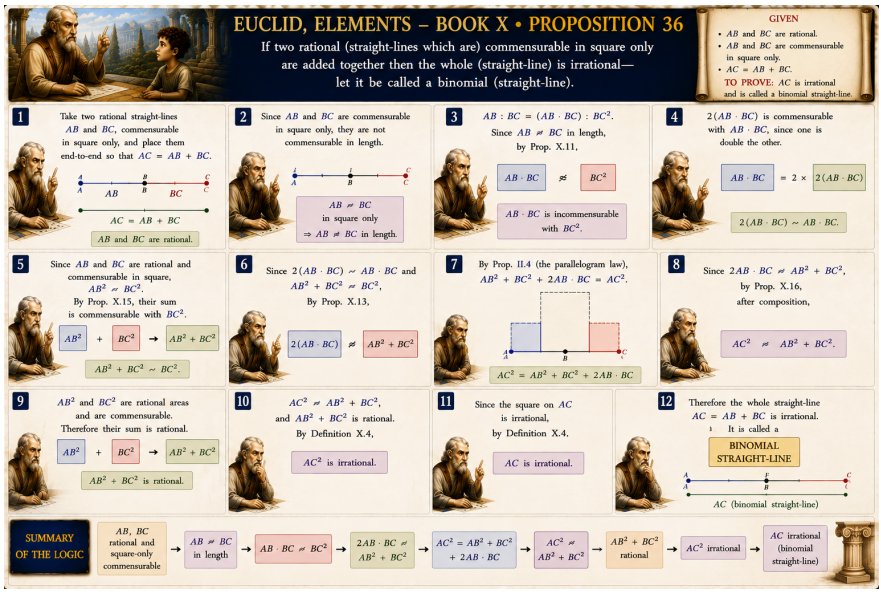
And the sum of the (squares) on AB and BC (is) rational.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 36 established.



2.37 Proposition 37

Statement

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational—let it be called a first bimedial (straight-line).

Proof Actions

1. For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a rational (area), be laid down together.

2. I say that the whole (straight-line), AC, is irrational.

3. For since AB is incommensurable in length with BC, (the sum of) the (squares) on AB and BC is thus also incommensurable with twice the (rectangle contained) by AB and BC [see previous proposition].

4. And, via composition, (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC— that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC [Prop. 10.16].

5. And the (rectangle contained) by AB and BC (is) rational—for AB and BC were assumed to enclose a rational (area).

6. Thus, the (square) on AC (is) irrational.

7. Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).

8. (Which is) the very thing it was required to show. k3/4.

9. The first bimedial and the corresponding first apotome of a medial, √ k (1 + k) x2 + k (1 -k)2 = 0. roots of the quartic x4 -2

Complete Cinematic Logic

For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a rational (area), be laid down together.

⇓

And, via composition, (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC— that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC [Prop. 10.16].

⇓

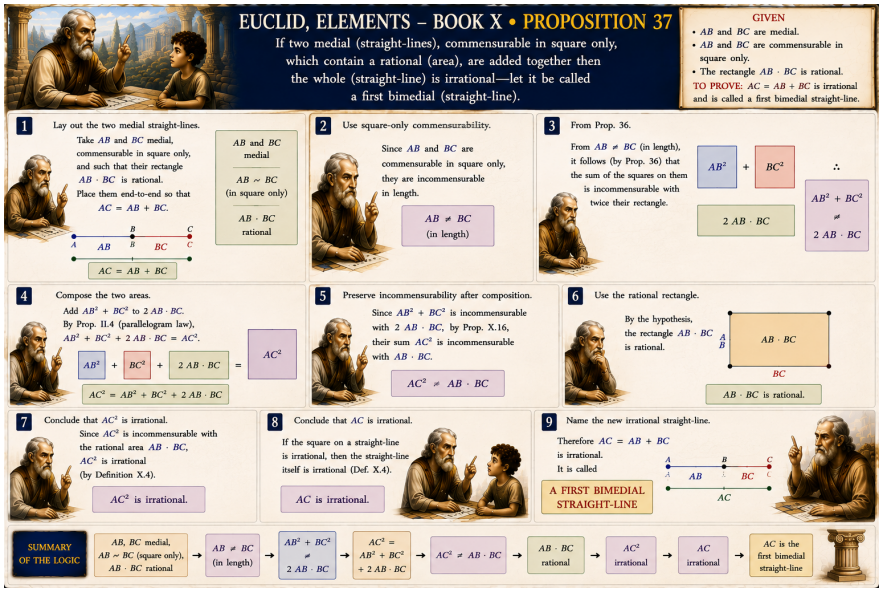
Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).

⇓

The first bimedial and the corresponding first apotome of a medial, √ k (1 + k) x2 + k (1 -k)2 = 0. roots of the quartic x4 -2

⇓

Conclusion: Proposition 37 established.



2.38 Proposition 38

Statement

If two medial (straight-lines), commensurable in square then the whole (straight-line) is irrational—let it be called a second bimedial (straight-line).

Proof Actions

1. For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a medial (area), be laid down together [Prop. 10.28].
2. I say that AC is irrational.
3. For let the rational (straight-line) DE be laid down, and let (the rectangle) DF, equal to the (square) on AC, have been applied to DE, making DG as breadth [Prop. 1.44].
4. And since the (square) on AC is equal to (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC [Prop. 2.4], so let (the rectangle) EH, equal to (the sum of) the squares on AB and BC, have been applied to DE.
5. The remainder HF is thus equal to twice the (rectangle contained) by AB and BC.
6. And since AB and BC are each medial, (the sum of) the squares on AB and BC is thus also medial.
7. And twice the (rectangle contained) by AB and BC was also assumed (to be) medial.
8. And EH is equal to (the sum of) the squares on AB and BC, and FH (is) equal to twice the (rectangle contained) by AB and BC.
9. Thus, EH and HF (are) each medial.
10. And they were applied to the rational (straight-line) DE.
11. Thus, DH and HG are each rational, and incommensurable in length with DE [Prop. 10.22].

12. Therefore, since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11].

13. But, the sum of the squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6].

14. Thus, the sum of the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.13].

15. But, EH is equal to (the sum of) the squares on AB and BC, and HF is equal to twice the (rectangle) contained by AB and BC.

16. Thus, EH is incommensurable with HF.

17. Hence, DH is also incommensurable in length with HG [Props. 6.1, 10.11].

18. Thus, DH and HG are rational (straight-lines which are) commensurable in square only.

19. Hence, DG is irrational [Prop. 10.36]. And DE (is) rational.

20. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20].

21. The area DF is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4].

22. And AC is the square-root of DF.

23. AC is thus irrational—let it be called a second bimedial (straight-line).§ (Which is) the very thing it was required to show.

24. Props. 10.15, 10.23. +k′1/2/k1/4.

25. The second bimedial and the corresponding second apotome √ k] x2 + 10.75), are the positive roots of the quartic x4 -2 [(k + k′)/

Complete Cinematic Logic

For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a medial (area), be laid down together [Prop. 10.28].

⇓

Thus, EH and HF (are) each medial.

⇓

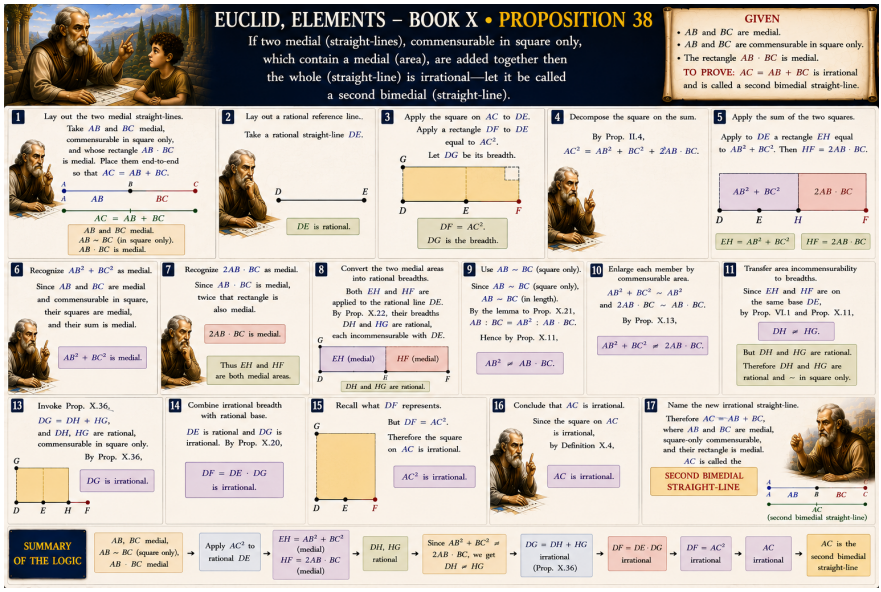
Hence, DH is also incommensurable in length with HG [Props. 6.1, 10.11].

⇓

The second bimedial and the corresponding second apotome √ k] x2 + 10.75), are the positive roots of the quartic x4 -2 [(k + k′)/

⇓

Conclusion: Proposition 38 established.



2.39 Proposition 39

Statement

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational—let it be called a major (straight-line).

Proof Actions

1. For let the two straight-lines, AB and BC, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33].

2. I say that AC is irrational.

3. For since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is [thus] also medial [Props. 10.6, 10.23 corr.].

4. And the sum of the (squares) on AB and BC (is) rational.

5. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the sum of the (squares) on AB and BC [Def. 10.4].

6. Hence, (the sum of) the squares on AB and BC, plus twice the (rectangle contained) by AB and BC—that is, the (square) on AC [Prop. 2.4]—is also incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16] [and the sum of the (squares) on AB and BC (is) rational].

7. Thus, the (square) on AC is irrational.

8. Hence, AC is also irrational [Def. 10.4]—let it be called a major (straight-line).

9. (Which is) the very thing it was required to show. q k2)1/2]/2 + [1 -k/(1 + k2)1/2]/2.

10. The major and the corresponding -k/(1 + k2)1/2]/2 (see Prop. 10.76), are the positive roots of the quartic

Complete Cinematic Logic

For let the two straight-lines, AB and BC, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33].

⇓

And the sum of the (squares) on AB and BC (is) rational.

⇓

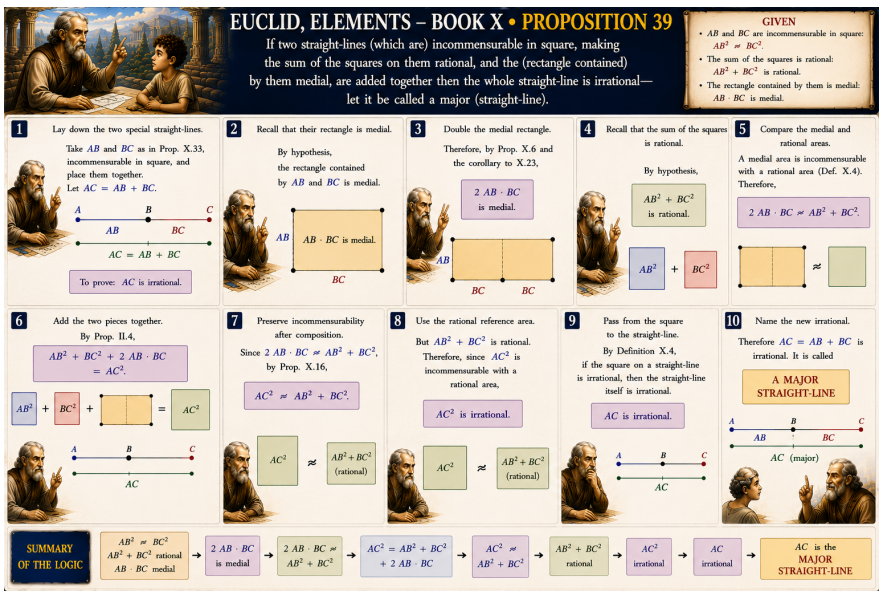
Thus, the (square) on AC is irrational.

⇓

The major and the corresponding -k/(1 + k2)1/2]/2 (see Prop. 10.76), are the positive roots of the quartic

⇓

Conclusion: Proposition 39 established.



2.40 Proposition 40

Statement

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational—let it be called the square-root of a rational plus a medial (area).

Proof Actions

1. For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34].

2. I say that AC is irrational.

3. For since the sum of the (squares) on AB and BC is medial, and twice the (rectangle contained) by AB and BC (is) rational, the sum of the (squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC.

4. Hence, the (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.16].

5. And twice the (rectangle contained) by AB and BC (is) rational.

6. The (square) on AC (is) thus irrational.

7. Thus, AC (is) irrational [Def. 10.4]—let it be called the square-root of a rational plus a medial (area).

8. (Which is) the very thing it was required to show. q q [(1 + k2)1/2 + k]/[2 (1 + k2)]+ [(1 + k2)1/2 -k]/[2 (1 + k2)]. as q q [(1 + k2)1/2 + k]/[2 (1 + k2)]- [(1 + k2)1/2 -k]/[2 (1 + k2)] expressible as x2 + k2/(1 + k2)2 = 0.

Complete Cinematic Logic

For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34].

⇓

For since the sum of the (squares) on AB and BC is medial, and twice the (rectangle contained) by AB and BC (is) rational, the sum of the (squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC.

⇓

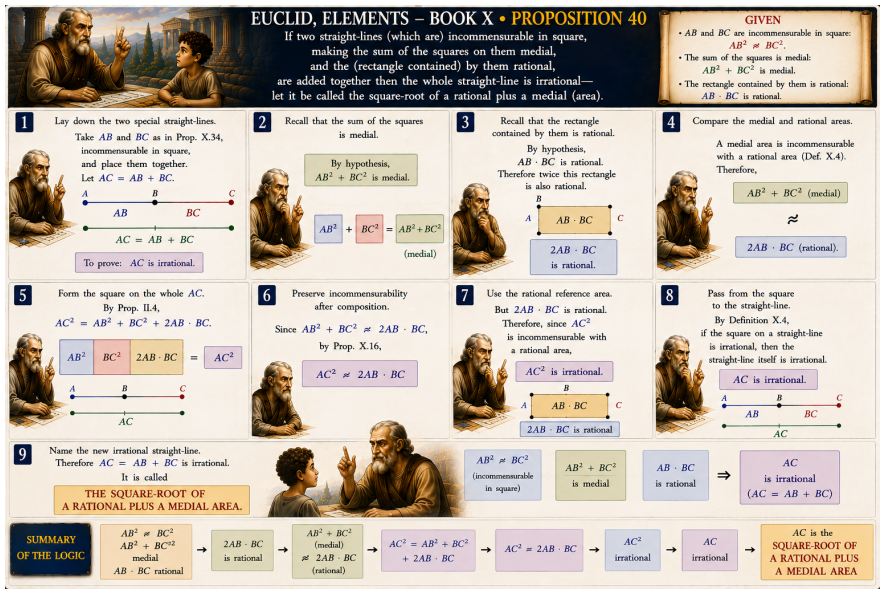
The (square) on AC (is) thus irrational.

⇓

(Which is) the very thing it was required to show. q q [(1 + k2)1/2 + k]/[2 (1 + k2)]+ [(1 + k2)1/2 -k]/[2 (1 + k2)]. as q q [(1 + k2)1/2 + k]/[2 (1 + k2)]- [(1 + k2)1/2 -k]/[2 (1 + k2)] expressible as x2 + k2/(1 + k2)2 = 0.

⇓

Conclusion: Proposition 40 established.



2.41 Proposition 41

Statement

> *If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).*

Proof Actions

1. For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35].

2. I say that AC is irrational.

3. Let the rational (straight-line) DE be laid out, and let (the rectangle) DF, equal to (the sum of) the (squares) on AB and BC, and (the rectangle) GH, equal to twice the (rectangle contained) by AB and BC, have been applied to DE.

4. Thus, the whole of DH is equal to the square on AC [Prop. 2.4].

5. And since the sum of the (squares) on AB and BC is medial, and is equal to DF, DF is thus also medial.

6. And it is applied to the rational (straight-line) DE.

7. Thus, DG is rational, and incommen325 surable in length with DE [Prop. 10.22].

8. So, for the same (reasons), GK is also rational, and incommensurable in length with GF—that is to say, DE.

9. And since (the sum of) the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DF is incommensurable with GH.

10. Hence, DG is also incommensurable (in length) with GK [Props. 6.1, 10.11]. And they are rational.

11. Thus, DG and GK are rational (straight-lines which are) commensurable in square only.

12. Thus, DK is irrational, and that (straight-line which is) And DE (is) rational. called binomial [Prop. 10.36].

13. Thus, DH is irrational, and its square-root is irrational [Def. 10.4].

14. And AC (is) the square-root of HD.

15. Thus, AC is irrational—let it be called the square-root of (the sum of) two medial (areas).

16. (Which is) the very thing it was required to show. « „q q as k′1/4 [1 + k/(1 + k2)1/2]/2 + [1 -k/(1 + k2)1/2]/2 . « „q q expressible as k′1/4 [1 + k/(1 + k2)1/2]/2 - [1 -k/(1 + k2)1/2]/2 + k′ k2/(1 + k2) = 0.

17. Lemma We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.

18. Let the straight-line AB be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) C and D.

19. And let AC be assumed (to be) greater than DB.

20. I say that (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB.

21. For let AB have been cut in half at E.

22. And since AC is greater than DB, let DC have been subtracted from both.

23. Thus, the remainder AD is greater than the remainder CB. And AE (is) equal to EB.

24. Thus, DE (is) less than EC.

25. Thus, points C and D are not equally far from the point of bisection.

26. And since the (rectangle contained) by AC and CB, plus the (square) on EC, is equal to the (square) on EB [Prop. 2.5], but, moreover, the (rectangle contained) by AD and DB, plus the (square) on DE, is also equal to the (square) on EB [Prop. 2.5], the (rectangle contained) by AC and CB, plus the (square) on EC, is thus equal to the (rectangle contained) by AD and DB, plus the (square) on DE.

27. And, of these, the (square) on DE is less than the (square) on EC.

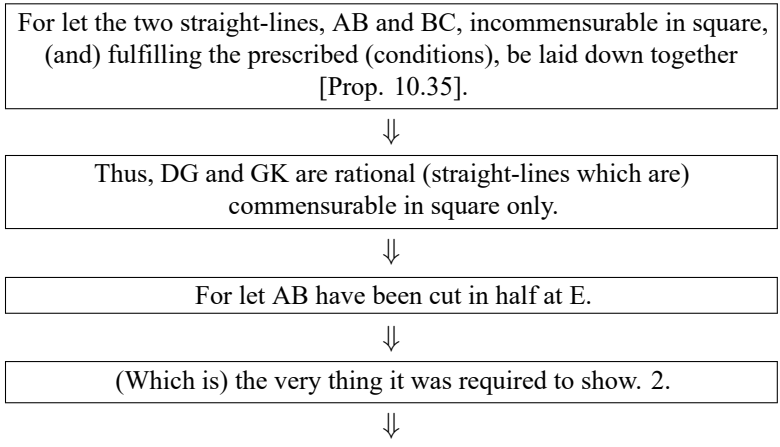
28. And, thus, the remaining (rectangle contained) by AC and CB is less than the (rectangle contained) by AD and DB.

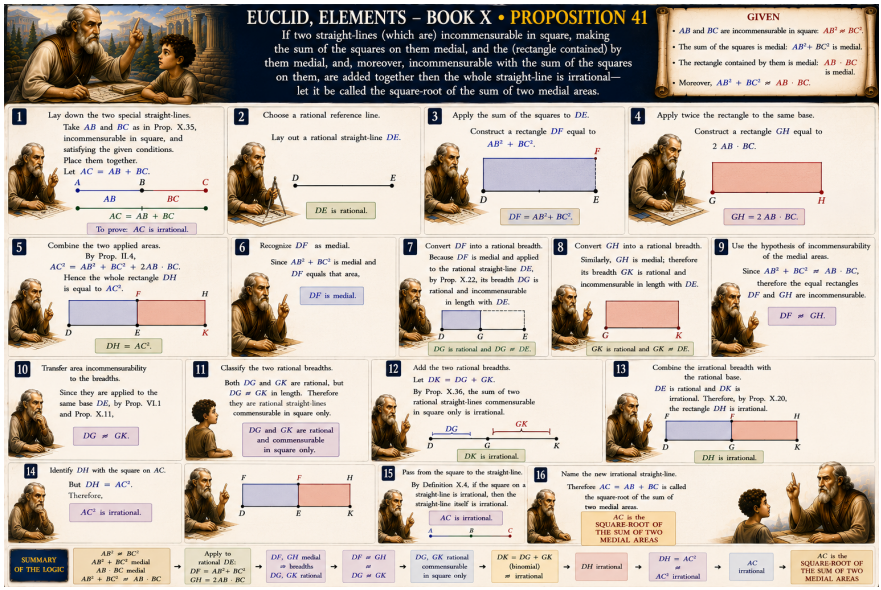
29. And, hence, twice the (rectangle contained) by AC and CB is less than twice the (rectangle contained) by AD and DB.

30. And thus the remaining sum of the (squares) on AC and CB is greater than the sum of the (squares) on AD and DB.

31. (Which is) the very thing it was required to show. 2.

Complete Cinematic Logic





2.42 Proposition 42

Statement

A binomial (straight-line) can be divided into its (component) terms at one point only.

Proof Actions

1. Let AB be a binomial (straight-line) which has been divided into its (component) terms at C.

2. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36].

3. I say that AB cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

4. For, if possible, let it also have been divided at D, such that AD and DB are also rational (straight-lines which are) commensurable in square only.

5. So, (it is) clear that AC is not the same as DB.

6. For, if possible, let it be (the same).

7. So, AD will also be the same as CB.

8. And as AC will be to CB, so BD (will be) to DA.

9. And AB will (thus) also be divided at D in the same (manner) as the division at C.

10. The very opposite was assumed.

11. Thus, AC is not the same as DB.

12. So, on account of this, points C and D are not equally far from the point of bisection.

13. Thus, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount)—on account of both (the sum of) the (squares) on AC and CB, plus twice the (rectangle contained) by AC and CB, and (the sum of) the (squares) on AD and DB, plus twice the (rectangle contained) by AD and DB, being equal to the (square) on AB [Prop. 2.4].

14. But, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by a rational (area).

15. For (they are) both rational Thus, twice the (rectangle contained) by AD (areas). and DB also differs from twice the (rectangle contained) by AC and CB by a rational (area, despite both) being medial (areas) [Prop. 10.21].

16. The very thing is absurd.

17. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].

18. Thus, a binomial (straight-line) cannot be divided (into its component terms) at different points.

19. Thus, (it can be so divided) at one point only.

20. (Which is) the very thing it was required to show. $k'' = k$ and $k''' = k'$.

21. Likewise, k1/2 + k′1/2 = k″1/2 + k‴1/2 has only k‴ = k).

Complete Cinematic Logic

Let AB be a binomial (straight-line) which has been divided into its (component) terms at C.

⇓

And as AC will be to CB, so BD (will be) to DA.

⇓

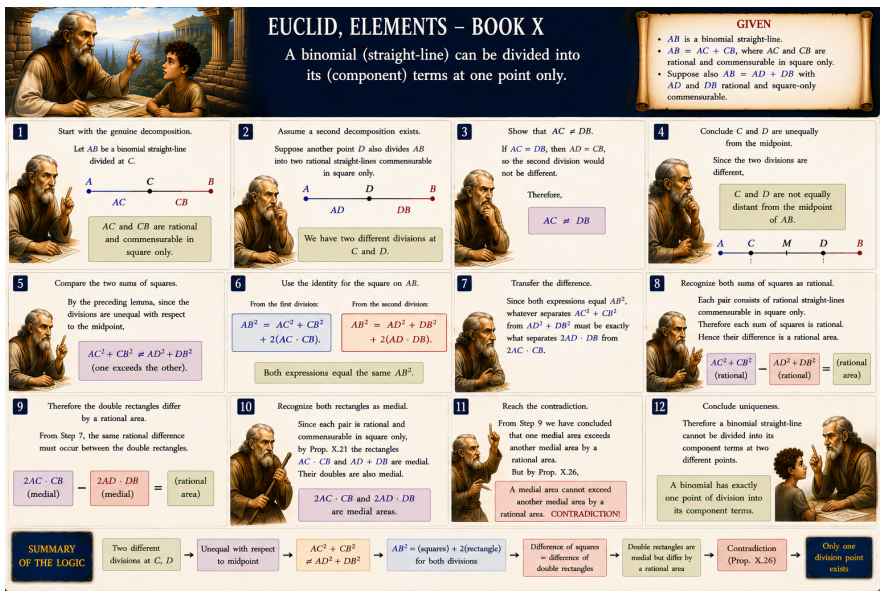
For (they are) both rational Thus, twice the (rectangle contained) by AD (areas). and DB also differs from twice the (rectangle contained) by AC and CB by a rational (area, despite both) being medial (areas) [Prop. 10.21].

⇓

Likewise, k1/2 + k′1/2 = k″1/2 + k‴1/2 has only k‴ = k).

⇓

Conclusion: Proposition 42 established.



2.43 Proposition 43

Statement

A first bimedial (straight-line) can be divided (into its component terms) at one point only.

Proof Actions

1. Let AB be a first bimedial (straight-line) which has been divided at C, such that AC and CB are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37].

2. I say that AB cannot be (so) divided at another point.

3. For, if possible, let it also have been divided at D, such that AD and DB are also medial (straight-lines), commensurable in square only, (and) containing a rational (area).

4. Since, therefore, by whatever (amount) twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by this (same amount) [Prop. 10.41 lem.].

5. And twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB by a rational (area).

6. For (they are) both rational (areas).

7. (The sum of) the (squares) on AC and CB thus differs from (the sum of) the (squares) on AD and DB by a rational (area, despite both) being medial (areas).

8. The very thing is absurd [Prop. 10.26].

9. Thus, a first bimedial (straight-line) cannot be divided into its (component) terms at different points.

10. Thus, (it can be so divided) at one point only.

11. (Which is) the very thing it was required to show. i.e., $k' = k$.

Complete Cinematic Logic

Let AB be a first bimedial (straight-line) which has been divided at C, such that AC and CB are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37].

⇓

Since, therefore, by whatever (amount) twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by this (same amount) [Prop. 10.41 lem.].

⇓

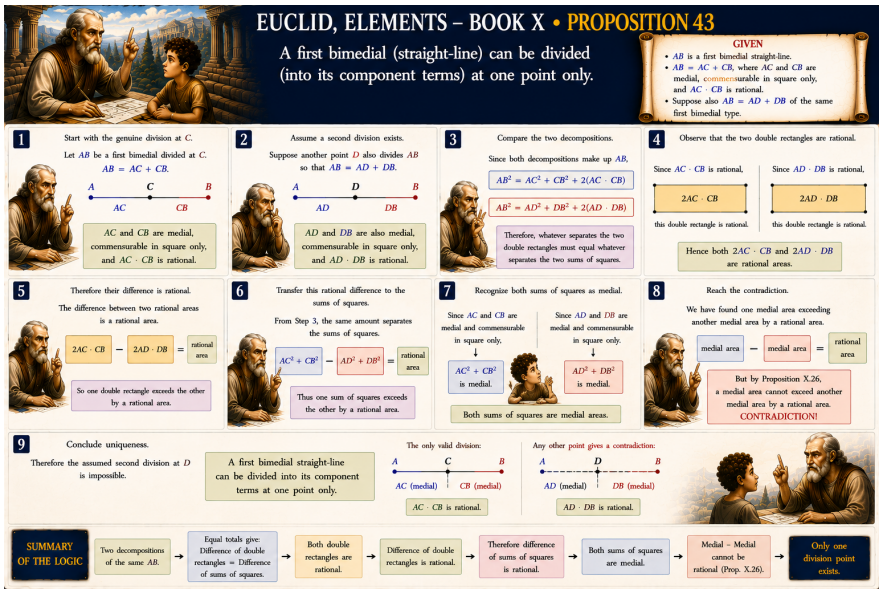
The very thing is absurd [Prop. 10.26].

⇓

(Which is) the very thing it was required to show. i.e., k′ = k.

⇓

Conclusion: Proposition 43 established.



2.44 Proposition 44

Statement

A second bimedial (straight-line) can be divided (into its component terms) at one point only.

Proof Actions

1. Let AB be a second bimedial (straight-line) which has been divided at C, so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38].

2. So, (it is) clear that C is not (located) at the point of bisection, since (AC and BC) are not commensurable in length.

3. I say that AB cannot be (so) divided at another point.

4. For, if possible, let it also have been (so) divided at D, so that AC is not the same as DB, but AC (is), by hypothesis, greater.

5. So, (it is) clear that (the sum of) the (squares) on AD and DB is also less than (the sum of) the (squares) on AC and CB, as we showed above [Prop. 10.41 lem.].

6. And AD and DB are medial (straight-lines), commensurable in square only, (and) containing a medial (area).

7. And let the rational (straightline) EF be laid down.

8. And let the rectangular parallelogram EK, equal to the (square) on AB, have been applied to EF.

9. And let EG, equal to (the sum of) the (squares) on AC and CB, have been cut off (from EK).

10. Thus, the remainder, HK, is equal to twice the (rectangle contained) by AC and CB [Prop. 2.4].

11. So, again, let EL, equal to (the sum of) the (squares) on AD and DB—which was shown (to be) less than (the sum of) the (squares) on AC and CB—have been cut off (from EK).

12. And, thus, the remainder, MK, (is) equal to twice the (rectangle contained) by AD and DB.

13. And since (the sum of) the (squares) on AC and CB is medial, EG (is) thus [also] medial.

14. And it is applied to the rational (straight-line) EF.

15. Thus, EH is rational, and incommensurable in length with EF [Prop. 10.22].

16. So, for the same (reasons), HN is also rational, and incommensurable in length with EF.

17. And since AC and CB are medial (straight-lines which are) commensurable in square And as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by AC and CB [Prop. 10.21 lem.].

18. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11].

19. But, (the sum of) the (squares) on AC and CB is commensurable with the (square) on AC.

20. For, AC and CB are commensurable in square [Prop. 10.15].

21. And twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6].

22. And thus (the sum of) the squares on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13].

23. But, EG is equal to (the sum of) the (squares) on AC and CB, and HK equal to twice the (rectangle contained) by AC and CB.

24. Thus, EG is incommensurable with HK.

25. Hence, EH is also incom329 mensurable in length with HN [Props. 6.1, 10.11].

26. And (they are) rational (straight-lines).

27. Thus, EH and HN are rational (straight-lines which are) commensurable in square only.

28. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36].

29. Thus, EN is a binomial (straightline) which has been divided (into its component terms) at H.

30. So, according to the same (reasoning), EM and MN can be shown (to be) rational (straight-lines which are) commensurable in square only.

31. And EN will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) H and M (which is absurd [Prop. 10.42]).

32. And EH is not the same as MN, since (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB.

33. But, (the sum of) the (squares) on AD and DB is greater than twice the (rectangle contained) by AD and DB [Prop. 10.59 lem.].

34. Thus, (the sum of) the (squares) on AC and CB—that is to say, EG—is also much greater than twice the (rectangle contained) by AD and DB—that is to say, MK.

35. Hence, EH is also greater than MN [Prop. 6.1].

36. Thus, EH is not the same as MN.

37. (Which is) the very thing it was required to show. one solution: i.e., $k'' = k$ and $k''' = k'$.

Complete Cinematic Logic

Let AB be a second bimedial (straight-line) which has been divided at C, so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38].

⇓

And since (the sum of) the (squares) on AC and CB is medial, EG (is) thus [also] medial.

⇓

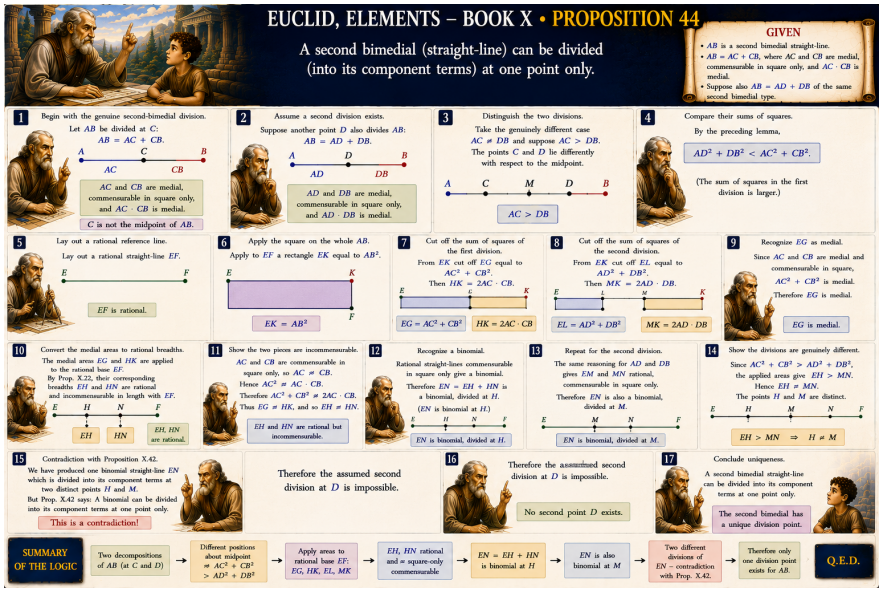
Hence, EH is also incom329 mensurable in length with HN [Props. 6.1, 10.11].

⇓

(Which is) the very thing it was required to show. one solution: i.e., k″ = k and k‴ = k′.

⇓

Conclusion: Proposition 44 established.



2.45 Proposition 45

Statement

A major (straight-line) can only be divided (into its component terms) at the same point.

Proof Actions

1. Let AB be a major (straight-line) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the squares on AC and CB rational, and the (rectangle contained) by AC and CD medial [Prop. 10.39].
2. I say that AB cannot be (so) divided at another point.
3. For, if possible, let it also have been divided at D, such that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB rational, and the (rectangle contained) by them medial.
4. And since, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount).
5. But, (the sum of) the (squares) on AC and CB exceeds (the sum of) the (squares) on AD and DB by a rational (area).
6. For (they are) both rational (areas).
7. Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area), (despite both) being medial (areas).
8. The very thing is impossible [Prop. 10.26].
9. Thus, a major (straight-line) cannot be divided (into its component terms) at different points.
10. Thus, it can only be (so) divided at the same (point).
11. (Which is) the very thing it was required to show. q q [1 + k′/(1 + k′2)1/2]/2 + [1 -k′/(1 + k′2)1/2]/2 has only one =

Complete Cinematic Logic

Let AB be a major (straight-line) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the squares on AC and CB rational, and the (rectangle contained) by AC and CD medial [Prop. 10.39].

⇓

And since, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount).

⇓

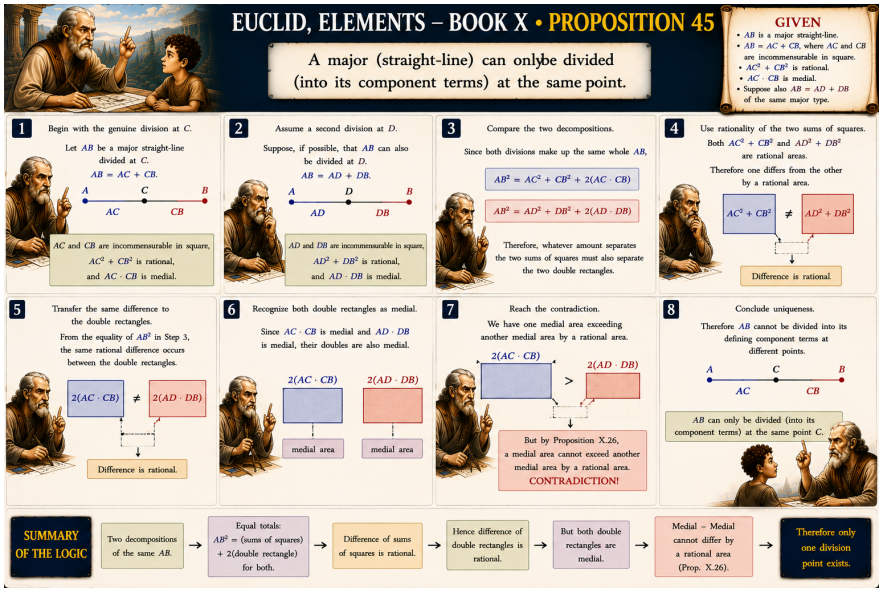
The very thing is impossible [Prop. 10.26].

⇓

(Which is) the very thing it was required to show. q q [1 + k′/(1 + k′2)1/2]/2 + [1 -k′/(1 + k′2)1/2]/2 has only one =

⇓

Conclusion: Proposition 45 established.



2.46 Proposition 46

Statement

The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only.

Proof Actions

1. Let AB be the square-root of a rational plus a medial (area) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and twice the (rectangle contained) by AC and CB rational [Prop. 10.40].

2. I say that AB cannot be (so) divided at another point.

3. For, if possible, let it also have been divided at D, so that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB medial, and twice the (rectangle contained) by AD and DB rational.

4. Therefore, since by whatever (amount) twice the (rectangle contained) by AC and CB differs from twice the (rectangle contained) by AD and DB, (the sum of) the (squares) on AD and DB also differs from (the sum of) the (squares) on AC and CB by this (same amount).

5. And twice the (rectangle contained) by AC and CB exceeds twice the (rectangle contained) by AD and DB by a rational (area).

6. (The sum of) the (squares) on AD and DB thus also exceeds (the sum of) the (squares) on AC and CB by a rational (area), (despite both) being medial (areas).

7. The very thing is impossible [Prop. 10.26].

8. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points.

9. Thus, it can be (so) divided at one point (only).

10. (Which is) the very thing it was required to show. q (1 + k2)] = [(1 + k′2)1/2 + k′]/[2 (1 + k′2)]

Complete Cinematic Logic

Let AB be the square-root of a rational plus a medial (area) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and twice the (rectangle contained) by AC and CB rational [Prop. 10.40].

⇓

Therefore, since by whatever (amount) twice the (rectangle contained) by AC and CB differs from twice the (rectangle contained) by AD and DB, (the sum of) the (squares) on AD and DB also differs from (the sum of) the (squares) on AC and CB by this (same amount).

⇓

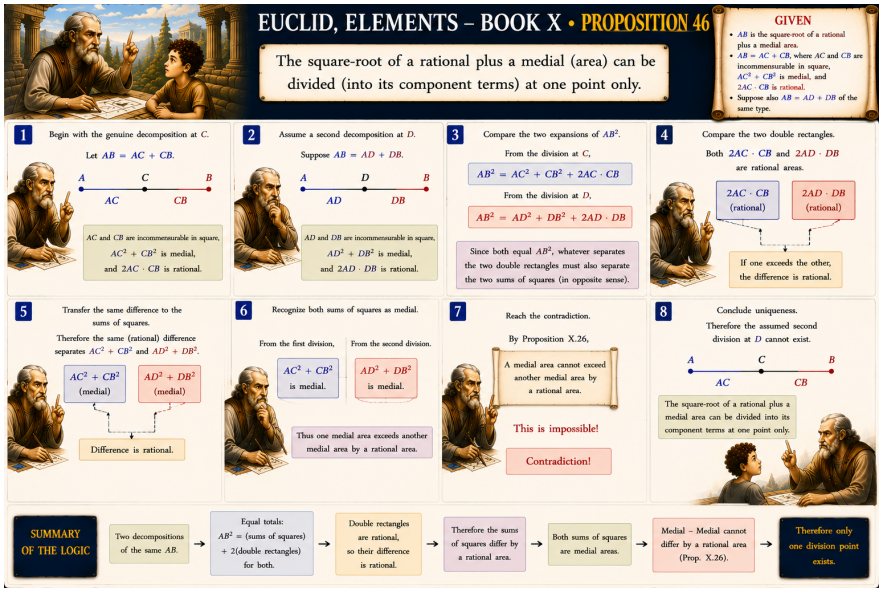
The very thing is impossible [Prop. 10.26].

⇓

(Which is) the very thing it was required to show. q (1 + k2)] = [(1 + k′2)1/2 + k′]/[2 (1 + k′2)]

⇓

Conclusion: Proposition 46 established.



2.47 Proposition 47

Statement

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only.

Proof Actions

1. Let AB be [the square-root of (the sum of) two medial (areas)] which has been divided at C, such that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and the (rectangle contained) by AC and CB medial, and, moreover, incommensurable with the sum of the (squares) on (AC and CB) [Prop. 10.41].

2. I say that AB cannot be divided at another point fulfilling the prescribed (conditions).

3. For, if possible, let it have been divided at D, such that AC is again manifestly not the same as DB, but AC (is), by hypothesis, greater.

4. And let the rational (straight-line) EF be laid down.

5. And let EG, equal to (the sum of) the (squares) on AC and CB, and HK, equal to twice the (rectangle contained) by AC and CB, have been applied to EF.

6. Thus, the whole of EK is equal to the square on AB [Prop. 2.4].

7. So, again, let EL, equal to (the sum of) the (squares) on AD and DB, have been applied to EF.

8. Thus, the remainder—twice the (rectangle contained) by AD and DB—is equal to the remainder, MK.

9. And since the sum of the (squares) on AC and CB was assumed (to be) medial, EG is also medial.

10. And it is applied to the rational (straight-line) EF.

11. HE is thus rational, and incommensurable in length with EF [Prop. 10.22].

12. So, for the same (reasons), HN is also rational, and incommensurable in length with EF.

13. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is thus also incommensurable with GN.

14. Hence, EH is also incommensurable with HN [Props. 6.1, 10.11].

15. And they are (both) rational (straight-lines).

16. Thus, EH and HN are rational (straight-lines which are) commensurable in square only.

17. Thus, EN is a binomial (straightline) which has been divided (into its component terms) at H [Prop. 10.36].

18. So, similarly, we can show that it has also been (so) divided at M.

19. And EH is not the same as MN.

20. Thus, a binomial (straight-line) has been divided (into its component terms) at different points.

21. The very thing is absurd [Prop. 10.42].

22. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into its component terms) at different points.

23. Thus, it can be (so) divided at one [point] only. q k2)1/2]/2 = k‴1/4 [1 + k″/(1 + k″2)1/2]/2 and k‴ = k′. Definitions II 5.

24. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line). 6.

25. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line). 7.

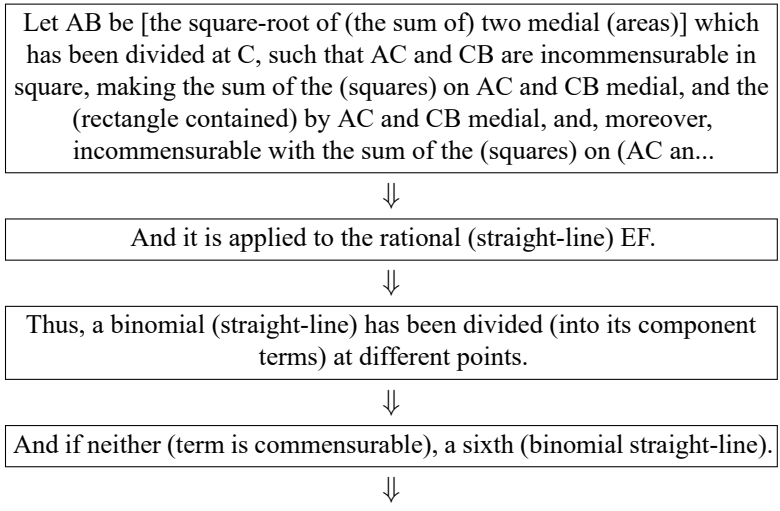
26. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line). 8.

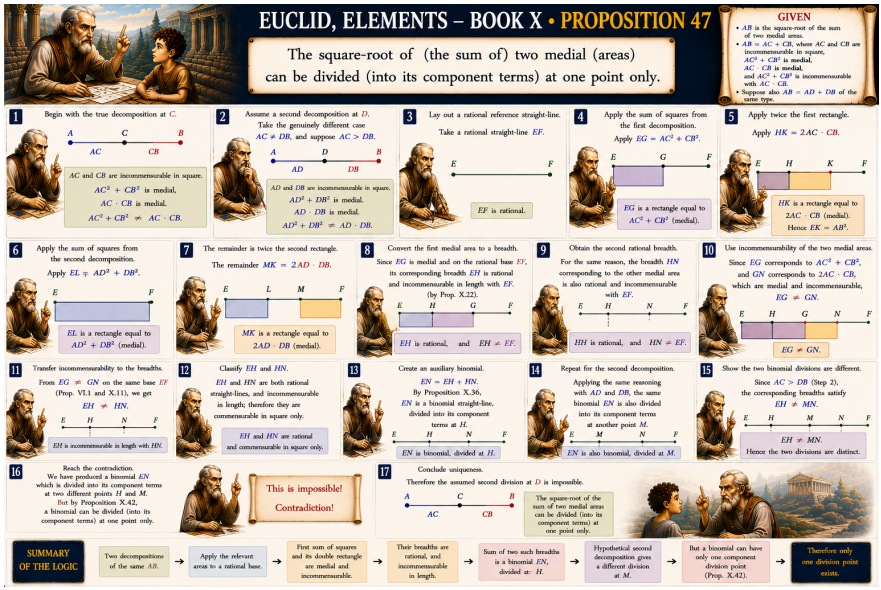
27. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line). 9.

28. And if the lesser (term is commensurable), a fifth (binomial straight-line). 10.

29. And if neither (term is commensurable), a sixth (binomial straight-line).

Complete Cinematic Logic





2.48 Proposition 48

Statement

To find a first binomial (straight-line).

Proof Actions

1. Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].

2. And let some rational (straight-line) D be laid down.

3. And let EF be commensurable in length with D.

4. EF is thus also rational [Def. 10.3].

5. And let it have been contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.].

6. And AB has to AC the ratio which (some) number (has) to (some) num333 ber.

7. Thus, the (square) on EF also has to the (square) on FG the ratio which (some) number (has) to (some) number.

8. Hence, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. And EF is rational.

9. Thus, FG (is) also rational.

10. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, thus the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either.

11. Thus, EF is incommensurable in length with FG [Prop 10.9].

12. EF and FG are thus rational (straight-lines which are) commensurable in square only.

13. Thus, EG is a binomial (straight-line) [Prop. 10.36].

14. I say that (it is) also a first (binomial straight-line).

15. For since as the number BA is to AC, so the (square) on EF (is) to the (square) on FG, and BA (is) greater than AC, the (square) on EF (is) thus also greater than the (square) on FG [Prop. 5.14].

16. Therefore, let (the sum of) the (squares) on FG and H be equal to the (square) on EF.

17. And since as BA is to AC, so the (square) on EF (is) to the (square) on FG, thus, via conversion, as AB is to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.].

18. And AB has to BC the ratio which (some) square number (has) to (some) square number.

19. Thus, the (square) on EF also has to the (square) on H the ratio which (some) square number (has) to (some) square number.

20. Thus, EF is commensurable in length with H [Prop. 10.9].

21. Thus, the square on EF is greater than (the square on) FG by the (square) on (some straight-line) commensurable (in length) with (EF).

22. And EF and FG are rational (straight-lines).

23. And EF (is) commensurable in length with D.

24. Thus, EG is a first binomial (straight-line) [Def. 10.5].

25. (Which is) the very thing it was required to show. √ 1 -k′ 2.

26. This, and the first apotome, whose straight-line is k + k k′ 2 = 0.

Complete Cinematic Logic

Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].

⇓

Thus, FG (is) also rational.

⇓

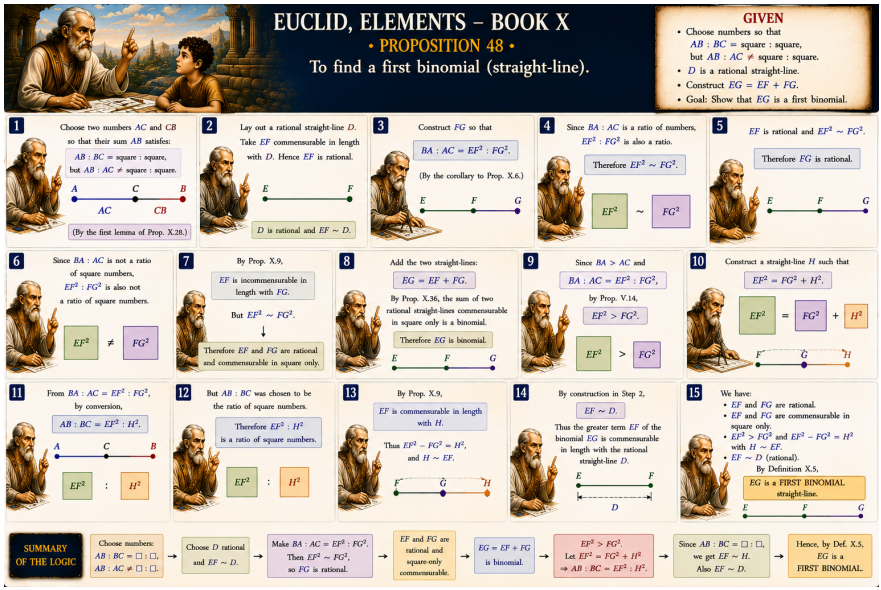
And AB has to BC the ratio which (some) square number (has) to (some) square number.

⇓

This, and the first apotome, whose straight-line is k + k k′ 2 = 0.

⇓

Conclusion: Proposition 48 established.



2.49 Proposition 49

Statement

To find a second binomial (straight-line).

Proof Actions

1. Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].

2. And let the rational (straight-line) D be laid down.

3. And let EF be commensurable in length with D.

4. EF is thus a rational (straight-line).

5. So, let it also have been contrived that as the number CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.].

6. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6].

7. Thus, FG is also a rational (straightline).

8. And since the number CA does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either.

9. Thus, EF is incommensurable in length with FG [Prop. 10.9].

10. EF and FG are thus rational (straight-lines which are) commensurable in square only.

11. Thus, EG is a binomial (straightline) [Prop. 10.36].

12. So, we must show that (it is) also a second (binomial straight-line).

13. For since, inversely, as the number BA is to AC, so the (square) on GF (is) to the (square) on FE [Prop. 5.7 corr.], and BA (is) greater than AC, the (square) on GF (is) thus [also] greater than the (square) on FE [Prop. 5.14].

14. Let (the sum of) the (squares) on EF and H be equal to the (square) on GF.

15. Thus, via conversion, as AB is to BC, so the (square) on FG (is) to the (square) on H [Prop. 5.19 corr.].

16. But, AB has to BC the ratio which (some) square number (has) to (some) square number.

17. Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number.

18. Thus, FG is commensurable in length with H [Prop. 10.9].

19. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) commensurable in length with (FG).

20. And FG and FE are rational (straight-lines which are) commensurable in square only.

21. And the lesser term EF is commensurable in length with the rational (straightline) D (previously) laid down.

22. Thus, EG is a second binomial (straight-line) [Def. 10.6].

23. (Which is) the very thing it was required to show. $\sqrt{1 - k'^2} + k$.

24. This, and the second apotome, binomial straight-line is $k/\sqrt{1 - k'^2})\, x + k2\,[k'^2/(1 - k'^2)] = 0.\ k/$

Complete Cinematic Logic

Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].

⇓

Thus, EF is incommensurable in length with FG [Prop. 10.9].

⇓

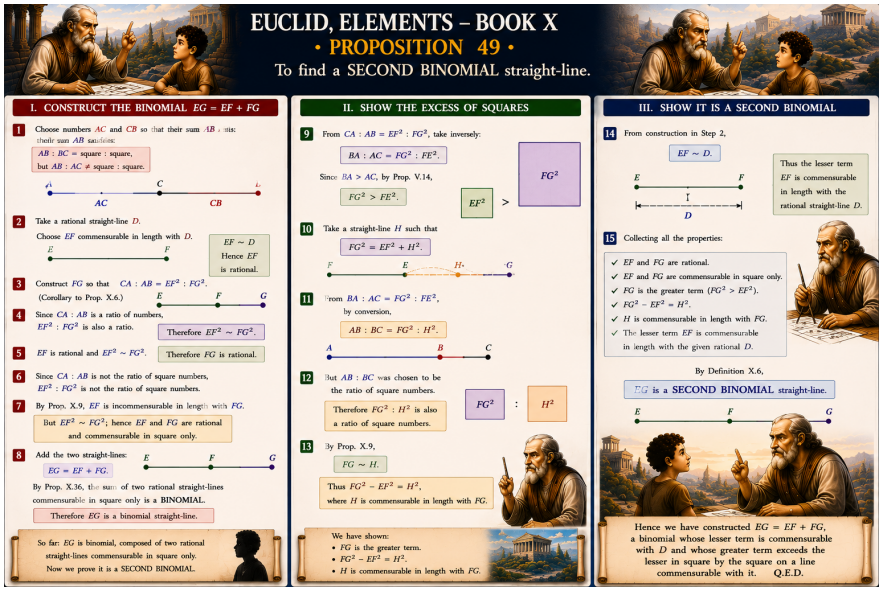
Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number.

⇓

This, and the second apotome, binomial straight-line is $k/\sqrt{1 - k'^2})\, x + k2\,[k'^2/(1 - k'^2)] = 0.\ k/$

⇓

Conclusion: Proposition 49 established.



2.50 Proposition 50

Statement

To find a third binomial (straight-line).

Proof Actions

1. Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number.

2. And let some other nonsquare number D also be laid down, and let it not have to each of BA and AC the ratio which (some) square number (has) to (some) square number.

3. And let some rational straight-line E be laid down, and let it have been contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.].

4. Thus, the (square) on E is commensurable with the (square) on FG [Prop. 10.6].

5. And E is a rational (straight-line).

6. Thus, FG is also a rational (straight-line).

7. And since D does not have to AB the ratio which (some) square number has to (some) square number, the (square) on E does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number E is thus incommensurable in length with FG either. [Prop. 10.9].

8. So, again, let it have been contrived that as the number BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].

9. Thus, the (square) on FG is commensurable with the (square) on GH [Prop. 10.6].

10. And FG (is) a rational (straight-line).

11. Thus, GH (is) also a rational (straight-line).

12. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on HG the ratio which (some) square number (has) to (some) square number either.

13. Thus, FG is incommensurable in length with GH FG and GH are thus rational (straight- [Prop. 10.9]. lines which are) commensurable in square only.

14. Thus, FH is a binomial (straight-line) [Prop. 10.36].

15. So, I say that (it is) also a third (binomial straight-line).

16. For since as D is to AB, so the (square) on E (is) to the (square) on FG, and as BA (is) to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D (is) to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22].

17. And D does not have to AC the ratio which (some) square number (has) Thus, the (square) on E to (some) square number. does not have to the

(square) on GH the ratio which (some) square number (has) to (some) square number either.

18. Thus, E is incommensurable in length with GH [Prop. 10.9].

19. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14].

20. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG.

21. Thus, via conversion, as AB [is] to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.].

22. And AB has to BC the ratio which (some) square number (has) to (some) square number.

23. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number.

24. Thus, FG [is] commensurable in length with K [Prop. 10.9].

25. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG).

26. And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with E.

27. Thus, FH is a third binomial (straight-line) [Def. 10.7].

28. (Which is) the very thing it was required to show. √ binomial straight-line is k1/2 (1 + 1 -k′ 2).

29. This, and the third apotome, -2 k1/2 x + k k′ 2 = 0.

Complete Cinematic Logic

Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number.

⇓

And FG (is) a rational (straight-line).

⇓

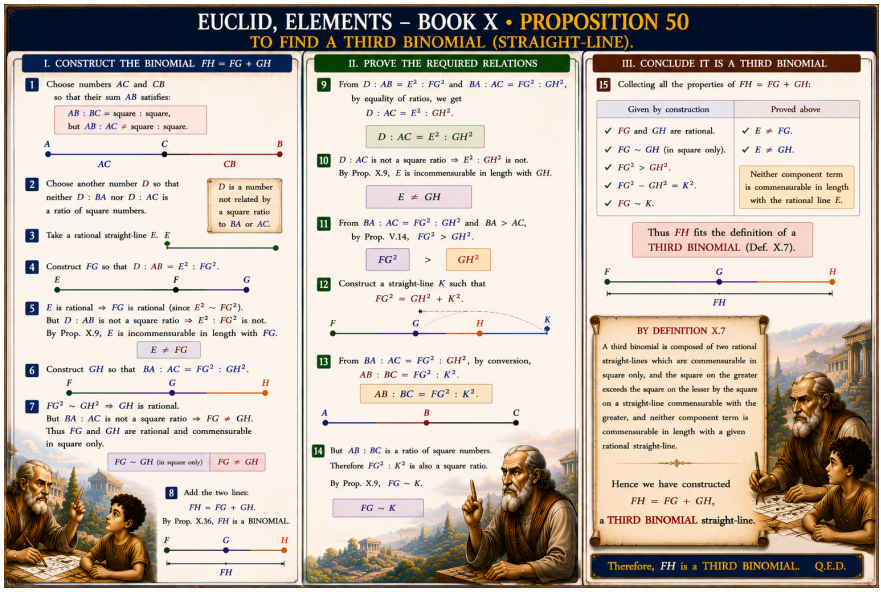
Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG.

⇓

This, and the third apotome, -2 k1/2 x + k k′ 2 = 0.

⇓

Conclusion: Proposition 50 established.



2.51 Proposition 51

Statement

To find a fourth binomial (straight-line).

Proof Actions

1. Let the two numbers AC and CB be laid down such that AB does not have to BC, or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].
2. And let the rational (straight-line) D be laid down.
3. And let EF be commensurable in length with D.
4. Thus, EF is also a rational (straight-line).
5. And let it have been contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.].
6. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6].
7. Thus, FG is also a rational (straightline).
8. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) Thus, EF is incommensurable square number either. in length with FG [Prop. 10.9].
9. Thus, EF and FG are rational (straight-lines which are) commensurable in square only.
10. Hence, EG is a binomial (straight-line) [Prop. 10.36].
11. So, I say that (it is) also a fourth (binomial straight-line).
12. For since as BA is to AC, so the (square) on EF (is) to the (square) on FG [and BA (is) greater than AC], the (square) on EF (is) thus greater than the (square) on FG [Prop. 5.14].

13. Therefore, let (the sum of) the squares on FG and H be equal to the (square) on EF.

14. Thus, via conversion, as the number AB (is) to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.].

15. And AB does not have to BC the ratio which (some) square number (has) to (some) square number.

16. Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

17. Thus, EF is incommensurable in length with H [Prop. 10.9].

18. Thus, the square on EF is greater than (the square on) GF by the (square) on (some straight-line) incommensurable (in length) with (EF).

19. And EF and FG are rational (straight-lines which are) commensurable in square only.

20. And EF is commensurable in length with D.

21. Thus, EG is a fourth binomial (straight-line) [Def. 10.8].

22. (Which is) the very thing it was required to show. binomial straight-line is $k (1 + 1/\sqrt{1 + k'})$.

23. This, and the fourth apotome, $k x + k2 k'/(1 + k') = 0$.

Complete Cinematic Logic

Let the two numbers AC and CB be laid down such that AB does not have to BC, or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I].

⇓

And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) Thus, EF is incommensurable square number either. in length with FG [Prop. 10.9].

⇓

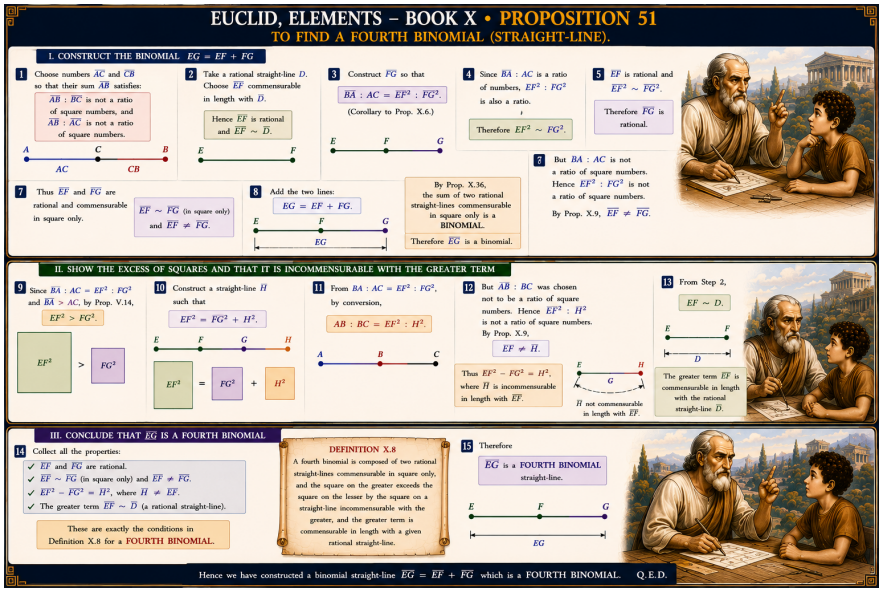
Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

⇓

This, and the fourth apotome, $k\,x + k2\,k'/(1 + k') = 0$.

⇓

Conclusion: Proposition 51 established.



2.52 Proposition 52

Statement

To find a fifth binomial straight-line.

Proof Actions

1. Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.].
2. And let some rational straight-line D be laid down.
3. And let EF be commensurable [in length] with D.
4. Thus, EF (is) a rational (straightline).
5. And let it have been contrived that as CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.].
6. And CA does not have to AB the ra338 tio which (some) square number (has) to (some) square number.
7. Thus, the (square) on EF does not have to the (square) on FG the ratio which (some) square number Thus, EF and (has) to (some) square number either.
8. FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9].
9. Thus, EG is a binomial (straight-line) [Prop. 10.36].
10. So, I say that (it is) also a fifth (binomial straight-line).
11. For since as CA is to AB, so the (square) on EF (is) to the (square) on FG, inversely, as BA (is) to AC, so the (square) on FG (is) to the (square) on FE [Prop. 5.7 corr.].
12. Thus, the (square) on GF (is) greater than the (square) on FE [Prop. 5.14].
13. Therefore, let (the sum of) the (squares) on EF and H be equal to the (square) on GF.

14. Thus, via conversion, as the number AB is to BC, so the (square) on GF (is) to the (square) on H [Prop. 5.19 corr.].

15. And AB does not have to BC the ratio which (some) square number (has) to (some) square number.

16. Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

17. Thus, FG is incommensurable in length with H [Prop. 10.9].

18. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) incommensurable (in length) with (FG).

19. And GF and FE are rational (straight-lines which are) commensurable in square only.

20. And the lesser term EF is commensurable in length with the rational (straight-line previously) laid down, D.

21. Thus, EG is a fifth binomial (straight-line).

22. (Which is) the very thing it was required to show. straight-line is k ($\sqrt{1}$ + k′ + 1).

23. This, and the fifth apotome, whose k′ x + k2 k′ = 0.

Complete Cinematic Logic

Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.].

⇓

FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9].

⇓

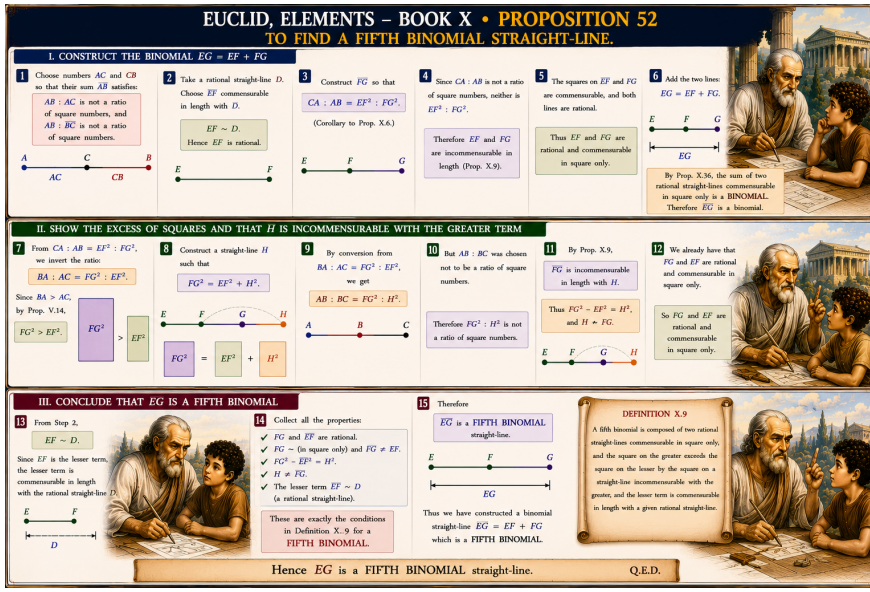
Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

⇓

This, and the fifth apotome, whose k′ x + k2 k′ = 0.

⇓

Conclusion: Proposition 52 established.



2.53 Proposition 53

Statement

To find a sixth binomial (straight-line).

Proof Actions

1. Let the two numbers AC and CB be laid down such that AB does not have to each of them the ratio which (some) square number (has) to (some) square number.

2. And let D also be another number, which is not square, and does not have to each of BA and AC the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I].

3. And let some rational straightline E be laid down.

4. And let it have been contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.].

5. Thus, the (square) on E (is) commensurable with the (square) on FG [Prop. 10.6]. And E is rational.

6. Thus, FG (is) also rational.

7. And since D does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on E thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either.

8. Thus, E (is) incommensurable in length with FG [Prop. 10.9].

9. So, again, let it have be contrived that as BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].

10. The (square) on FG (is) thus commensurable with the (square) on HG The (square) on HG (is) thus rational. [Prop. 10.6]. Thus, HG (is) rational.

11. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either.

12. Thus, FG is incommensurable in length with GH [Prop. 10.9].

13. Thus, FG and GH are rational (straight-lines which are) commensurable in square only.

14. Thus, FH is a binomial (straight-line) [Prop. 10.36].

15. So, we must show that (it is) also a sixth (binomial straight-line).

16. For since as D is to AB, so the (square) on E (is) to the (square) on FG, and also as BA is to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D is to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22].

17. And D does not have to AC the ratio which (some) square number (has) Thus, the (square) on E to (some) square number. does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either.

18. E is thus incommensurable in length with GH [Prop. 10.9].

19. And (E) was also shown (to be) incommensurable (in length) with FG.

20. Thus, FG and GH are each incommensurable in length with E.

21. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14].

22. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG.

23. Thus, via conversion, as AB (is) to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.].

24. And AB does not have to BC the ratio which (some) square number (has) to (some) square number.

25. Hence, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either.

26. Thus, FG is incommensurable in length with K [Prop. 10.9].

27. The square on FG is thus greater than (the square on) GH by the (square) on (some straight-line which is) incom340 mensurable (in length) with (FG).

28. And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) E (previously) laid down.

29. Thus, FH is a sixth binomial (straight-line) [Def. 10.10].

30. (Which is) the very thing it was required to show. √ √ k′.

31. This, and the sixth apotome, whose binomial straight-line is k + = 0.

32. Lemma Let AB and BC be two squares, and let them be laid down such that DB is straight-on to BE.

33. FB is, thus, also straight-on to BG.

34. And let the parallelogram AC I say that AC is a square, and have been completed. that DG is the mean proportional to AB and BC, and, moreover, DC is the mean proportional to AC and CB.

35. For since DB is equal to BF, and BE to BG, the whole of DE is thus equal to the whole of FG.

36. But DE is equal to each of AH and KC, and FG is equal to each of AK and HC [Prop. 1.34].

37. Thus, AH and KC are also equal to AK and HC, respectively.

38. Thus, the parallelogram AC is equilateral.

39. And (it is) also right-angled. Thus, AC is a square.

40. And since as FB is to BG, so DB (is) to BE, but as FB (is) to BG, so AB (is) to DG, and as DB (is) to BE, so DG (is) to BC [Prop. 6.1], thus also as AB (is) to DG, so DG (is) to BC [Prop. 5.11].

41. Thus, DG is the mean proportional to AB and BC.

42. So I also say that DC [is] the mean proportional to AC and CB.

43. For since as AD is to DK, so KG (is) to GC.

44. For [they are] respectively equal.

45. And, via composition, as AK (is) to KD, so KC (is) to CG [Prop. 5.18].

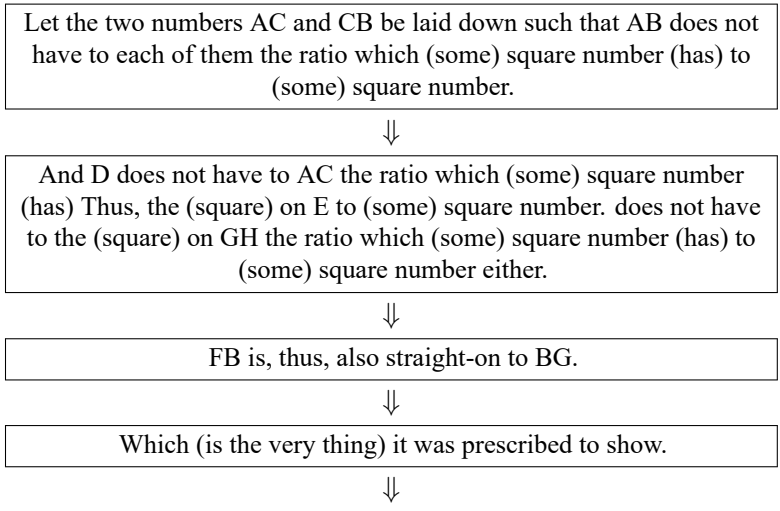
46. But as AK (is) to KD, so AC (is) to CD, and as KC (is) to CG, so DC (is) to CB [Prop. 6.1].

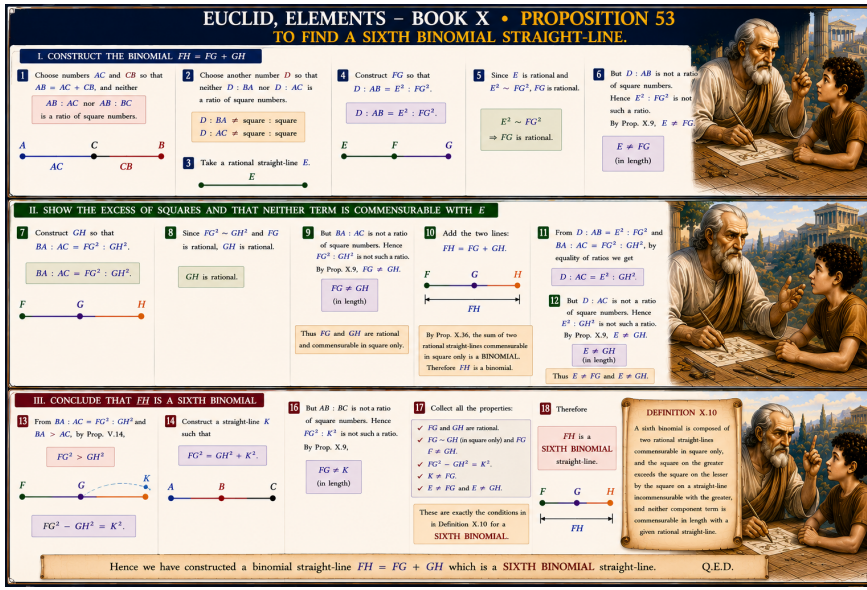
47. Thus, also, as AC (is) to DC, so DC (is) to BC [Prop. 5.11].

48. Thus, DC is the mean proportional to AC and CB.

49. Which (is the very thing) it was prescribed to show.

Complete Cinematic Logic





2.54 Proposition 54

Statement

> *If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial.*

Proof Actions

1. For let the area AC be contained by the rational (straight-line) AB and by the first binomial (straightline) AD.

2. I say that square-root of area AC is the irrational (straight-line which is) called binomial.

3. For since AD is a first binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term.

4. So, (it is) clear that AE and ED are rational (straight-lines which are) commensurable in square only, and that the square on AE is greater than

(the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and that AE is commensurable (in length) with the rational (straight-line) AB (first) laid out [Def. 10.5].

5. So, let ED have been cut in half at point F.

6. And since the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term)—that is to say, the (square) on EF—falling short by a square figure, is applied to the greater (term) AE, then it divides it into (terms which are) commensurable (in length) [Prop 10.17].

7. Therefore, let the (rectangle contained) by AG and GE, equal to the (square) on EF, have been applied to AE.

8. AG is thus commensurable in length with EG.

9. And let GH, EK, and FL have been drawn from (points) G, E, and F (respectively), parallel to either of AB or CD.

10. And let the square SN, equal to the parallelogram AH, have been constructed, and (the square) NQ, equal to (the parallelogram) GK [Prop. 2.14].

11. And let MN be laid down so as to be straight-on to NO.

12. RN is thus also straight-on to NP.

13. And let the parallelogram SQ have been completed.

14. SQ is thus a square [Prop. 10.53 lem.].

15. And since the (rectangle contained) by AG and GE is equal to the (square) on EF, thus as AG is to EF, so FE (is) to EG [Prop. 6.17].

16. And thus as AH (is) to EL, (so) EL (is) to KG [Prop. 6.1].

17. Thus, EL is the mean proportional to AH and GK.

18. But, AH is equal to SN, and GK (is) equal to NQ.

19. EL is thus the mean proportional to SN and NQ.

20. And MR is also the mean proportional to the same— (namely), SN and NQ [Prop. 10.53 lem.]. EL is thus equal to MR.

21. Hence, it is also equal to PO [Prop. 1.43].

22. And AH plus GK is equal to SN plus NQ.

23. Thus, the whole of AC is equal to the whole of SQ—that is to say, to the square on MO.

24. Thus, MO (is) the square-root of (area) AC.

25. I say that MO is a binomial (straight-line).

26. For since AG is commensurable (in length) with GE, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15].

27. And AE was also assumed (to be) commensurable (in length) with AB.

28. Thus, AG and GE are also commensurable (in length) with AB And AB is rational.

29. AG and GE are [Prop. 10.12].

30. Thus, AH and GK are each thus each also rational. rational (areas), and AH is commensurable with GK [Prop. 10.19].

31. But, AH is equal to SN, and GK to NQ.

32. SN and NQ—that is to say, the (squares) on MN and NO (respectively)—are thus also rational and commenAnd since AE is incommensurable in length surable. with ED, but AE is commensurable (in length) with AG, and DE (is) commensurable (in length) with EF, AG (is) thus also incommensurable (in length) with EF [Prop. 10.13].

33. Hence, AH is also incommensurable with EL [Props. 6.1, 10.11].

34. But, AH is equal to SN, and EL to MR.

35. Thus, SN is also incommensurable with MR.

36. But, as SN (is) to MR, (so) PN (is) to NR PN is thus incommensurable (in length) [Prop. 6.1]. with NR [Prop. 10.11].

37. And PN (is) equal to MN, and NR to NO.

38. Thus, MN is incommensurable (in length) with NO.

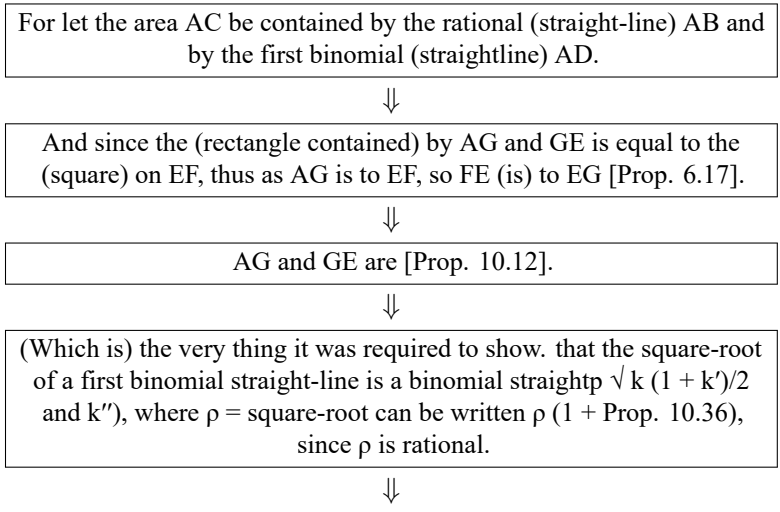
39. And the (square) on MN is commensurable with the (square) on NO, and each (is) rational.

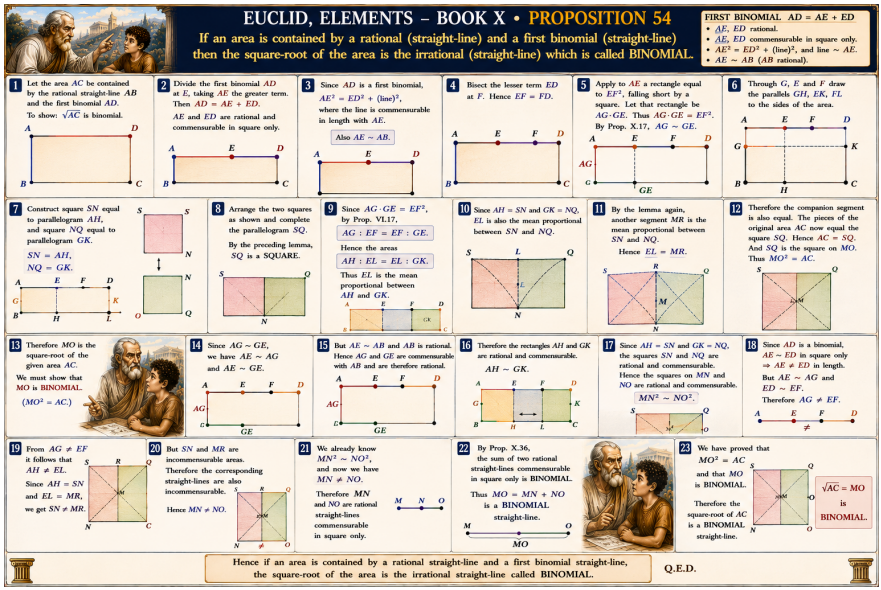
40. MN and NO are thus rational (straight-lines which are) commensurable in square only.

41. Thus, MO is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of AC.

42. (Which is) the very thing it was required to show. that the square-root of a first binomial straight-line is a binomial straightp $\sqrt{\ }$ k (1 + k′)/2 and k″), where ρ = square-root can be written ρ (1 + Prop. 10.36), since ρ is rational.

Complete Cinematic Logic





2.55 Proposition 55

Statement

If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedial.

Proof Actions

1. For let the area ABCD be contained by the rational (straight-line) AB and by the second binomial (straightline) AD.

2. I say that the square-root of area AC is a first bimedial (straight-line).

3. For since AD is a second binomial (straight-line), let it have been divided into its (component) terms at E, such that AE is the greater term.

4. Thus, AE and ED are rational (straight-lines which are) commensurable in square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and the lesser term ED is commensurable in length with AB [Def. 10.6].

5. Let ED have been cut in half at F.

6. And let the (rectangle contained) by AGE, equal to the (square) on EF, have been applied to AE, falling short by a square figAG (is) thus commensurable in length with GE ure.

7. And let GH, EK, and FL have been [Prop. 10.17]. drawn through (points) G, E, and F (respectively), parallel to AB and CD.

8. And let the square SN, equal to the parallelogram AH, have been constructed, and the square NQ, equal to GK.

9. And let MN be laid down so as to be straight-on to NO.

10. Thus, RN [is] also straight-on to NP.

11. And let the square SQ have been completed.

12. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that MR is the mean proportional to SN and NQ, and (is) equal to EL, and that MO is the square-root of the area AC.

13. So, we must show that MO is a first bimedial (straight-line).

14. Since AE is incommensurable in length with ED, and ED (is) commensurable (in length) with AB, AE (is) thus incommensurable (in length) with AB And since AG is commensurable (in [Prop. 10.13]. length) with EG, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15].

15. But, AE is incommensurable in length with AB.

16. Thus, AG and GE are also (both) incommensurable (in length) with AB [Prop. 10.13].

17. Thus, BA, AG, and (BA, and) GE are (pairs of) rational (straight-lines which are) commensurable in square only.

18. And, hence, each of AH and GK Hence, each of SN is a medial (area) [Prop. 10.21]. and NQ is also a medial (area).

19. Thus, MN and NO are medial (straight-lines).

20. And since AG (is) commensurable in length with GE, AH is also commensurable with GK—that is to say, SN with NQ—that is to say, the

(square) on MN with the (square) on NO [hence, MN and NO are commensurable in square] [Props. 6.1, 10.11].

21. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and ED commensurable (in length) with EF, AG (is) thus incommensurable (in length) with EF [Prop. 10.13].

22. Hence, AH is also incommensurable with EL—that is to say, SN with MR—that is to say, PN with NR—that is to say, MN is incommensurable in length with NO [Props. 6.1, 10.11].

23. But MN and NO have also been shown to be medial (straight-lines) which are commensurable in square.

24. Thus, MN and NO are medial (straightlines which are) commensurable in square only.

25. So, I say that they also contain a rational (area).

26. For since DE was assumed (to be) commensurable (in length) with each of AB and EF, EF (is) thus also commensurable with EK [Prop. 10.12].

27. And they (are) each rational.

28. Thus, EL— that is to say, MR—(is) rational [Prop. 10.19].

29. And MR is the (rectangle contained) by MNO.

30. And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedial [Prop. 10.37].

31. Thus, MO is a first bimedial (straight-line).

32. (Which is) the very thing it was required to show. that the square-root of a second binomial straight-line is a first bimedial -k′ 2 + k whose square-root can be written ρ (k″1/4 + k″3/4), where length of a first bimedial straight-line (see Prop. 10.37), since ρ is rational.

Complete Cinematic Logic

For let the area ABCD be contained by the rational (straight-line) AB and by the second binomial (straightline) AD.

⇓

And let the square SQ have been completed.

⇓

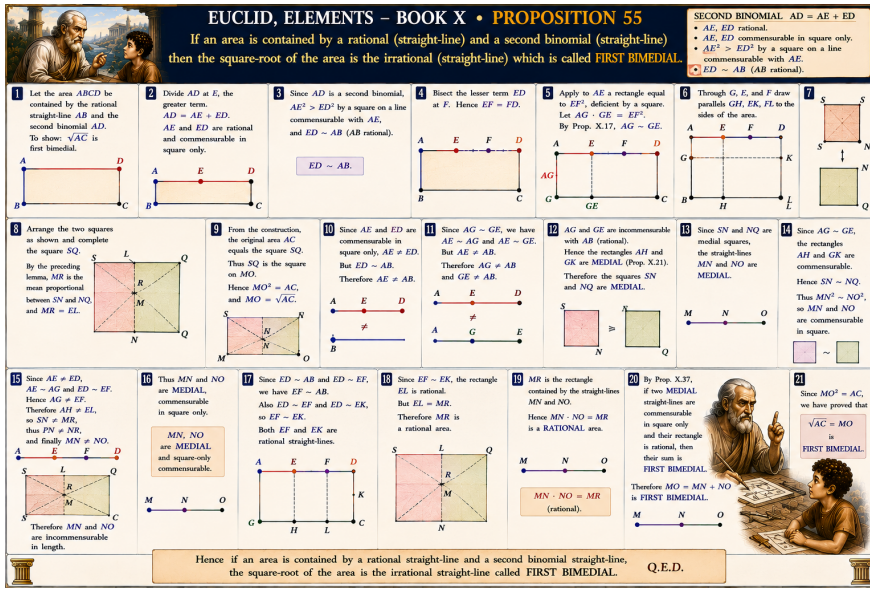
Hence, AH is also incommensurable with EL—that is to say, SN with MR—that is to say, PN with NR—that is to say, MN is incommensurable in length with NO [Props. 6.1, 10.11].

⇓

(Which is) the very thing it was required to show. that the square-root of a second binomial straight-line is a first bimedial -k′ 2 + k whose square-root can be written ρ (k″1/4 + k″3/4), where length of a first bimedial straight-line (see Prop. 10.37), since ρ is rational.

⇓

Conclusion: Proposition 55 established.



2.56 Proposition 56

Statement

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedial.

Proof Actions

1. For let the area ABCD be contained by the rational (straight-line) AB and by the third binomial (straightline) AD, which has been divided into its (component) terms at E, of which AE is the greater.

2. I say that the square-root of area AC is the irrational (straight-line which is) called second bimedial.

3. For let the same construction be made as previously.

4. And since AD is a third binomial (straight-line), AE and ED are thus rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and neither of AE and ED [is] commensurable in length with AB [Def. 10.7].

5. So, similarly to that which has been previously demonstrated, we can show that MO is the square-root of area AC, and MN and NO are medial (straight-lines which are) commensurable in square only. Hence, MO is bimedial.

6. So, we must show that (it is) also second (bimedial).

7. [And] since DE is incommensurable in length with AB—that is to say, with EK—and DE (is) commensurable (in length) with EF, EF is thus incommensurable in length with EK [Prop. 10.13].

8. And they are (both) rational (straight-lines).

9. Thus, FE and EK are rational (straight-lines which are) commensurable in square only.

10. EL—that is to say, MR—[is] thus medial [Prop. 10.21].

11. And it is contained by MNO.

12. Thus, the (rectangle contained) by MNO is medial.

13. Thus, MO is a second bimedial (straight-line) [Prop. 10.38].

14. (Which is) the very thing it was required to show. that the square-root of a third binomial straight-line is a second bimedial 1 -k′ 2) whose square-root can be written ρ (k1/4 + k″1/2/k1/4), where second bimedial straight-line (see Prop. 10.38), since ρ is rational.

Complete Cinematic Logic

For let the area ABCD be contained by the rational (straight-line) AB and by the third binomial (straightline) AD, which has been divided into its (component) terms at E, of which AE is the greater.

⇓

So, similarly to that which has been previously demonstrated, we can show that MO is the square-root of area AC, and MN and NO are medial (straight-lines which are) commensurable in square only. Hence, MO is bimedial.

⇓

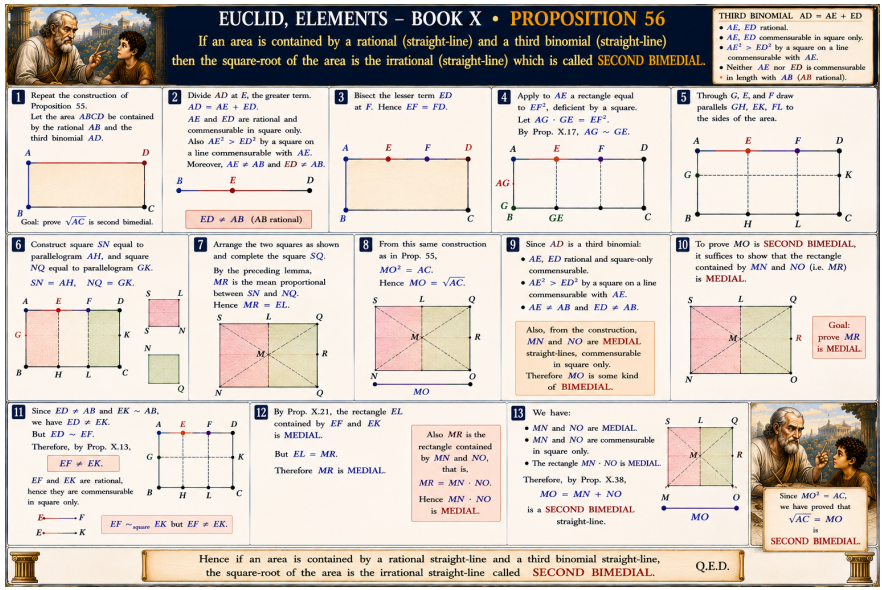
EL—that is to say, MR—[is] thus medial [Prop. 10.21].

⇓

(Which is) the very thing it was required to show. that the square-root of a third binomial straight-line is a second bimedial 1 -k′ 2) whose square-root can be written ρ (k1/4 + k″1/2/k1/4), where second bimedial straight-line (see Prop. 10.38), since ρ is rational.

⇓

Conclusion: Proposition 56 established.



2.57 Proposition 57

Statement

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major.

Proof Actions

1. For let the area AC be contained by the rational (straight-line) AB and the fourth binomial (straight-line) AD, which has been divided into its (component) terms at E, of which let AE be the greater.

2. I say that the squareroot of AC is the irrational (straight-line which is) called major.

3. For since AD is a fourth binomial (straight-line), AE and ED are thus rational (straight-lines which are) com346 mensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on

(some straight-line) incommensurable (in length) with (AE), and AE [is] commensurable in length with AB [Def. 10.8].

4. Let DE have been cut in half at F, and let the parallelogram (contained by) AG and GE, equal to the (square) on EF, (and falling short by a square figure) have been applied to AE.

5. AG is thus incommensurable in length with GE [Prop. 10.18].

6. Let GH, EK, and FL have been drawn parallel to AB, and let the rest (of the construction) have been made the same as the (proposition) before this.

7. So, it is clear that MO is the square-root of area AC.

8. So, we must show that MO is the irrational (straight-line which is) called major.

9. Since AG is incommensurable in length with EG, AH is also incommensurable with GK—that is to say, SN with NQ [Props. 6.1, 10.11].

10. Thus, MN and NO are incommensurable in square.

11. And since AE is commensurable in length with AB, AK is rational [Prop. 10.19].

12. And it is equal to the (sum of the squares) on MN and NO.

13. Thus, the sum of the (squares) on MN and NO [is] also rational.

14. And since DE [is] incommensurable in length with AB [Prop. 10.13]—that is to say, with EK—but DE is commensurable (in length) with EF, EF (is) thus incommensurable in length with EK [Prop. 10.13].

15. Thus, EK and EF are rational (straightlines which are) commensurable in square only.

16. LE— that is to say, MR—(is) thus medial [Prop. 10.21].

17. And it is contained by MN and NO.

18. The (rectangle contained) by MN and NO is thus medial.

19. And the [sum] of the (squares) on MN and NO (is) rational, and MN and NO are incommensurable in square.

20. And if two straightlines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

21. Thus, MO is the irrational (straight-line which is) And (it is) the square-root of area AC. called major.

22. (Which is) the very thing it was required to show. that the square-root of a fourth binomial straight-line is a major straightq [1 + k″/(1 + k″ 2)1/2]/2 + + k′) whose square-root can be written ρ length of a major straight-line (see Prop. 10.39), since ρ is rational.

Complete Cinematic Logic

For let the area AC be contained by the rational (straight-line) AB and the fourth binomial (straight-line) AD, which has been divided into its (component) terms at E, of which let AE be the greater.

⇓

So, we must show that MO is the irrational (straight-line which is) called major.

⇓

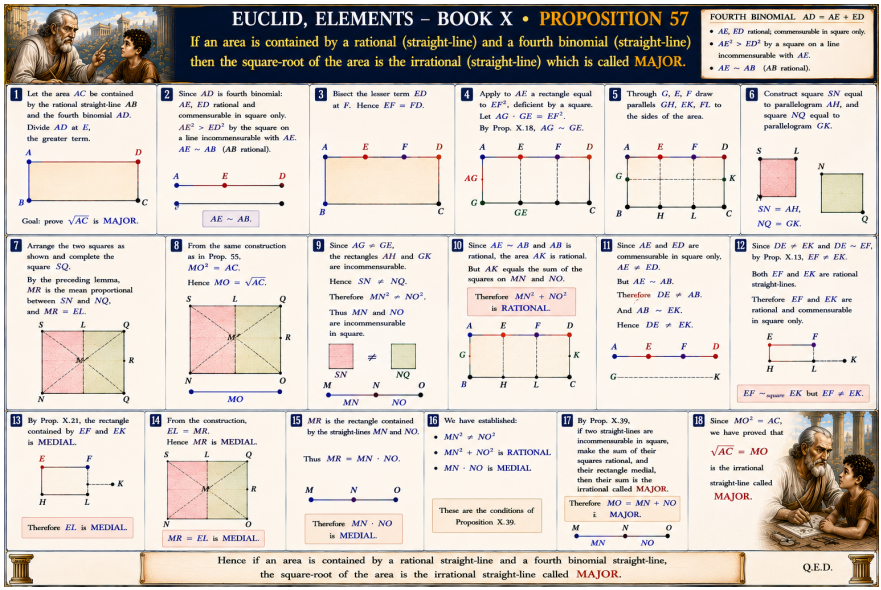
Thus, EK and EF are rational (straightlines which are) commensurable in square only.

⇓

(Which is) the very thing it was required to show. that the square-root of a fourth binomial straight-line is a major straightq [1 + k″/(1 + k″ 2)1/2]/2 + + k′) whose square-root can be written ρ length of a major straight-line (see Prop. 10.39), since ρ is rational.

⇓

Conclusion: Proposition 57 established.



2.58 Proposition 58

Statement

If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).

Proof Actions

1. For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term.

2. [So] I say that the square-root of area AC is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).

3. For let the same construction be made as that shown previously.

4. So, (it is) clear that MO is the square-root of area AC.

5. So, we must show that MO is the square-root of a rational plus a medial (area).

6. For since AG is incommensurable (in length) with GE [Prop. 10.18], AH is thus also incommensurable with HE—that is to say, the (square) on MN with the (square) on NO [Props. 6.1, 10.11].

7. Thus, MN and NO are incommensurable in square.

8. And since AD is a fifth binomial (straight-line), and ED [is] its lesser segment, ED (is) thus commensurable in length with AB But, AE is incommensurable (in length) [Def. 10.9]. with ED.

9. Thus, AB is also incommensurable in length with AE [BA and AE are rational (straight-lines which are) commensurable in square only] [Prop. 10.13].

10. Thus, AK—that is to say, the sum of the (squares) on MN and NO—is medial [Prop. 10.21].

11. And since DE is commensurable in length with AB—that is to say, with EK—but, DE is commensurable (in length) with EF, EF is thus also commensurable (in length) with EK [Prop. 10.12]. And EK (is) rational.

12. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—(is) also rational [Prop. 10.19].

13. MN and NO are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

14. Thus, MO is the square-root of a rational plus a medial (area) [Prop. 10.40].

15. And (it is) the square-root of area AC.

16. (Which is) the very thing it was required to show. that the square-root of a fifth binomial straight-line is the square root of k ($\sqrt{1 + k' + 1}$) whose square-root can be written p k ($1 + k'' 2$) and $k'' 2 = k'$.

17. This is the length of ($1 + k'' 2$)], where $\rho = \rho$ is rational.

Complete Cinematic Logic

For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term.

⇓

For since AG is incommensurable (in length) with GE [Prop. 10.18], AH is thus also incommensurable with HE—that is to say, the (square) on MN with the (square) on NO [Props. 6.1, 10.11].

⇓

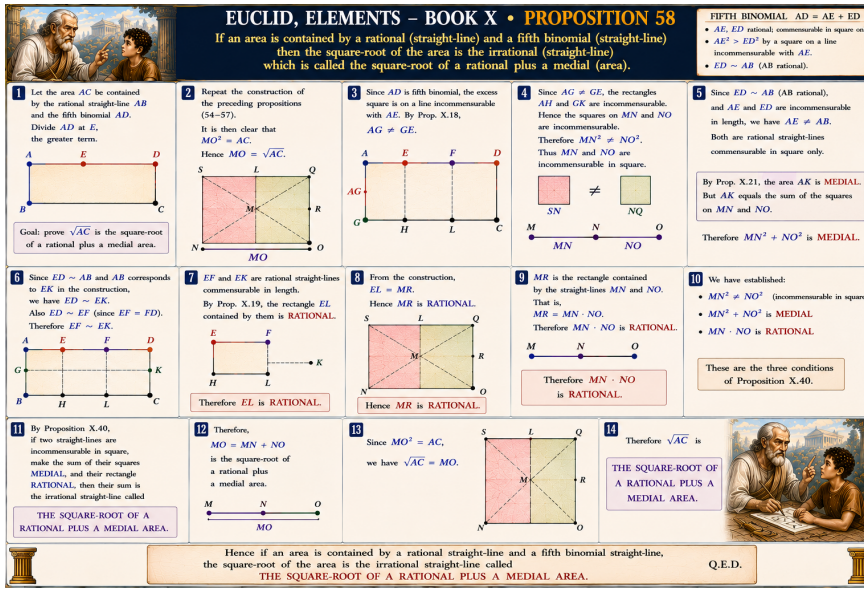
Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—(is) also rational [Prop. 10.19].

⇓

This is the length of (1 + k″ 2)], where ρ = ρ is rational.

⇓

Conclusion: Proposition 58 established.



2.59 Proposition 59

Statement

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas).

Proof Actions

1. For let the area ABCD be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term.

2. So, I say that the square-root of AC is the square-root of (the sum of) two medial (areas).

3. [For] let the same construction be made as that shown previously.

4. So, (it is) clear that MO is the square-root of AC, and that MN is incommensurable in square with NO.

5. And since EA is incommensurable in length with AB [Def. 10.10], EA and AB are thus rational (straightlines which are) commensurable in square only.

6. Thus, AK—that is to say, the sum of the (squares) on MN and NO—is medial [Prop. 10.21].

7. Again, since ED is incommensurable in length with AB [Def. 10.10], FE is thus also incommensurable (in length) with EK [Prop. 10.13].

8. Thus, FE and EK are rational (straightlines which are) commensurable in square only.

9. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—is medial [Prop. 10.21].

10. And since AE is incommensurable (in length) with EF, AK is also incommensurable with EL [Props. 6.1, 10.11].

11. But, AK is the sum of the (squares) on MN and NO, and EL is the (rectangle contained) by MNO.

12. Thus, the sum of the (squares) on MNO is incommensurable with the (rectangle contained) by MNO.

13. And each of them is medial.

14. And MN and NO are incommensurable in square.

15. Thus, MO is the square-root of (the sum of) two medial (areas) [Prop. 10.41].

16. And (it is) the square-root of AC.

17. (Which is) the very thing it was required to show. that the square-root of a sixth binomial straight-line is the square root of $\sqrt{\sqrt{k'}}$ whose square-root can be written $k + k'' 2 = (k - k')/k'$.

18. This is the length of the square-root of the sum of Lemma If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).

19. Let AB be a straight-line, and let it have been cut unequally at C, and let AC be greater (than CB).

20. I say that (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB.

21. For let AB have been cut in half at D.

22. Therefore, since a straight-line has been cut into equal (parts) at D, and into unequal (parts) at C, the (rectangle contained) by AC and CB, plus the (square) on CD, is thus equal to the (square) on AD [Prop. 2.5].

23. Hence, the (rectangle contained) by AC and CB is less than the (square) on AD.

24. Thus, twice the (rectangle contained) by AC and CB is less than double the (square) on AD.

25. But, (the sum of) the (squares) on AC and CB [is] double (the sum of) the (squares) on AD and DC [Prop. 2.9].

26. Thus, (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB.

27. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the area ABCD be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term.

⇓

And since AE is incommensurable (in length) with EF, AK is also incommensurable with EL [Props. 6.1, 10.11].

⇓

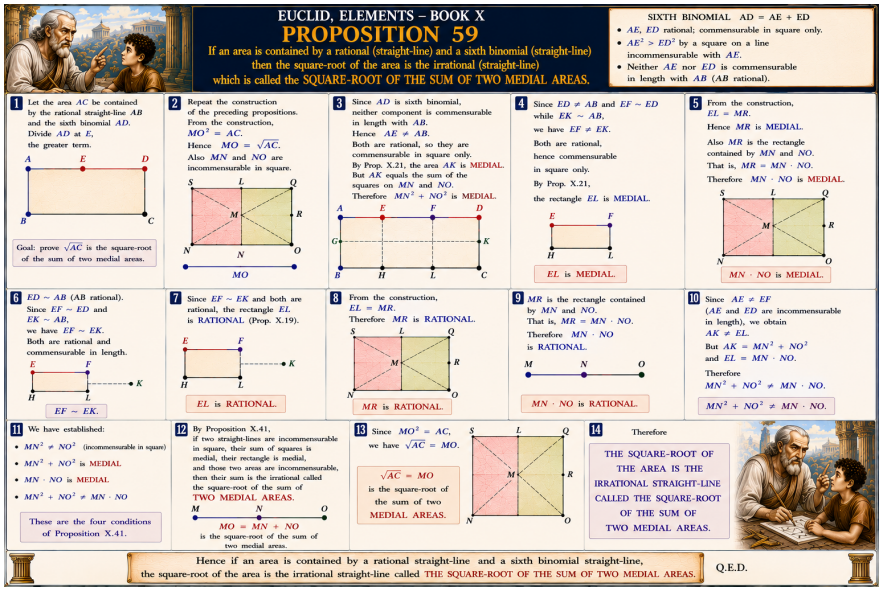
Let AB be a straight-line, and let it have been cut unequally at C, and let AC be greater (than CB).

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 59 established.



2.60 Proposition 60

Statement

> *The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line).*

Proof Actions

1. Let AB be a binomial (straight-line), having been divided into its (component) terms at C, such that AC is the greater term.

2. And let the rational (straight-line) DE be laid down.

3. And let the (rectangle) DEFG, equal to the (square) on AB, have been applied to DE, producing DG as breadth.

4. I say that DG is a first binomial (straightline).

5. For let DH, equal to the (square) on AC, and KL, equal to the (square) on BC, have been applied to DE.

6. Thus, the remaining twice the (rectangle contained) by AC and CB is equal to MF [Prop. 2.4].

7. Let MG have been cut in half at N, and let NO have been drawn parallel [to each of ML and GF].

8. MO and NF are thus each equal to once the (rectangle contained) by ACB.

9. And since AB is a binomial (straight-line), having been divided into its (component) terms at C, AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36].

10. Thus, the (squares) on AC and CB are rational, and commensurable with one another.

11. And hence the sum of the (squares) on AC and CB (is rational) [Prop. 10.15], and is equal to DL. Thus, DL is rational.

12. And it is applied to the rational (straightline) DE.

13. DM is thus rational, and commensurable in length with DE [Prop. 10.20].

14. Again, since AC and CB are rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by AC and CB—that is to say, MF—is thus medial [Prop. 10.21].

15. And it is applied to the rational (straight-line) ML.

16. MG is thus also rational, and incommensurable in length with ML—that is to say, with DE [Prop. 10.22].

17. And MD is also rational, and commensurable in length with DE.

18. Thus, DM is incommensurable in length with MG [Prop. 10.13]. And they are rational.

19. DM and MG are thus rational (straight-lines which are) commensurable Thus, DG is a binomial (straight-line) in square only. [Prop. 10.36].

20. So, we must show that (it is) also a first (binomial straight-line).

21. Since the (rectangle contained) by ACB is the mean proportional to the squares on AC and CB [Prop. 10.53 lem.], MO is thus also the mean proportional to DH and KL.

22. Thus, as DH is to MO, so MO (is) to KL—that is to say, as DK (is) to MN, (so) MN (is) to MK [Prop. 6.1].

23. Thus, the (rectangle contained) by DK and KM is equal to the (square) on MN [Prop. 6.17].

24. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL.

25. Hence, DK is also commensurable with KM [Props. 6.1, 10.11].

26. And since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59 lem.], DL (is) thus also greater than MF.

27. Hence, DM is also greater than MG [Props. 6.1, 5.14].

28. And the (rectangle contained) by DK and KM is equal to the (square) on MN—that is to say, to one quarter the (square) on MG.

29. And DK (is) commensurable (in length) with KM.

30. And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17].

31. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straightline) commensurable (in length) with (DM).

32. And DM and MG are rational.

33. And DM, which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) DE.

34. Thus, DG is a first binomial (straight-line) [Def. 10.5].

35. (Which is) the very thing it was required to show. 10.54.

Complete Cinematic Logic

Let AB be a binomial (straight-line), having been divided into its (component) terms at C, such that AC is the greater term.

⇓

And it is applied to the rational (straightline) DE.

⇓

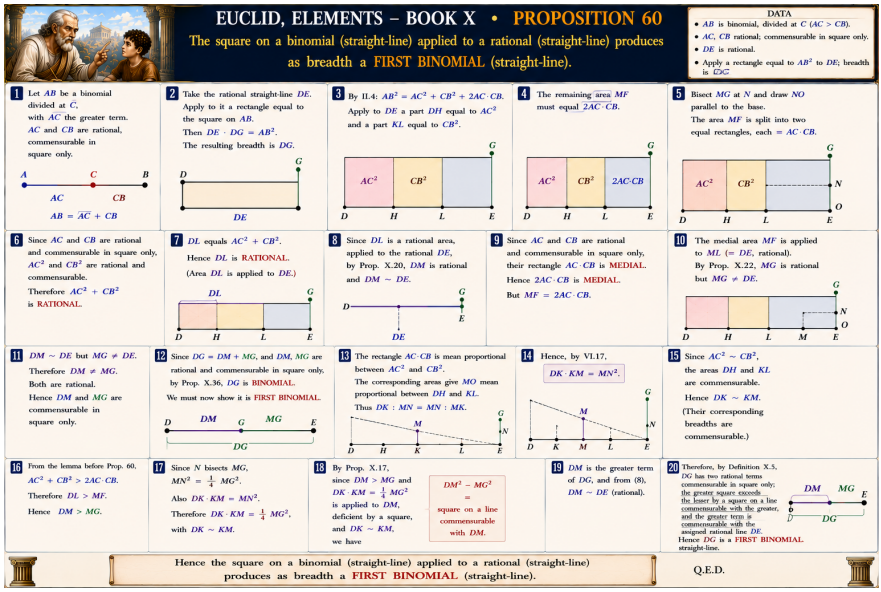
And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL.

⇓

(Which is) the very thing it was required to show. 10.54.

⇓

Conclusion: Proposition 60 established.



2.61 Proposition 61

Statement

The square on a first bimedial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line).

Proof Actions

1. Let AB be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, of which AC (is) the greater.
2. And let the rational (straight-line) DE be laid down.
3. And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth.
4. I say that DG is a second binomial (straight-line).
5. For let the same construction have been made as And since AB is a in the (proposition) before this. first bimedial (straight-line), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a rational (area) Hence, the (squares) on AC and CB [Prop. 10.37].
6. Thus, DL is medial are also medial [Prop. 10.21].
7. [Props. 10.15, 10.23 corr.].
8. And it has been applied to the rational (straight-line) DE.
9. MD is thus rational, and incommensurable in length with DE [Prop. 10.22].
10. Again, since twice the (rectangle contained) by AC and CB is rational, MF is also rational.
11. And it is applied to the rational (straight-line) ML.
12. Thus, MG [is] also rational, and commensurable in length with ML—that is to say, with DE [Prop. 10.20].

13. DM is thus incommensurable in length with MG [Prop. 10.13]. And they are rational.

14. DM and MG are thus rational, and commensu352 rable in square only.

15. DG is thus a binomial (straight-line) [Prop. 10.36].

16. So, we must show that (it is) also a second (binomial straight-line).

17. For since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59], DL (is) thus also greater than MF.

18. Hence, DM (is) also (greater) than MG [Prop. 6.1].

19. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL.

20. Hence, DK is also commensurable (in length) with KM [Props. 6.1, 10.11].

21. And the (rectangle contained) by DKM is equal to the (square) on MN.

22. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17].

23. And MG is commensurable in length with DE.

24. Thus, DG is a second binomial (straight-line) [Def. 10.6]. Prop. 10.55.

Complete Cinematic Logic

Let AB be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, of which AC (is) the greater.

⇓

MD is thus rational, and incommensurable in length with DE [Prop. 10.22].

⇓

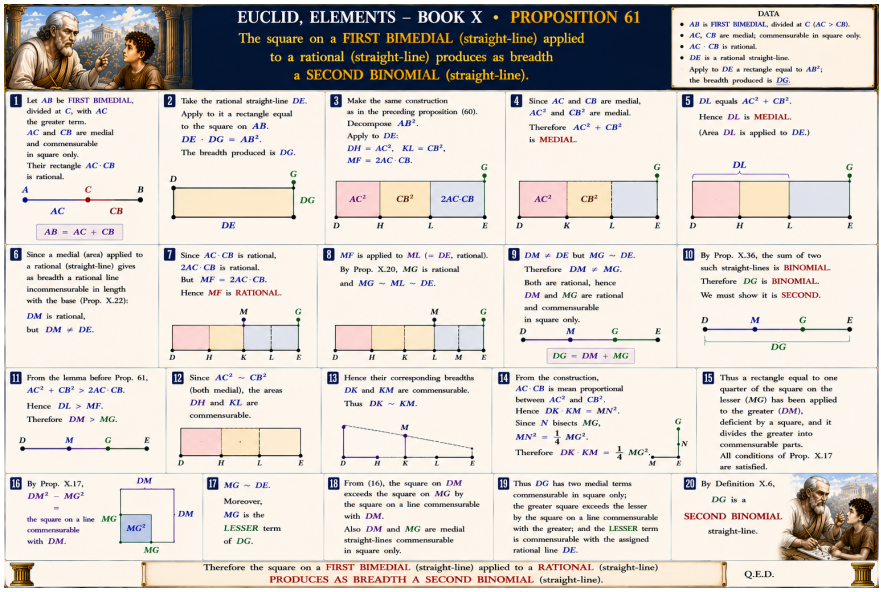
For since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59], DL (is) thus also greater than MF.

⇓

Thus, DG is a second binomial (straight-line) [Def. 10.6]. Prop. 10.55.

⇓

Conclusion: Proposition 61 established.



2.62 Proposition 62

Statement

The square on a second bimedial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line).

Proof Actions

1. Let AB be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, such that AC is the greater segment.

2. And let DE be some rational (straight-line).

3. And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth.

4. I say that DG is a third binomial (straight-line).

5. Let the same construction be made as that shown preAnd since AB is a second bimedial (straightviously. line), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a medial (area) [Prop. 10.38].

6. Hence, the sum of the (squares) on AC and CB is also medial [Props. 10.15, 10.23 corr.]. And it is equal to DL.

7. Thus, DL (is) also medial.

8. And it is applied to the rational (straight-line) DE.

9. MD is thus also rational, and in353 commensurable in length with DE [Prop. 10.22].

10. So, for the same (reasons), MG is also rational, and incommensurable in length with ML—that is to say, with DE.

11. Thus, DM and MG are each rational, and incommensurable in length with DE.

12. And since AC is incommensurable in length with CB, and as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by ACB [Prop. 10.21 lem.], the (square) on AC (is) also incommensurable with the (rectangle contained) by ACB [Prop. 10.11].

13. And hence the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by ACB—that is to say, DL with MF [Props. 10.12, 10.13].

14. Hence, DM is also incommensurable (in length) with MG [Props. 6.1, 10.11]. And they are rational.

15. DG is thus a binomial (straight-line) [Prop. 10.36].

16. [So] we must show that (it is) also a third (binomial straight-line).

17. So, similarly to the previous (propositions), we can conclude that DM is greater than MG, and DK (is) commensurable (in length) with KM.

18. And the (rectangle contained) by DKM is equal to the (square) on MN.

19. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17].

20. And neither of DM and MG is commensurable in length with DE.

21. Thus, DG is a third binomial (straight-line) [Def. 10.7].

22. (Which is) the very thing it was required to show. Prop. 10.56.

Complete Cinematic Logic

Let AB be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, such that AC is the greater segment.

⇓

And it is applied to the rational (straight-line) DE.

⇓

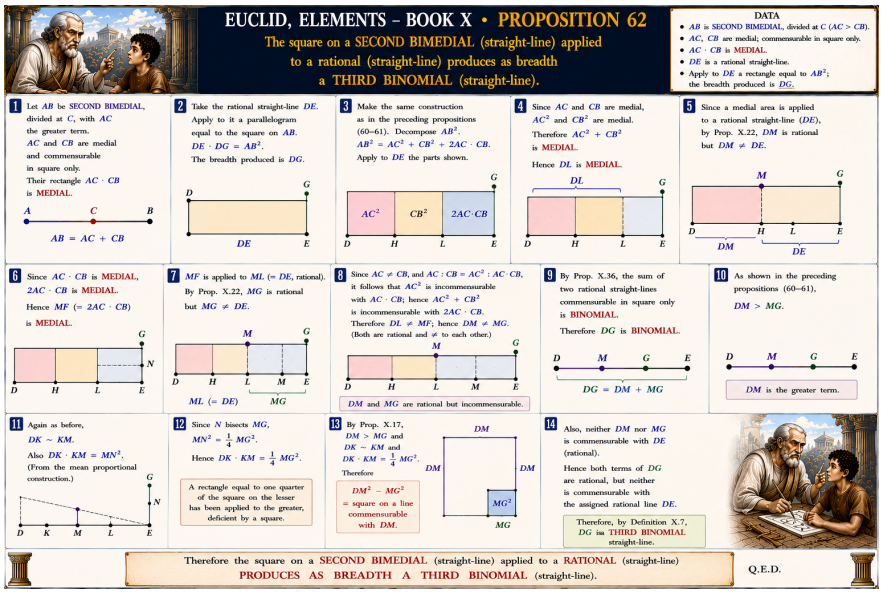
DG is thus a binomial (straight-line) [Prop. 10.36].

⇓

(Which is) the very thing it was required to show. Prop. 10.56.

⇓

Conclusion: Proposition 62 established.



2.63 Proposition 63

Statement

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line).

Proof Actions

1. Let AB be a major (straight-line) having been divided at C, such that AC is greater than CB, and (let) DE (be) a rational (straight-line).

2. And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth.

3. I say that DG is a fourth binomial (straight-line).

4. Let the same construction be made as that shown pre354 And since AB is a major (straight-line), havviously. ing been divided at C, AC and CB are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39].

5. Therefore, since the sum of the (squares) on AC and CB is rational, DL is thus rational.

6. Thus, DM (is) also rational, and commensurable in length with DE [Prop. 10.20].

7. Again, since twice the (rectangle contained) by AC and CB—that is to say, MF—is medial, and is (applied to) the rational (straight-line) ML, MG is thus also rational, and incommensurable in length with DE [Prop. 10.22].

8. DM is thus also incommensurable in length with MG [Prop. 10.13].

9. DM and MG are thus rational (straight-lines which are) commensurable Thus, DG is a binomial (straight-line) in square only. [Prop. 10.36].

10. [So] we must show that (it is) also a fourth (binomial straight-line).

11. So, similarly to the previous (propositions), we can show that DM is greater than MG, and that the (rectangle contained) by DKM is equal to the (square) on MN.

12. Therefore, since the (square) on AC is incommensurable with the (square) on CB, DH is also incommensurable with KL.

13. Hence, DK is also incommensurable with KM [Props. 6.1, 10.11].

14. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18].

15. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM).

16. And DM and MG are rational (straight-lines which are) commensurable in square only.

17. And DM is commensurable (in length) with the (previously) laid down rational (straight-line) DE.

18. Thus, DG is a fourth binomial (straight-line) [Def. 10.8].

19. (Which is) the very thing it was required to show. 10.57.

Complete Cinematic Logic

Let AB be a major (straight-line) having been divided at C, such that AC is greater than CB, and (let) DE (be) a rational (straight-line).

⇓

Again, since twice the (rectangle contained) by AC and CB—that is to say, MF—is medial, and is (applied to) the rational (straight-line) ML, MG is thus also rational, and incommensurable in length with DE [Prop. 10.22].

⇓

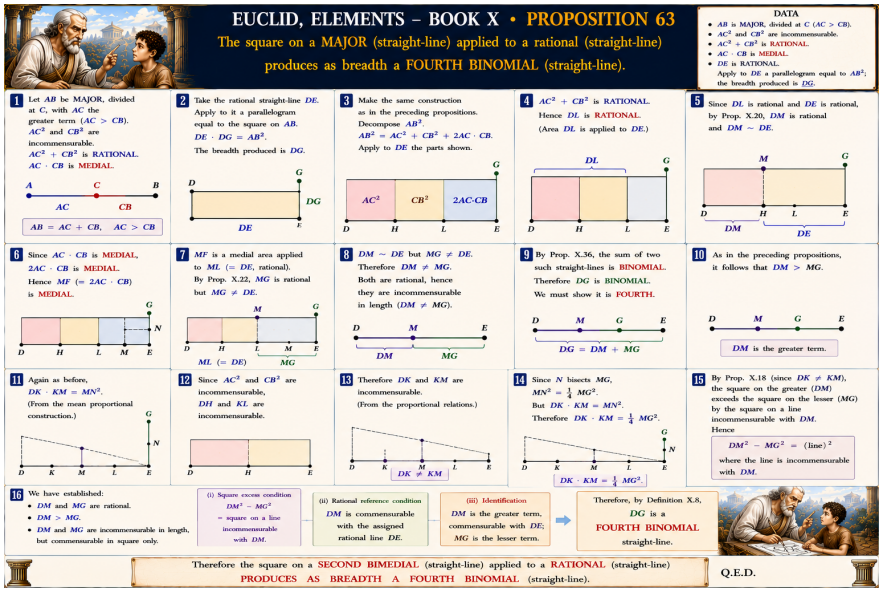
Hence, DK is also incommensurable with KM [Props. 6.1, 10.11].

⇓

(Which is) the very thing it was required to show. 10.57.

⇓

Conclusion: Proposition 63 established.



2.64 Proposition 64

Statement

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line).

Proof Actions

1. Let AB be the square-root of a rational plus a medial (area) having been divided into its (component) straightlines at C, such that AC is greater.
2. And let the rational (straight-line) DE be laid down.
3. And let the (parallelogram) DF, equal to the (square) on AB, have been ap355 plied to DE, producing DG as breadth.
4. I say that DG is a fifth binomial straight-line.
5. Let the same construction be made as in the (propositions) before this.
6. Therefore, since AB is the square-root of a rational plus a medial (area), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40].
7. Therefore, since the sum of the (squares) on AC and CB is medial, DL is thus medial.
8. Hence, DM is rational and incommensurable in length with DE [Prop. 10.22].
9. Again, since twice the (rectangle contained) by ACB— that is to say, MF—is rational, MG (is) thus rational and commensurable (in length) with DE [Prop. 10.20].
10. DM (is) thus incommensurable (in length) with MG [Prop. 10.13].
11. Thus, DM and MG are rational (straightlines which are) commensurable in square only.
12. Thus, DG is a binomial (straight-line) [Prop. 10.36].

13. So, I say that (it is) also a fifth (binomial straight-line).

14. For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by DKM is equal to the (square) on MN, and DK (is) incommensurable in length with KM.

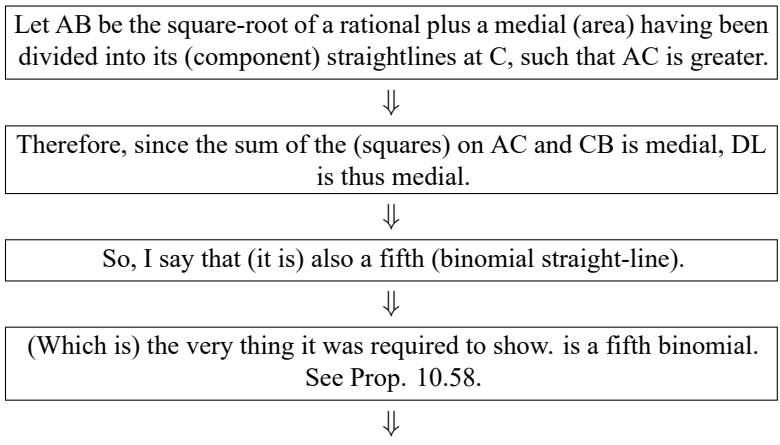
15. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM) [Prop. 10.18].

16. And DM and MG are [rational] (straight-lines which are) commensurable in square only, and the lesser MG is commensurable in length with DE.

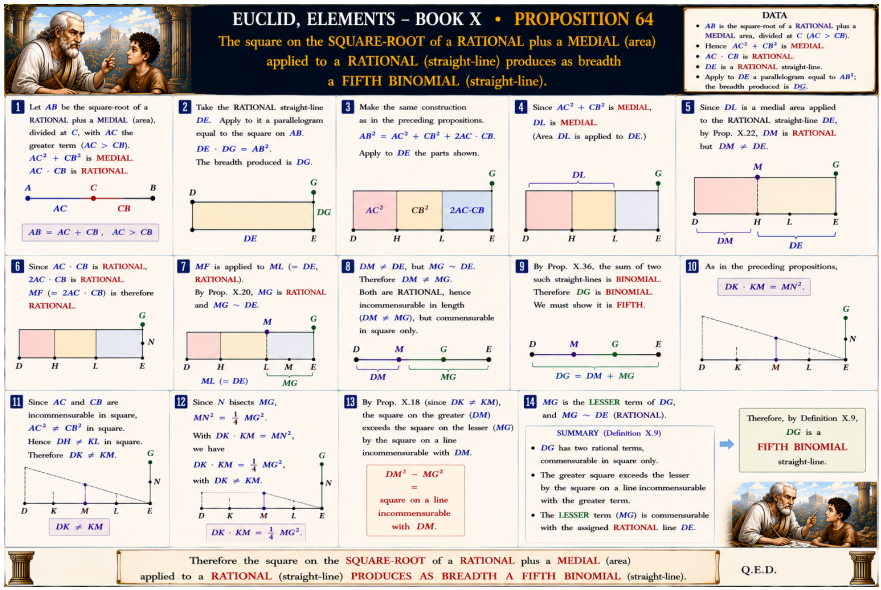
17. Thus, DG is a fifth binomial (straight-line) [Def. 10.9].

18. (Which is) the very thing it was required to show. is a fifth binomial. See Prop. 10.58.

Complete Cinematic Logic



Conclusion: Proposition 64 established.



2.65 Proposition 65

Statement

The square on the square-root of (the sum of) two medial (areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line).

Proof Actions

1. Let AB be the square-root of (the sum of) two medial (areas), having been divided at C.

2. And let DE be a rational (straight-line).

3. And let the (parallelogram) DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth.

4. I say that DG is a sixth binomial (straight-line).

5. For let the same construction be made as in the preAnd since AB is the square-root vious (propositions). of (the sum of) two medial (areas), hav-

ing been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41].

6. Hence, according to what has been previously demonstrated, DL and MF are each medial.

7. And they are applied to the rational (straight-line) DE.

8. Thus, DM and MG are each rational, and incommensurable in length with DE [Prop. 10.22].

9. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, DL is thus incommensurable with MF.

10. Thus, DM (is) also incommensurable (in length) with MG [Props. 6.1, 10.11].

11. DM and MG are thus rational (straight-lines which are) commensurable Thus, DG is a binomial (straight-line) in square only. [Prop. 10.36].

12. So, I say that (it is) also a sixth (binomial straight-line).

13. So, similarly (to the previous propositions), we can again show that the (rectangle contained) by DKM is equal to the (square) on MN, and that DK is incommensurable in length with KM.

14. And so, for the same (reasons), the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable in length with (DM) [Prop. 10.18].

15. And neither of DM and MG is commensurable in length with the (previously) laid down rational (straight-line) DE.

16. Thus, DG is a sixth binomial (straight-line) [Def. 10.10].

17. (Which is) the very thing it was required to show. binomial. See Prop. 10.59.

Complete Cinematic Logic

Let AB be the square-root of (the sum of) two medial (areas), having been divided at C.

⇓

Hence, according to what has been previously demonstrated, DL and MF are each medial.

⇓

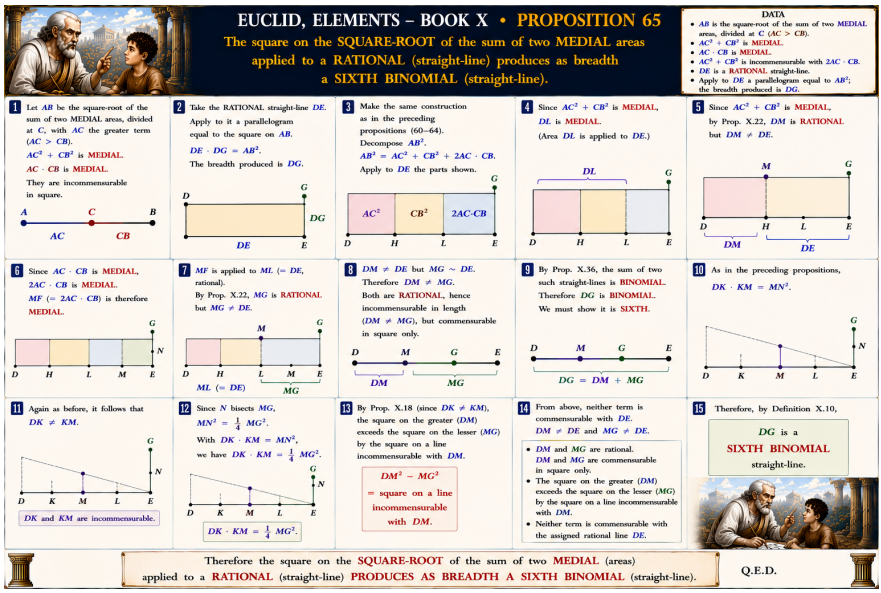
So, I say that (it is) also a sixth (binomial straight-line).

⇓

(Which is) the very thing it was required to show. binomial. See Prop. 10.59.

⇓

Conclusion: Proposition 65 established.



2.66 Proposition 66

Statement

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.

Proof Actions

1. Let AB be a binomial (straight-line), and let CD be commensurable in length with AB.
2. I say that CD is a binomial (straight-line), and (is) the same in order as AB.
3. For since AB is a binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term.
4. AE and EB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36].
5. Let it have been contrived that as AB (is) to CD, so AE (is) to CF [Prop. 6.12].
6. Thus, the remainder EB is also to the remainder FD, as AB (is) to CD [Props. 6.16, 5.19 corr.].
7. And AB (is) commensurable in length with CD.
8. Thus, AE is also commensurable (in length) with CF, and EB with FD [Prop. 10.11].
9. And AE and EB are rational.
10. Thus, CF and FD are also rational.
11. And as AE is to CF, (so) EB (is) to FD Thus, alternately, as AE is to EB, (so) [Prop. 5.11].
12. CF (is) to FD [Prop. 5.16].
13. And AE and EB [are] commensurable in square only.

14. Thus, CF and FD are also commensurable in square only [Prop. 10.11]. And they are rational.

15. CD is thus a binomial (straight-line) [Prop. 10.36].

16. So, I say that it is the same in order as AB.

17. For the square on AE is greater than (the square on) EB by the (square) on (some straight-line) either commensurable or incommensurable (in length) with (AE).

18. Therefore, if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14].

19. And if AE is commensurable (in length) with (some previously) laid down rational (straight-line) then CF will also be commensurable (in length) with it [Prop. 10.12].

20. And, on account of this, AB and CD are each first binomial (straightlines) [Def. 10.5]—that is to say, the same in order.

21. And if EB is commensurable (in length) with the (previously) laid down rational (straight-line) then FD is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, (CD) will be the same in order as AB.

22. For each of them will be second binomial (straightlines) [Def. 10.6].

23. And if neither of AE and EB is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of CF and FD will be commensurable (in length) with it [Prop. 10.13], and each (of AB and CD) is a third (binomial straight-line) And if the square on AE is greater than [Def. 10.7].

24. (the square on) EB by the (square) on (some straightline) incommensurable (in length) with (AE) then the square on CF is also greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14].

25. And if AE is commensurable (in length) with the (previously) laid down rational (straight-line) then CF is also commensurable (in length) with it

[Prop. 10.12], and each (of AB and CD) is a fourth (binomial straight-line) [Def. 10.8].

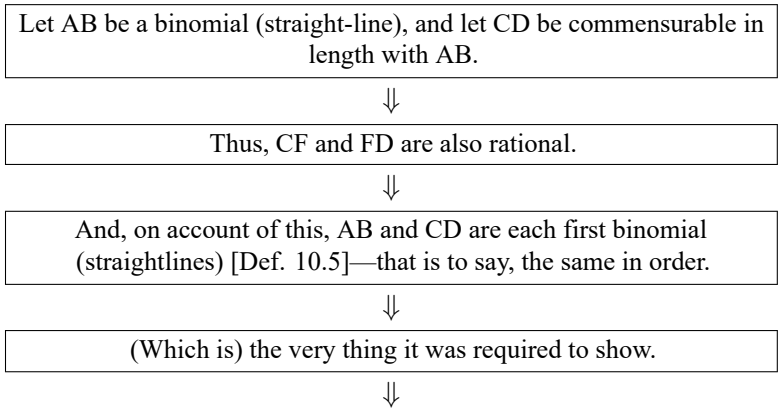
26. And if EB (is commensurable in length with the previously laid down rational straight-line) then FD (is) also (commensurable in length with it), and each (of AB and CD) will be a fifth (binomial straight-line) [Def. 10.9].

27. And if neither of AE and EB (is commensurable in length with the previously laid down rational straight-line) then also neither of CF and FD is commensurable (in length) with the laid down rational (straight-line), and each (of AB and CD) will be a sixth (binomial straight-line) [Def. 10.10].

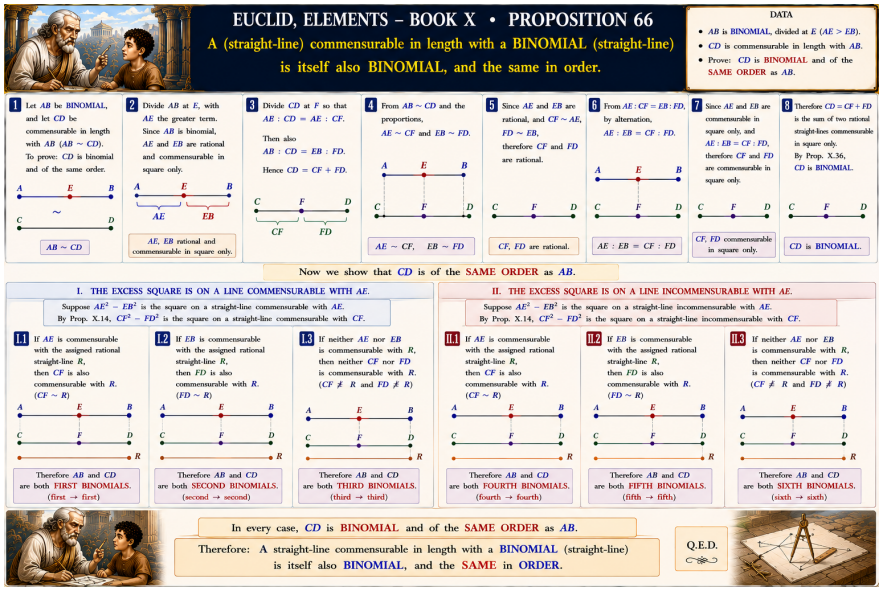
28. Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straightline), and the same in order.

29. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 66 established.



2.67 Proposition 67

Statement

A (straight-line) commensurable in length with a bimedial (straight-line) is itself also bimedial, and the same in order.

Proof Actions

1. Let AB be a bimedial (straight-line), and let CD be commensurable in length with AB.

2. I say that CD is bimedial, and the same in order as AB.

3. For since AB is a bimedial (straight-line), let it have been divided into its (component) medial (straight-lines) at E.

4. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.37, 10.38].

5. And let it have been contrived that as AB (is) to CD, (so) AE (is) to CF [Prop. 6.12].

6. And thus as the remainder EB is to the remainder FD, so AB (is) to CD [Props. 5.19 corr., 6.16].

7. And AB (is) commensurable in length with CD.

8. Thus, AE and EB are also commensurable (in length) with CF and FD, respectively [Prop. 10.11].

9. And AE and EB (are) medial.

10. Thus, CF and FD (are) also medial [Prop. 10.23].

11. And since as AE is to EB, (so) CF (is) to FD, and AE and EB are commensurable in square only, CF and FD are [thus] also commensurable in square only [Prop. 10.11].

12. And they were also shown (to be) medial.

13. Thus, CD is a bimedial (straight-line).

14. So, I say that it is also the same in order as AB.

15. For since as AE is to EB, (so) CF (is) to FD, thus also as the (square) on AE (is) to the (rectangle contained) by AEB, so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.].

16. Alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AEB (is) to the (rectangle contained) by CFD [Prop. 5.16].

17. And the (square) on AE (is) commensurable with the (square) on CF.

18. Thus, the (rectangle contained) by AEB (is) also commensurable with the (rectangle contained) by CFD [Prop. 10.11].

19. Therefore, either the (rectangle contained) by AEB is rational, and the (rectangle contained) by CFD is rational [and, on account of this, (AE and CD) are first bimedial (straight-lines)], or (the rectangle contained by AEB is) medial, and (the rectangle contained by CFD is) medial, and (AB and CD) are each second (bimedial straight-lines) [Props. 10.23, 10.37, 10.38].

20. And, on account of this, CD will be the same in order as AB.

21. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let AB be a bimedial (straight-line), and let CD be commensurable in length with AB.

⇓

Thus, AE and EB are also commensurable (in length) with CF and FD, respectively [Prop. 10.11].

⇓

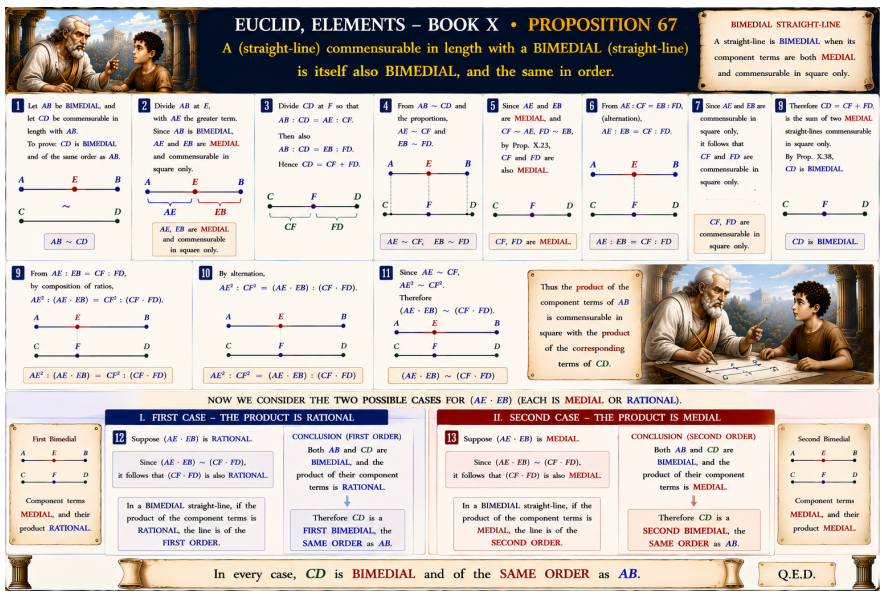
For since as AE is to EB, (so) CF (is) to FD, thus also as the (square) on AE (is) to the (rectangle contained) by AEB, so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.].

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 67 established.



2.68 Proposition 68

Statement

A (straight-line) commensurable (in length) with a major (straight-line) is itself also major.

Proof Actions

1. Let AB be a major (straight-line), and let CD be commensurable (in length) with AB.

2. I say that CD is a major (straight-line).

3. Let AB have been divided (into its component terms) at E.

4. AE and EB are thus incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39].

5. And let (the) same (things) have been contrived as in the previous (propositions).

6. And since as AB is to CD, so AE (is) to CF and EB to FD, thus also as AE (is) to CF, so EB (is) to FD [Prop. 5.11].

7. And AB (is) commensurable (in length) with CD.

8. Thus, AE and EB (are) also commensurable (in length) with CF and FD, respectively [Prop. 10.11].

9. And since as AE is to CF, so EB (is) to FD, also, alternately, as AE (is) to EB, so CF (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE, so CD (is) to DF [Prop. 5.18].

10. And thus as the (square) on AB (is) to the (square) on BE, so the (square) on CD (is) to the (square) on DF [Prop. 6.20].

11. So, similarly, we can also show that as the (square) on AB (is) to the (square) on AE, so the (square) on CD (is) to the (square) on CF.

12. And thus as the (square) on AB (is) to (the sum of) the (squares) on AE and EB, so the (square) on CD (is) to (the sum of) the (squares) on CF and FD.

13. And thus, alternately, as the (square) on AB is to the (square) on CD, so (the sum of) the (squares) on AE and EB (is) to (the sum of) the (squares) on CF and FD [Prop. 5.16].

14. And the (square) on AB (is) commensurable with the (square) on CD.

15. Thus, (the sum of) the (squares) on AE and EB (is) also commensurable with (the sum of) the (squares) on CF and FD [Prop. 10.11].

16. And the (squares) on AE and EB (added) together are rational.

17. The (squares) on CF and FD (added) together (are) thus also rational.

18. So, similarly, twice the (rectangle contained) by AE and EB is also commensurable with twice the (rectangle contained) by CF and FD.

19. And twice the (rectangle contained) by AE and EB is medial.

20. Therefore, twice the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.].

21. CF and FD are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial.

22. The whole, CD, is thus that irrational (straight-line) called major [Prop. 10.39].

23. Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major.

24. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let AB be a major (straight-line), and let CD be commensurable (in length) with AB.

⇓

And since as AE is to CF, so EB (is) to FD, also, alternately, as AE (is) to EB, so CF (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE, so CD (is) to DF [Prop. 5.18].

⇓

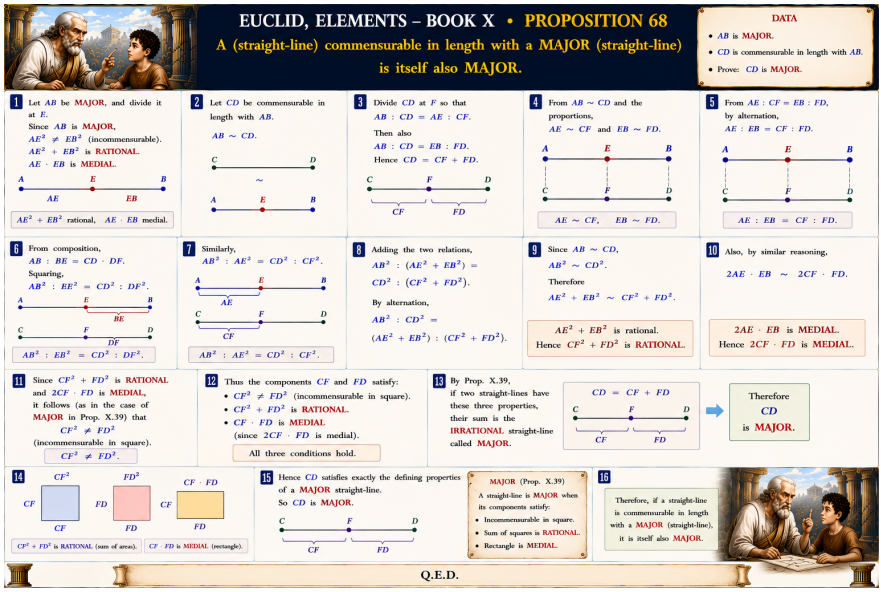
The (squares) on CF and FD (added) together (are) thus also rational.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 68 established.



2.69 Proposition 69

Statement

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).

Proof Actions

1. Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB.

2. We must show that CD is also the square-root of a rational plus a medial (area).

3. Let AB have been divided into its (component) straight-lines at E.

4. AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40].

5. And let the same construction have been made as in the previous (propositions).

6. So, similarly, we can show that CF and FD are also incommensurable in square, and that the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD.

7. And hence the sum of the squares on CF and FD is medial, and the (rectangle contained) by CF and FD (is) rational.

8. Thus, CD is the square-root of a rational plus a medial (area) [Prop. 10.40].

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB.

⇓

AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40].

⇓

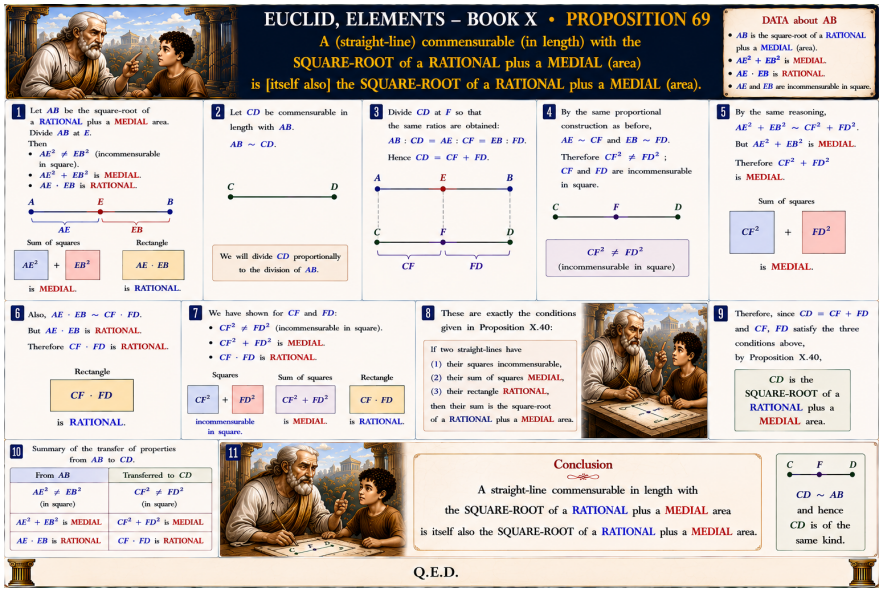
And hence the sum of the squares on CF and FD is medial, and the (rectangle contained) by CF and FD (is) rational.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 69 established.



2.70 Proposition 70

Statement

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).

Proof Actions

1. Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB.

2. We must show that CD is also the square-root of (the sum of) two medial (areas).

3. For since AB is the square-root of (the sum of) two medial (areas), let it have been divided into its (component) straight-lines at E.

4. Thus, AE and EB are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on AE and EB incommensurable with the (rectangle) contained by AE and EB [Prop. 10.41].

5. And let the same construction have been made as in the previous (propositions).

6. So, similarly, we can show that CF and FD are also incommensurable in square, and (that) the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD.

7. Hence, the sum of the squares on CF and FD is also medial, and the (rectangle contained) by CF and FD (is) medial, and, moreover, the sum of the squares on CF and FD (is) incommensurable with the (rectangle contained) by CF and FD.

8. Thus, CD is the square-root of (the sum of) two medial (areas) [Prop. 10.41].

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB.

⇓

Thus, AE and EB are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on AE and EB incommensurable with the (rectangle) contained by AE and EB [Prop. 10.41].

⇓

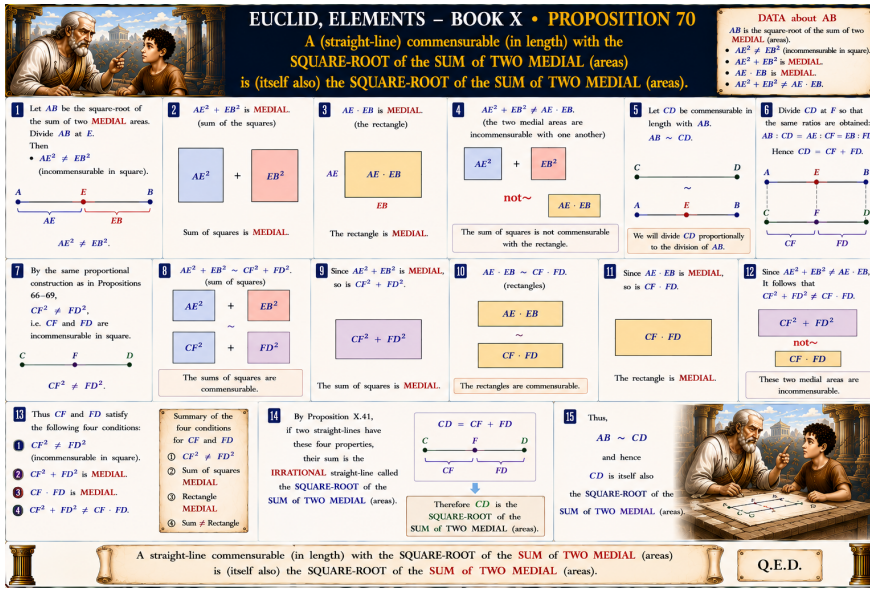
Hence, the sum of the squares on CF and FD is also medial, and the (rectangle contained) by CF and FD (is) medial, and, moreover, the sum of the squares on CF and FD (is) incommensurable with the (rectangle contained) by CF and FD.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 70 established.



2.71 Proposition 71

Statement

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the squareroots of the total area)—either a binomial, or a first bi362 medial, or a major, or the square-root of a rational plus a medial (area).

Proof Actions

1. Let AB be a rational (area), and CD a medial (area).
2. I say that the square-root of area AD is either binomial, or first bimedial, or major, or the square-root of a rational plus a medial (area).
3. For AB is either greater or less than CD.
4. Let it, first of all, be greater.
5. And let the rational (straight-line) EF be laid down.
6. And let (the rectangle) EG, equal to AB, have been applied to EF, producing EH as breadth.
7. And let (the recatangle) HI, equal to DC, have been applied to EF, producing HK as breadth.
8. And since AB is rational, and is equal to EG, EG is thus also rational.
9. And it has been applied to the [rational] (straight-line) EF, producing EH as breadth.
10. EH is thus rational, and commensurable in length with EF [Prop. 10.20].
11. Again, since CD is medial, and is equal to HI, HI is thus also medial.
12. And it is applied to the rational (straight-line) EF, producing HK as breadth.
13. HK is thus rational, and incommensurable in length with EF [Prop. 10.22].

14. And since CD is medial, and AB rational, AB is thus incommensurable with CD.

15. Hence, EG is also incommensurable with HI.

16. And as EG (is) to HI, so EH is to HK [Prop. 6.1].

17. Thus, EH is also incommensurable in length with HK [Prop. 10.11].

18. And they are both raThus, EH and HK are rational (straight-lines tional. which are) commensurable in square only.

19. EK is thus a binomial (straight-line), having been divided (into its component terms) at H [Prop. 10.36].

20. And since AB is greater than CD, and AB (is) equal to EG, and CD to HI, EG (is) thus also greater than HI.

21. Thus, EH is also greater than HK [Prop. 5.14].

22. Therefore, the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable in length with (EH), or by the (square) on (some straightline) incommensurable (in length with EH).

23. Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with EH).

24. And the greater (of the two components of EK) HE is commensurable (in length) with the (previously) laid down (straightline) EF.

25. EK is thus a first binomial (straight-line) [Def. 10.5]. And EF (is) rational.

26. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54].

27. Thus, the square-root of EI is a binomial (straight-line).

28. Hence the squareroot of AD is also a binomial (straight-line).

29. And, so, let the square on EH be greater than (the square on) HK by the (square) on (some straight-line) incommensurable (in length) with (EH).

30. And the greater (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF.

31. Thus, EK is a fourth binomial (straight-line) [Def. 10.8]. And EF (is) rational.

32. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57].

33. Thus, the square-root of area EI is a major (straight-line).

34. Hence, the square-root of AD is also major.

35. And so, let AB be less than CD.

36. Thus, EG is also less than HI.

37. Hence, EH is also less than HK [Props. 6.1, And the square on HK is greater than (the 5.14]. square on) EH either by the (square) on (some straightline) commensurable (in length) with (HK), or by the (square) on (some straight-line) incommensurable (in length) with (HK).

38. Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (HK).

39. And the lesser (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF.

40. Thus, EK is a second binomial (straight-line) [Def. 10.6]. And EF (is) rational.

41. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedial (straightline) [Prop. 10.55].

42. Thus, the square-root of area EI is a first bimedial (straight-line).

43. Hence, the square-root of AD is also a first bimedial (straight-line).

44. And so, let the square on HK be greater than (the square on) HE by the (square) on (some straight-line) incommensurable (in length) with (HK).

45. And the lesser (of the two components of EK) EH is commensurable (in length) with the (previously) laid down rational (straight-line) EF.

46. Thus, EK is a fifth binomial (straight-line) [Def. 10.9]. And EF (is) rational.

47. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58].

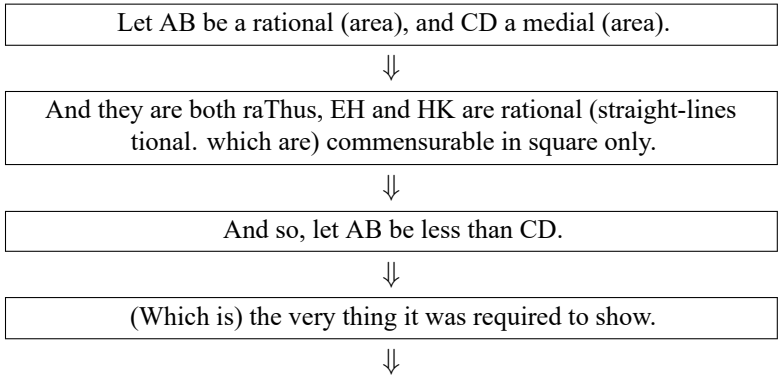
48. Thus, the square-root of area EI is the square-root of a rational plus a medial (area).

49. Hence, the square-root of area AD is also the square-root of a rational plus a medial (area).

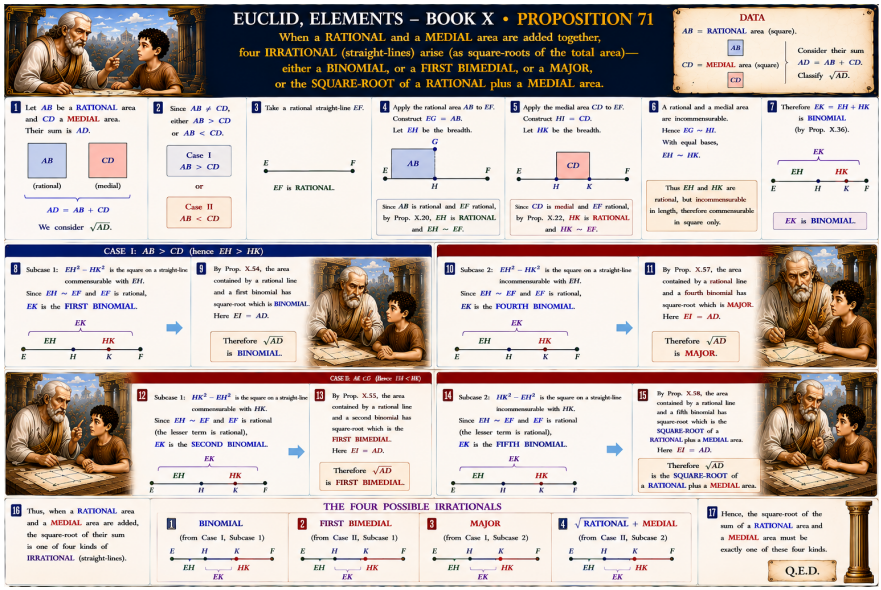
50. Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the squareroots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area).

51. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 71 established.



2.72 Proposition 72

Statement

> *When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the squareroot of (the sum of) two medial (areas).*

Proof Actions

1. For let the two medial (areas) AB and CD, (which are) incommensurable with one another, have been added together.

2. I say that the square-root of area AD is either a second bimedial, or the square-root of (the sum of) two medial (areas).

3. For AB is either greater than or less than CD.

4. By chance, let AB, first of all, be greater than CD.

5. And let the rational (straight-line) EF be laid down.

6. And let EG, equal to AB, have been applied to EF, producing EH as breadth, and HI, equal to CD, producing HK as breadth.

7. And since AB and CD are each medial, EG and HI (are) thus also each medial.

8. And they are applied to the rational straight-line FE, producing EH and HK (respectively) as breadth.

9. Thus, EH and HK are each rational (straight-lines which are) incommensurable in length with EF [Prop. 10.22].

10. And since AB is incommensurable with CD, and AB is equal to EG, and CD to HI, EG is thus also incommensurable with HI.

11. And as EG (is) to HI, so EH is to HK [Prop. 6.1].

12. EH is thus incommensurable in length with HK [Prop. 10.11].

13. Thus, EH and HK are rational (straight-lines which are) commensurable in square only.

14. EK is thus a binomial (straight-line) [Prop. 10.36].

15. And the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable (in length) with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH).

16. Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (EH).

17. And neither of EH or HK is commensurable in length with the (previously) laid down rational (straight-line) EF.

18. Thus, EK is a third binomial (straight-line) [Def. 10.7]. And EF (is) rational.

19. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second bimedial (straight-line) [Prop. 10.56].

20. Thus, the square-root of EI—that is to say, of AD— And so, let the square on EH is a second bimedial. be greater than (the square) on HK by the (square) on (some straight-line) incommensurable in length with (EH).

21. And EH and HK are each incommensurable in length with EF.

22. Thus, EK is a sixth binomial (straightline) [Def. 10.10].

23. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59].

24. Hence, the square-root of area AD is also the square-root of (the sum of) two medial (areas).

25. [So, similarly, we can show that, even if AB is less than CD, the square-root of area AD is either a second bimedial or the square-root of (the sum of) two medial (areas).] Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the squareroots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).

26. A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) nor (the same) as one another.

27. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straightline which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22].

28. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60].

29. And the (square) on a first bimedial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61].

30. And the (square) on a second bimedial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62].

31. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63].

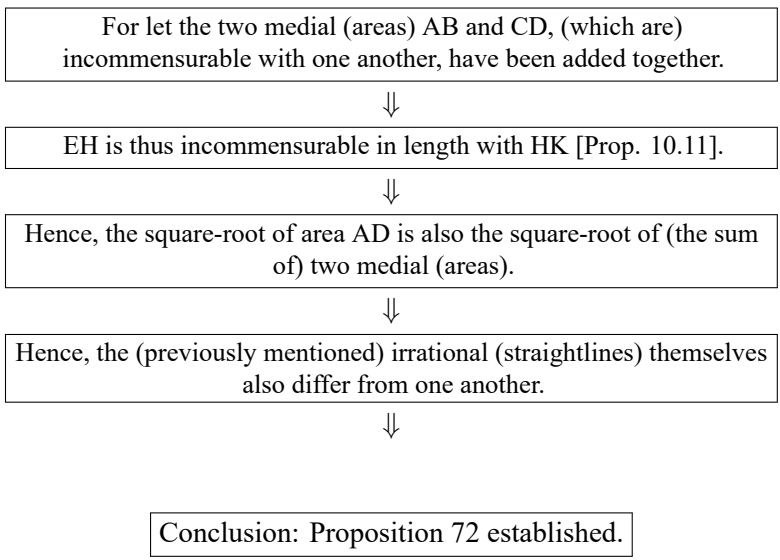
32. And the (square) on the square-root of a rational plus a medial (area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64].

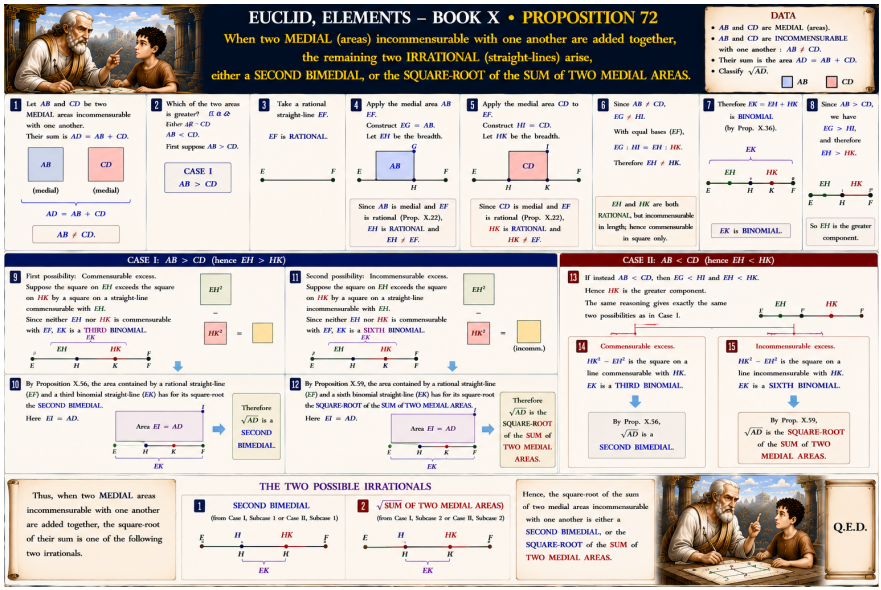
33. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65].

34. And the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational—and from one another, because they are not the same in order.

35. Hence, the (previously mentioned) irrational (straightlines) themselves also differ from one another.

Complete Cinematic Logic





2.73 Proposition 73

Statement

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.

Proof Actions

1. For let the rational (straight-line) BC, which commensurable in square only with the whole, have been subtracted from the rational (straight-line) AB.

2. I say that the remainder AC is that irrational (straight-line) called an apotome.

3. For since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC

[Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11].

4. But, the (sum of the) squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6].

5. And, inasmuch as the (sum of the squares) on AB and BC is equal to twice the (rectangle contained) by AB and BC plus the (square) on CA [Prop. 2.7], the (sum of the squares) on AB and BC is thus also incommensurable with the remaining (square) on AC [Props. 10.13, 10.16].

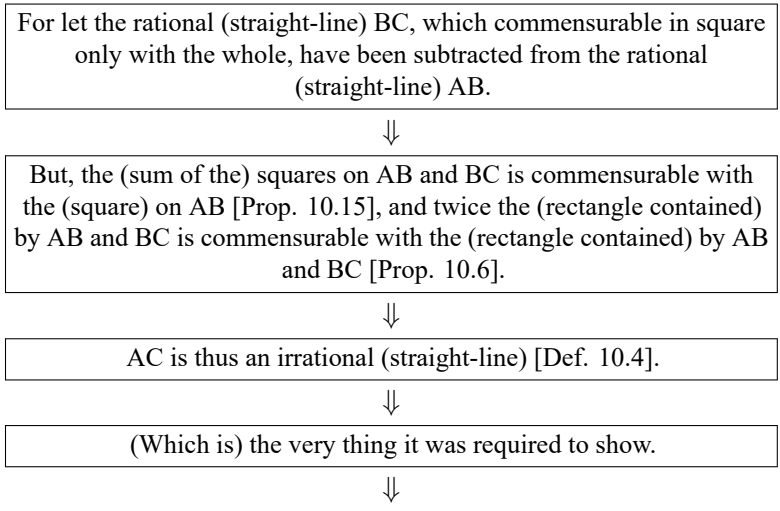
6. And the (sum of the squares) on AB and BC is rational.

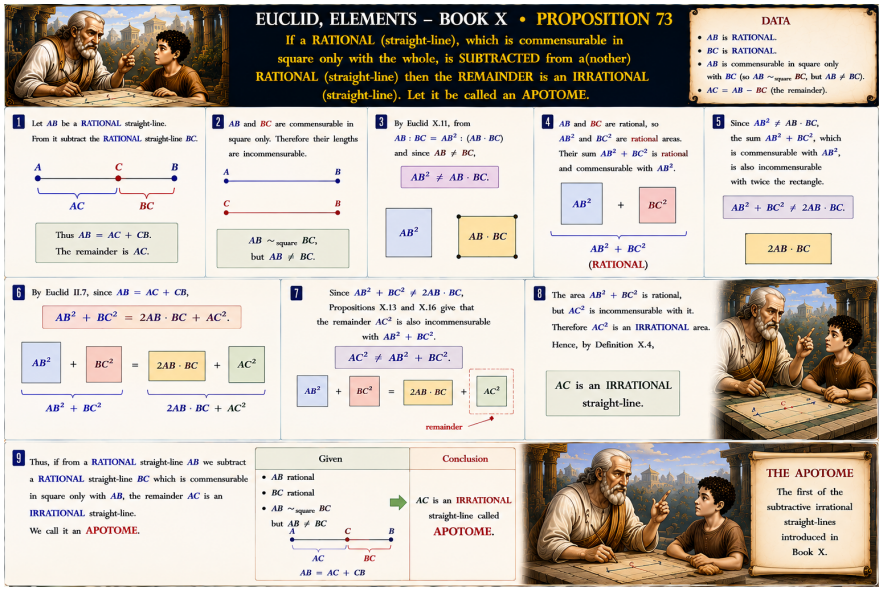
7. AC is thus an irrational (straight-line) [Def. 10.4].

8. And let it be called an apotome.

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic





2.74 Proposition 74

Statement

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a first apotome of a medial (straight-line).

Proof Actions

1. For let the medial (straight-line) BC, which is commensurable in square only with AB, and which makes with AB the rational (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.27].

2. I say that the remainder AC is an irrational (straight-line).

3. Let it be called the first apotome of a medial (straight-line).

4. For since AB and BC are medial (straight-lines), the (sum of the squares) on AB and BC is also medial.

5. And twice the (rectangle contained) by AB and BC (is) rational.

6. The (sum of the squares) on AB and BC (is) thus incommensurable with twice the (rectangle contained) by AB and BC.

7. Thus, twice the (rectangle contained) by AB and BC is also incommensurable with the remaining (square) on AC [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16].

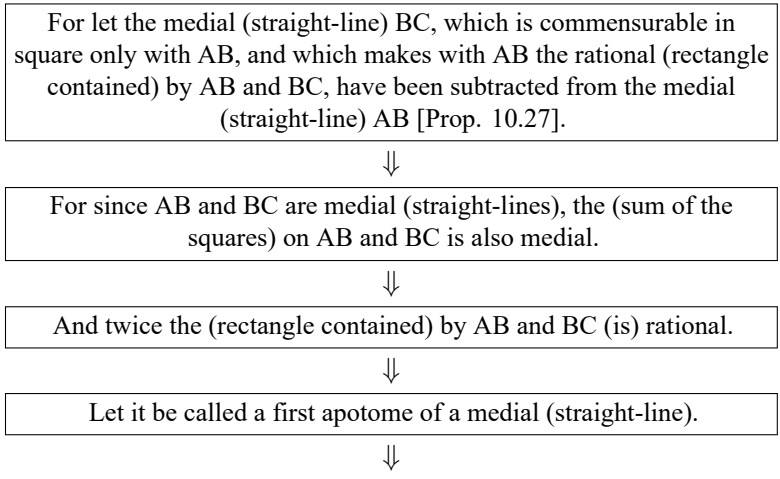
8. And twice the (rectangle contained) by AB and BC (is) rational.

9. Thus, the (square) on AC is irrational.

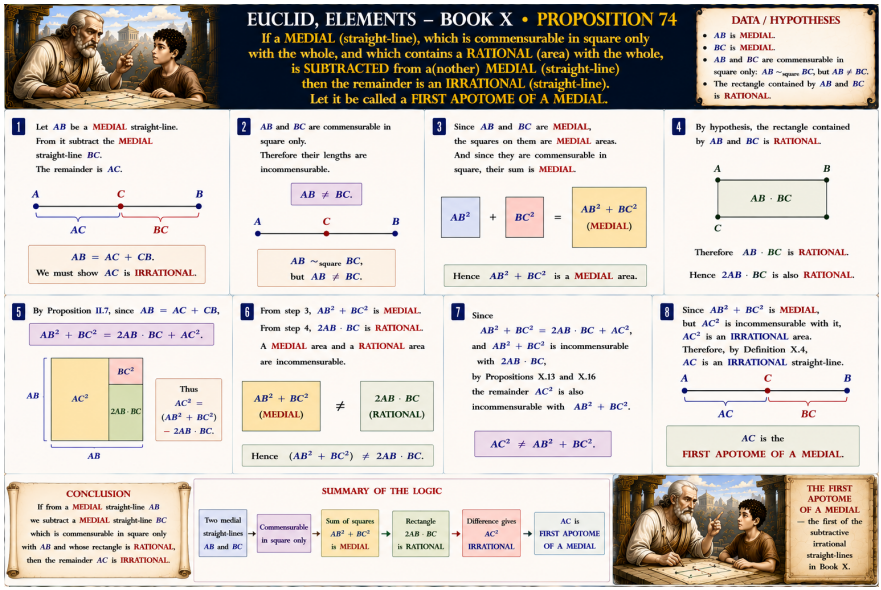
10. Thus, AC is an irrational (straight-line) [Def. 10.4].

11. Let it be called a first apotome of a medial (straight-line).

Complete Cinematic Logic



Conclusion: Proposition 74 established.



2.75 Proposition 75

Statement

> *If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).*

Proof Actions

1. For let the medial (straight-line) CB, which is commensurable in square only with the whole, AB, and which contains with the whole, AB, the medial (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.28].
2. I say that the remainder AC is an irrational (straight-line).
3. Let it be called a second apotome of a medial (straight-line).
4. For let the rational (straight-line) DI be laid down.

5. And let DE, equal to the (sum of the squares) on AB and BC, have been applied to DI, producing DG as breadth.

6. And let DH, equal to twice the (rectangle contained) by AB and BC, have been applied to DI, producing DF as breadth.

7. The remainder FE is thus equal to the (square) on AC [Prop. 2.7].

8. And since the (squares) on AB and BC are medial and commensurable (with one another), DE (is) thus also medial [Props. 10.15, 10.23 corr.].

9. And it is applied to the rational (straightline) DI, producing DG as breadth.

10. Thus, DG is rational, and incommensurable in length with DI [Prop. 10.22].

11. Again, since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is thus also medial [Prop. 10.23 corr.]. And it is equal to DH. Thus, DH is also medial.

12. And it has been applied to the rational (straight-line) DI, producing DF as breadth.

13. DF is thus rational, and incommensurable in length with DI [Prop. 10.22].

14. And since AB and BC are commensurable in square only, AB is thus incommensurable in length with BC.

15. Thus, the square on AB (is) also incommensurable with the (rectangle contained) by AB and BC [Props. 10.21 lem., 10.11].

16. But, the (sum of the squares) on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6].

17. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the (sum of the squares) on AB and BC [Prop. 10.13].

18. And DE is equal to the (sum of the squares) on AB and BC, and DH to twice the (rectangle contained) by AB and BC.

19. Thus, DE [is] incommensurable with DH.

20. And as DE (is) to DH, so GD (is) to DF [Prop. 6.1].

21. Thus, GD is incommensurable with DF [Prop. 10.11].

22. And they are both rational (straight-lines).

23. Thus, GD and DF are rational (straight-lines which are) commensurable in square only.

24. Thus, FG is an apotome [Prop. 10.73]. And DI (is) rational.

25. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational.

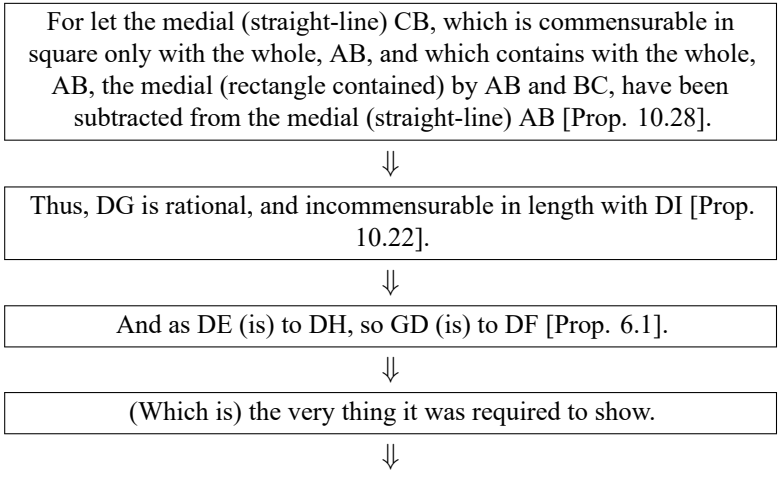
26. And AC is the square-root of FE.

27. Thus, AC is an irrational (straight-line) [Def. 10.4].

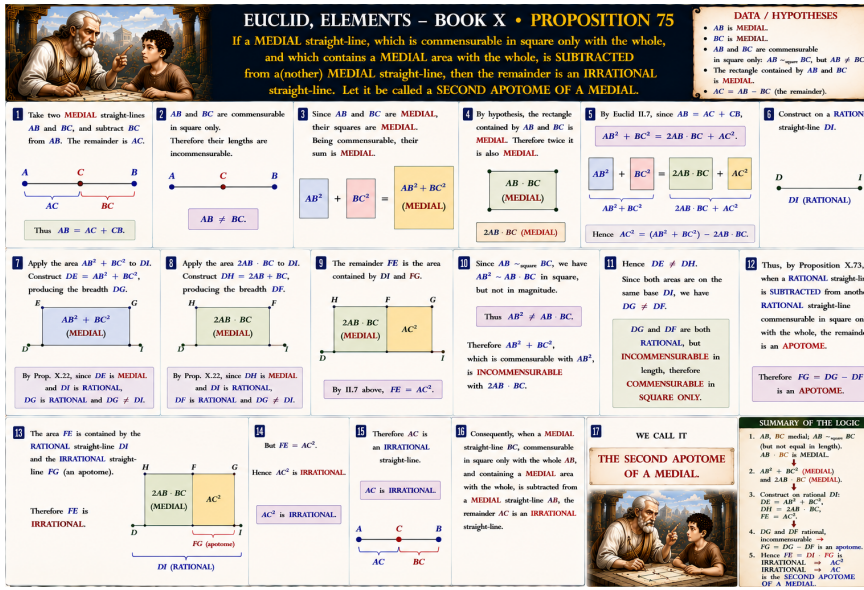
28. And let it be called the second apotome of a medial (straight-line).

29. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 75 established.



2.76 Proposition 76

Statement

> *If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called a minor (straight-line).*

Proof Actions

1. For let the straight-line BC, which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.33].

2. I say that the remainder AC is that irrational (straight-line) called minor.

3. For since the sum of the squares on AB and BC is rational, and twice the (rectangle contained) by AB and BC (is) medial, the (sum of the squares)

on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC.

4. And, via conversion, the (sum of the squares) on AB and BC is incommensurable with the remaining (square) on AC [Props. 2.7, 10.16].

5. And the (sum of the squares) on AB and BC (is) rational.

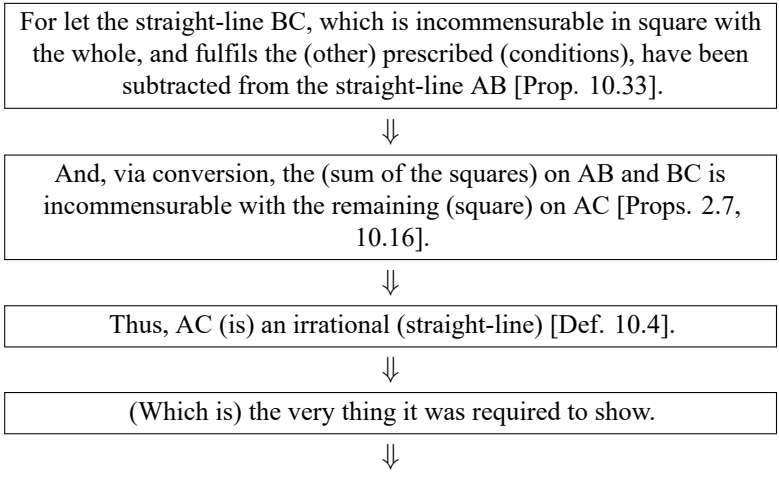
6. The (square) on AC (is) thus irrational.

7. Thus, AC (is) an irrational (straight-line) [Def. 10.4].

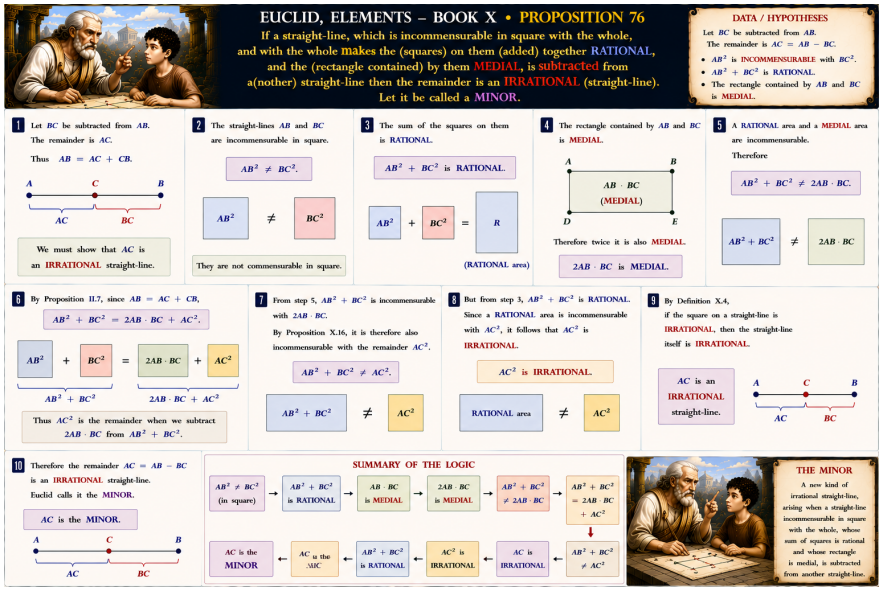
8. Let it be called a minor (straight-line).

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 76 established.



2.77 Proposition 77

Statement

> *If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a rational (area) a medial whole.*

Proof Actions

1. For let the straight-line BC, which is incommensurable in square with AB, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.34].

2. I say that the remainder AC is the aforementioned irrational (straight-line).

3. For since the sum of the squares on AB and BC is medial, and twice the

(rectangle contained) by AB and BC rational, the (sum of the squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC.

4. Thus, the remaining (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Props. 2.7, 10.16].

5. And twice the (rectangle contained) by AB and BC is rational.

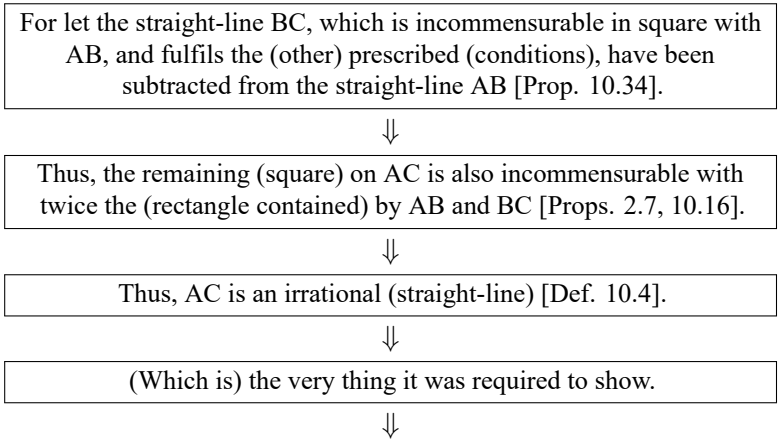
6. Thus, the (square) on AC is irrational.

7. Thus, AC is an irrational (straight-line) [Def. 10.4].

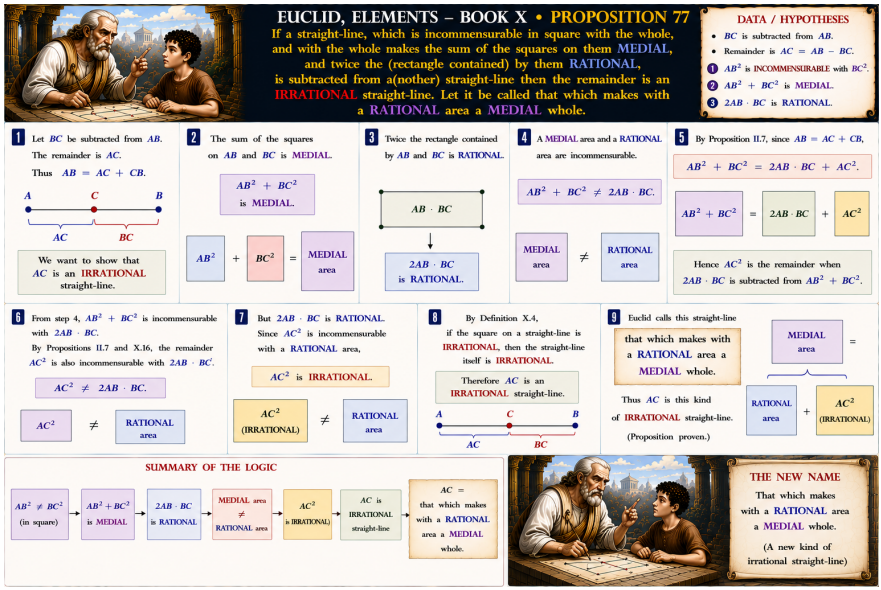
8. And let it be called that which makes with a rational (area) a medial whole.

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 77 established.



2.78 Proposition 78

Statement

> *If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a medial (area) a medial whole.*

Proof Actions

1. For let the straight-line BC, which is incommensurable in square AB, and fulfils the (other) prescribed (conditions), have been subtracted from the (straightline) AB [Prop. 10.35].

2. I say that the remainder AC is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

3. For let the rational (straight-line) DI be laid down.

4. And let DE, equal to the (sum of the squares) on AB and BC, have been applied to DI, producing DG as breadth.

5. And let DH, equal to twice the (rectangle contained) by AB and BC, have been subtracted (from DE) [producing DF as breadth].

6. Thus, the remainder FE is equal to the (square) on AC [Prop. 2.7].

7. Hence, AC is the square-root of FE.

8. And since the sum of the squares on AB and BC is medial, and is equal to DE, DE [is] thus medial.

9. And it is applied to the rational (straight-line) DI, producing DG as breadth.

10. Thus, DG is rational, and incommensurable in length with DI [Prop 10.22].

11. Again, since twice the (rectangle contained) by AB and BC is medial, and is equal to DH, DH is thus medial.

12. And it is applied to the rational (straight-line) DI, producing DF as breadth.

13. Thus, DF is also rational, and incommensurable in length with DI [Prop. 10.22].

14. And since the (sum of the squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DE (is) also incommensurable with DH.

15. And as DE (is) to DH, so DG also is to DF [Prop. 6.1].

16. Thus, DG (is) incommensurable (in length) with DF [Prop. 10.11].

17. And Thus, GD and DF are ratiothey are both rational. nal (straight-lines which are) commensurable in square only.

18. Thus, FG is an apotome [Prop. 10.73]. And FH (is) rational.

19. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational.

20. And AC is the squareroot of FE. Thus, AC is irrational.

21. Let it be called that which makes with a medial (area) a medial whole.

22. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the straight-line BC, which is incommensurable in square AB, and fulfils the (other) prescribed (conditions), have been subtracted from the (straightline) AB [Prop. 10.35].

⇓

And since the sum of the squares on AB and BC is medial, and is equal to DE, DE [is] thus medial.

⇓

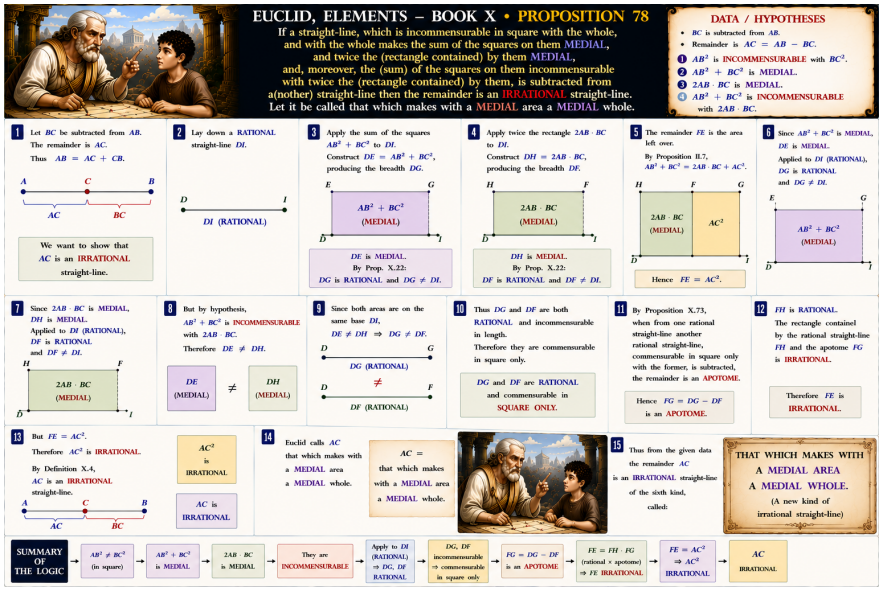
And as DE (is) to DH, so DG also is to DF [Prop. 6.1].

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 78 established.



2.79 Proposition 79

Statement

> *[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome.*

Proof Actions

1. Let AB be an apotome, with BC (so) attached to it.

2. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73].

3. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

4. For, if possible, let BD be (so) attached (to AB).

5. Thus, AD and DB are also rational (straight-lines which are) commensurable in square only [Prop. 10.73].

6. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area).

7. For both exceed by the same (area)— (namely), the (square) on AB [Prop. 2.7].

8. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB [also] exceeds twice the (rectangle contained) by AC and CB by this (same area).

9. And the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB by a rational (area).

10. For both (are) rational (areas).

11. Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area).

12. The very thing is impossible.

13. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

14. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

15. Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome.

16. (Which is) the very thing it was required to show. of plus signs.

Complete Cinematic Logic

Let AB be an apotome, with BC (so) attached to it.

⇓

And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area).

⇓

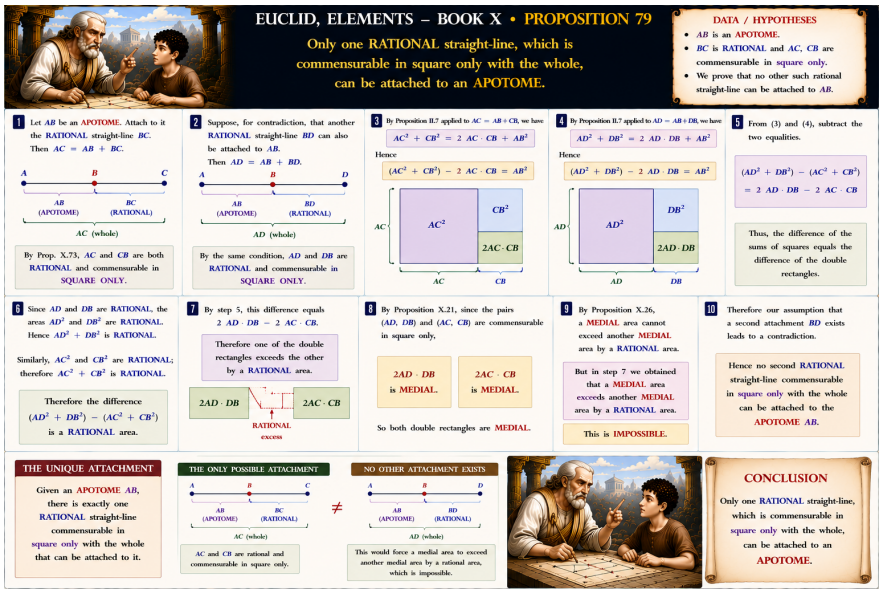
Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area).

⇓

(Which is) the very thing it was required to show. of plus signs.

⇓

Conclusion: Proposition 79 established.



2.80 Proposition 80

Statement

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).

Proof Actions

1. For let AB be a first apotome of a medial (straightline), and let BC be (so) attached to AB.

2. Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a rational (area)— (namely, that contained) by AC and CB [Prop. 10.74].

3. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to AB.

4. For, if possible, let DB also be (so) attached to AB.

5. Thus, AD and DB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that) contained by AD and DB [Prop. 10.74].

6. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area).

7. For [again] both exceed by the same (area)—(namely), the (square) on AB [Prop. 2.7].

8. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area).

9. And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area).

10. For both (are) rational (areas).

11. Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the) [squares] on AC and CB by a rational (area).

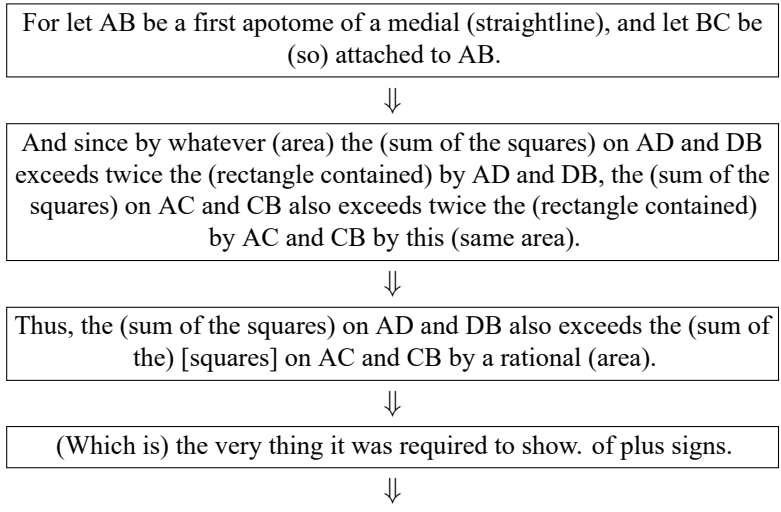
12. The very thing is impossible.

13. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

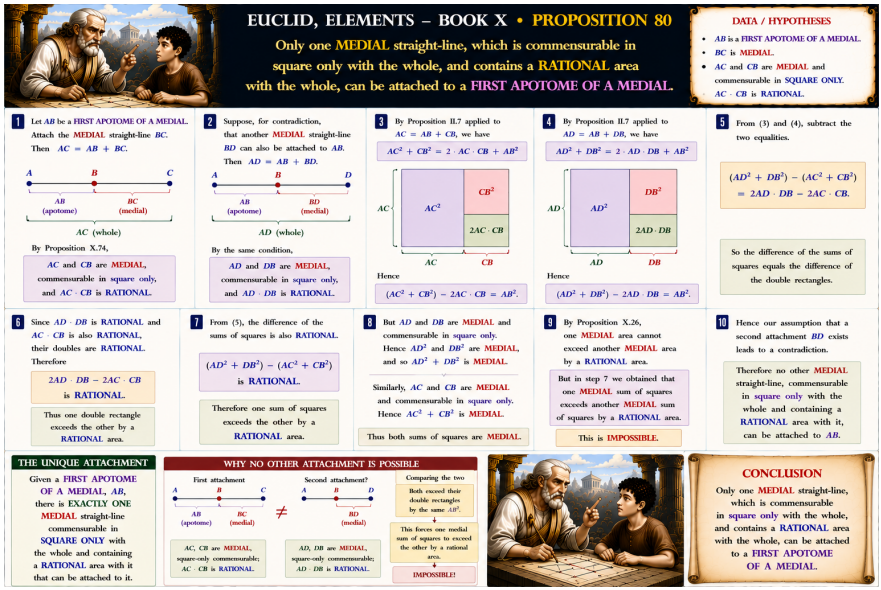
14. Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).

15. (Which is) the very thing it was required to show. of plus signs.

Complete Cinematic Logic



Conclusion: Proposition 80 established.



2.81 Proposition 81

Statement

> *Only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).*

Proof Actions

1. Let AB be a second apotome of a medial (straightline), with BC (so) attached to AB.

2. Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AC and CB [Prop. 10.75].

3. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to AB.

4. For, if possible, let BD be (so) attached.

5. Thus, AD and DB are also medial (straight-lines which are) commensurable in square only, containing a medial (area)— (namely, that contained) by AD and DB [Prop. 10.75].

6. And let the rational (straight-line) EF be laid down.

7. And let EG, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EM as breadth.

8. And let HG, equal to twice the (rectangle contained) by AC and CB, have been subtracted (from EG), producing HM as breadth.

9. The remainder EL is thus equal to the (square) on AB [Prop. 2.7].

10. Hence, AB is the square-root of EL.

11. So, again, let EI, equal to the (sum of the squares) on AD and DB have been applied to EF, producing EN as breadth.

12. And EL is also equal to the square on AB.

13. Thus, the remainder HI is equal to twice the (rectangle contained) by AD and DB [Prop. 2.7].

14. And since AC and CB are (both) medial (straight-lines), the (sum of the squares) on AC and CB is also medial. And it is equal to EG.

15. Thus, EG is also medial [Props. 10.15, 10.23 corr.].

16. And it is applied to the rational (straight-line) EF, producing EM as breadth.

17. Thus, EM is rational, and incommensurable in length with EF [Prop. 10.22].

18. Again, since the (rectangle contained) by AC and CB is medial, twice the (rectangle contained) by AC and CB is also medial [Prop. 10.23 corr.]. And it is equal to HG. Thus, HG is also medial.

19. And it is applied to the rational (straight-line) EF, producing HM as breadth.

20. Thus, HM is also rational, and incommensurable in length with EF [Prop. 10.22].

21. And since AC and CB are commensurable in square only, AC is thus incommensurable in length with CB.

22. And as AC (is) to CB, so the (square) on AC is to the (rectangle contained) by AC and CB [Prop. 10.21 corr.].

23. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11].

24. But, the (sum of the squares) on AC and CB is commensurable with the (square) on AC, and twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6].

25. Thus, the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13].

26. And EG is equal to the (sum of the squares) on AC and CB.

27. And GH is equal to twice the (rectangle contained) by AC and CB.

28. Thus, EG is incommensurable with HG.

29. And as EG (is) to HG, so EM is to HM [Prop. 6.1].

30. Thus, EM is incommensurable in length with MH [Prop. 10.11].

31. And they are both rational (straight-lines).

32. Thus, EM and MH are rational (straight-lines which are) commensurable in square only.

33. Thus, EH is an apotome [Prop. 10.73], and HM (is) attached to it.

34. So, similarly, we can show that HN (is) also (commensurable in square only with EN and is) attached to (EH).

35. Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome.

36. The very thing is impossible [Prop. 10.79].

37. Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).

38. (Which is) the very thing it was required to show. of plus signs.

Complete Cinematic Logic

Let AB be a second apotome of a medial (straightline), with BC (so) attached to AB.

⇓

Thus, the remainder HI is equal to twice the (rectangle contained) by AD and DB [Prop. 2.7].

⇓

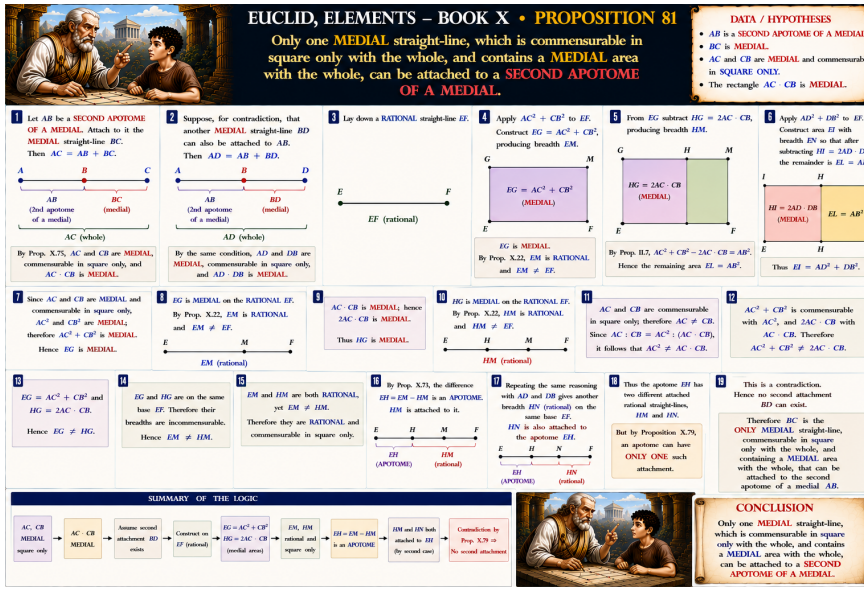
And EG is equal to the (sum of the squares) on AC and CB.

⇓

(Which is) the very thing it was required to show. of plus signs.

⇓

Conclusion: Proposition 81 established.



2.82 Proposition 82

Statement

> *Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).*

Proof Actions

1. Let AB be a minor (straight-line), and let BC be (so) attached to AB.
2. Thus, AC and CB are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76].
3. I say that another another straight-line fulfilling the same (conditions) cannot be attached to AB.
4. For, if possible, let BD be (so) attached (to AB).

5. Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76].

6. And since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area) [Prop. 2.7].

7. And the (sum of the) squares on AD and DB exceeds the (sum of the) squares on AC and CB by a rational (area).

8. For both are rational (areas).

9. Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area).

10. The very thing is impossible.

11. For both are medial (areas) [Prop. 10.26].

12. Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).

13. (Which is) the very thing it was required to show. of plus signs.

Complete Cinematic Logic

Let AB be a minor (straight-line), and let BC be (so) attached to AB.

⇓

Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76].

⇓

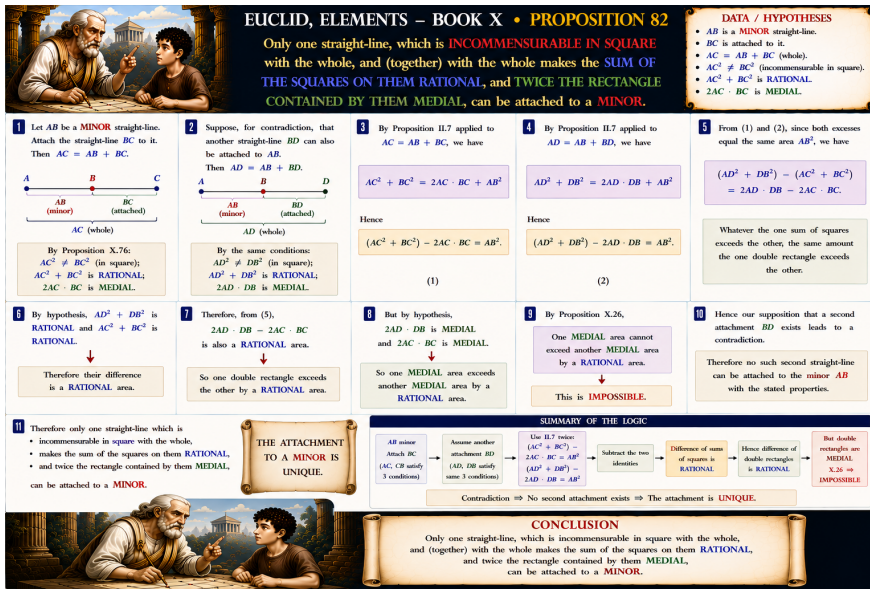
Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area).

⇓

(Which is) the very thing it was required to show. of plus signs.

⇓

Conclusion: Proposition 82 established.



2.83 Proposition 83

Statement

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole.

Proof Actions

1. Let AB be a (straight-line) which with a rational (area) makes a medial whole, and let BC be (so) attached to AB.

2. Thus, AC and CB are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77].

3. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to AB.

4. For, if possible, let BD be (so) attached (to AB).

5. Thus, AD and DB are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77].

6. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area).

7. And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area).

8. For they are (both) rational (areas).

9. Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the squares) on AC and CB by a rational (area).

10. The very thing is impossible.

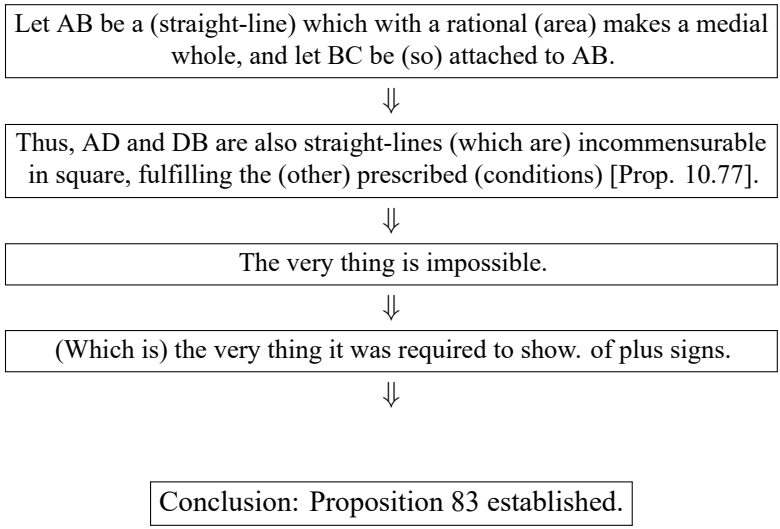
11. For both are medial (areas) [Prop. 10.26].

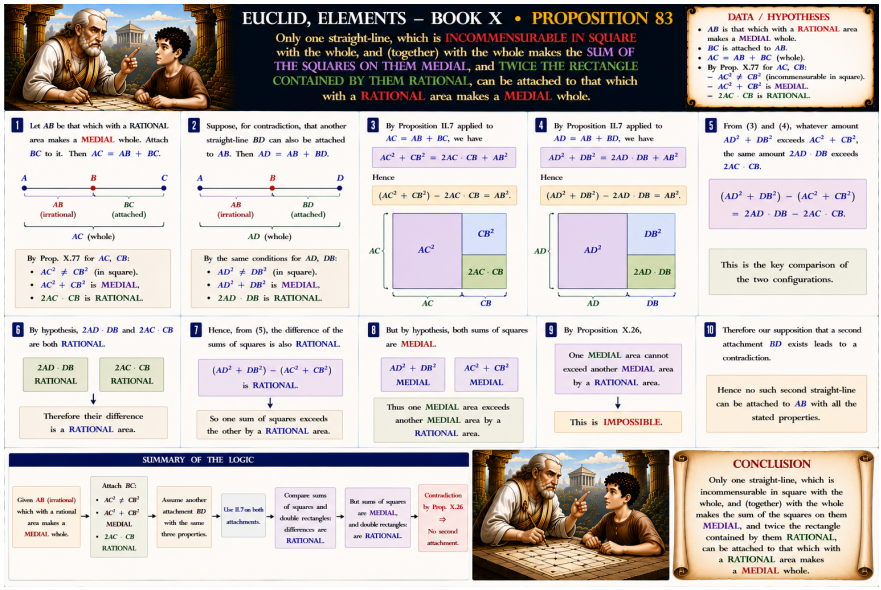
12. Thus, another straight-line cannot be attached to AB, which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole.

13. Thus, only one (such straight-line) can be (so) attached.

14. (Which is) the very thing it was required to show. of plus signs.

Complete Cinematic Logic





2.84 Proposition 84

Statement

> *Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole.*

Proof Actions

1. Let AB be a (straight-line) which with a medial (area) makes a medial whole, BC being (so) attached Thus, AC and CB are incommensurable in to it. square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78].

2. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to AB.

3. For, if possible, let BD be (so) attached.

4. Hence, AD and DB are also (straight-lines which are) incommensurable in square, making the squares on AD and DB (added) together medial, and twice the (rectangle contained) by AD and DB medial, and, moreover, the (sum of the squares) on AD and DB incommensurable with twice the (rectangle contained) by AD and DB [Prop. 10.78].

5. And let the rational (straight-line) EF be laid down.

6. And let EG, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EM as breadth.

7. And let HG, equal to twice the (rectangle contained) by AC and CB, have been applied to EF, producing HM as breadth.

8. Thus, the remaining (square) on AB is equal to EL [Prop. 2.7].

9. Thus, AB is the squareroot of EL.

10. Again, let EI, equal to the (sum of the squares) on AD and DB, have been applied to EF, producing EN as breadth.

11. And the (square) on AB is also equal to EL.

12. Thus, the remaining twice the (rectangle contained) by AD and DB [is] equal to HI [Prop. 2.7].

13. And since the sum of the (squares) on AC and CB is medial, and is equal to EG, EG is thus also medial.

14. And it is applied to the rational (straight-line) EF, producing EM as breadth.

15. EM is thus rational, and incommensurable in length with EF [Prop. 10.22].

16. Again, since twice the (rectangle contained) by AC and CB is medial, and is equal to HG, HG is thus also medial.

17. And it is applied to the rational (straight-line) EF, producing HM as breadth.

18. HM is thus rational, and incommensurable in length with EF [Prop. 10.22].

19. And since the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is also incommensurable with HG.

20. Thus, EM is also incommensurable in length with MH [Props. 6.1, 10.11].

21. And they are both rational (straight-lines).

22. Thus, EM and MH are rational (straight-lines which are) comThus, EH is an apotome mensurable in square only.

23. [Prop. 10.73], with HM attached to it.

24. So, similarly, we can show that EH is again an apotome, with HN attached to it.

25. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome.

26. The very thing was shown (to be) impossible [Prop. 10.79].

27. Thus, another straightline cannot be (so) attached to AB.

28. Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to AB.

29. (Which is) the very thing it was required to show. of plus signs. Definitions III 11.

30. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome. 12.

31. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the

(square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome. 13.

32. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome. 14.

33. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome. 15.

34. And if the attached (straight-line is commensurable), a fifth (apotome). 16.

35. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

Complete Cinematic Logic

Let AB be a (straight-line) which with a medial (area) makes a medial whole, BC being (so) attached Thus, AC and CB are incommensurable in to it. square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78].

⇓

Thus, the remaining twice the (rectangle contained) by AD and DB [is] equal to HI [Prop. 2.7].

⇓

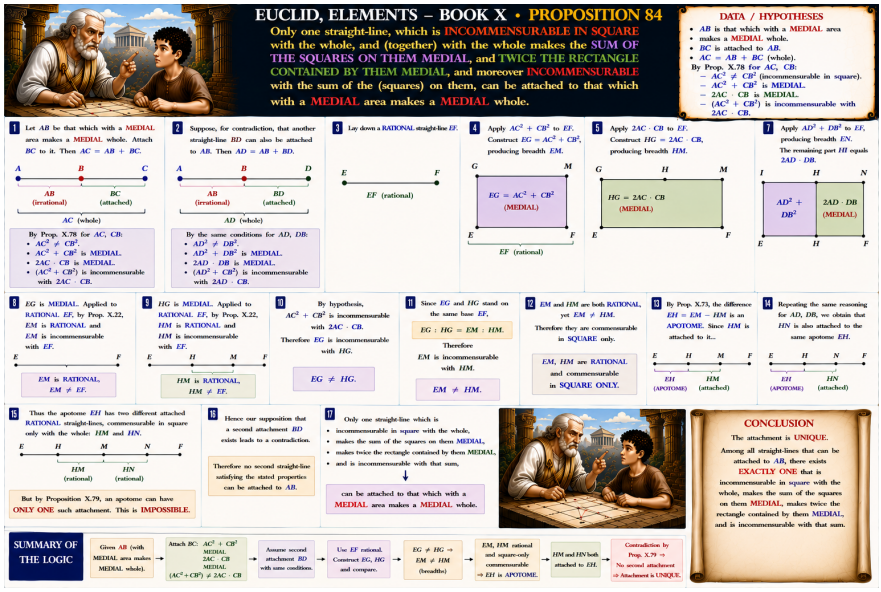
So, similarly, we can show that EH is again an apotome, with HN attached to it.

⇓

And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

⇓

Conclusion: Proposition 84 established.



2.85 Proposition 85

Statement

To find a first apotome.

Proof Actions

1. Let the rational (straight-line) A be laid down.
2. And let BG be commensurable in length with A.
3. BG is thus also a rational (straight-line).
4. And let two square numbers DE and EF be laid down, and let their difference FD be not square [Prop. 10.28 lem. I].
5. Thus, ED does not have to DF the ratio which (some) square number (has) to (some) square number.
6. And let it have been contrived that as ED (is) to DF, so the square on BG (is) to the square on GC [Prop. 10.6. corr.].
7. Thus, the (square) on BG is commensurable with the (square) on GC [Prop. 10.6].
8. And the (square) on BG (is) rational.
9. Thus, the (square) on GC (is) also rational.
10. Thus, GC is also rational.
11. And since ED does not have to DF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either.
12. Thus, BG is incommensurable in length with GC [Prop. 10.9].
13. And they are both rational (straight-lines).
14. Thus, BG and GC are rational (straight-lines which are) commensurable in square only.

15. Thus, BC is an apotome [Prop. 10.73].

16. So, I say that (it is) also a first (apotome).

17. Let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.].

18. And since as ED is to FD, so the (square) on BG (is) to the (square) on GC, thus, via conversion, as DE is to EF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.].

19. And DE has to EF the ratio which (some) square-number (has) to (some) square-number.

20. For each is a square (number).

21. Thus, the (square) on GB also has to the (square) on H the ratio which (some) square number (has) to (some) square number.

22. Thus, BG is commensurable in length with H [Prop. 10.9].

23. And the square on BG is greater than (the square on) GC by the (square) on H.

24. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG).

25. And the whole, BG, is commensurable in length with the (previously) laid down rational (straight-line) A.

26. Thus, BC is a first apotome [Def. 10.11].

27. Thus, the first apotome BC has been found.

28. (Which is) the very thing it was required to find.

Complete Cinematic Logic

Let the rational (straight-line) A be laid down.

⇓

Thus, GC is also rational.

⇓

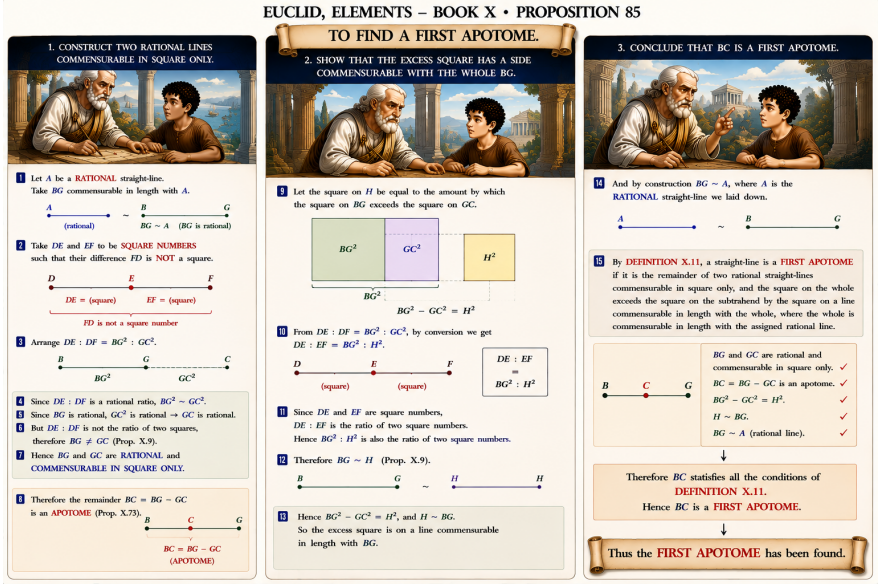
And DE has to EF the ratio which (some) square-number (has) to (some) square-number.

⇓

(Which is) the very thing it was required to find.

⇓

Conclusion: Proposition 85 established.



2.86 Proposition 86

Statement

To find a second apotome.

Proof Actions

1. Let the rational (straight-line) A, and GC (which is) commensurable in length with A, be laid down.
2. Thus, GC is a rational (straight-line).
3. And let the two square numbers DE and EF be laid down, and let their difference DF be not square [Prop. 10.28 lem. I].
4. And let it have been contrived that as FD (is) to DE, so the square on CG (is) to the square on GB [Prop. 10.6 corr.].
5. Thus, the square on CG is commensurable with the square on GB [Prop. 10.6].
6. And the (square) on CG (is) rational.
7. Thus, the (square) on GB [is] also rational.
8. Thus, BG is a rational (straight-line).
9. And since the square on GC does not have to the (square) on GB the ratio which (some) square number (has) to (some) square number, CG is incommensurable in length with GB [Prop. 10.9].
10. And they are both rational (straight-lines).
11. Thus, CG and GB are rational (straight-lines which are) commensurable in square only.
12. Thus, BC is an apotome [Prop. 10.73].
13. So, I say that it is also a second (apotome).
14. For let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.].

15. Therefore, since as the (square) on BG is to the (square) on GC, so the number ED (is) to the number DF, thus, also, via conversion, as the (square) on BG is to the (square) on H, so DE (is) to EF [Prop. 5.19 corr.].

16. And DE and EF are each square (numbers).

17. Thus, the (square) on BG has to the (square) on H the ratio which (some) square number (has) to (some) square number.

18. Thus, BG is commensurable in length with H [Prop. 10.9].

19. And the square on BG is greater than (the square on) GC by the (square) on H.

20. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG).

21. And the attachment CG is commensurable (in length) with the (previously) laid down rational (straight-line) A.

22. Thus, BC is a second apotome [Def. 10.12].

23. Thus, the second apotome BC has been found.

24. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let the rational (straight-line) A, and GC (which is) commensurable in length with A, be laid down.

⇓

And since the square on GC does not have to the (square) on GB the ratio which (some) square number (has) to (some) square number, CG is incommensurable in length with GB [Prop. 10.9].

⇓

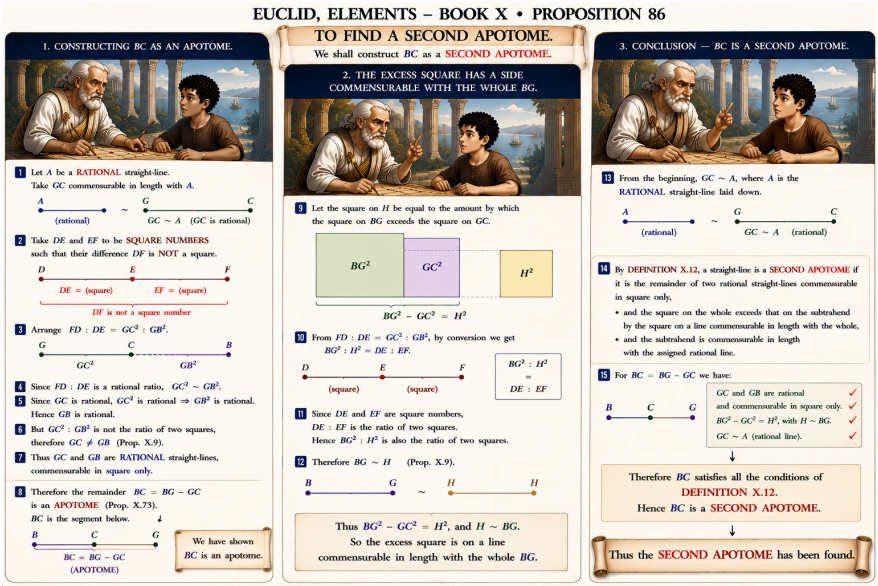
Thus, the (square) on BG has to the (square) on H the ratio which (some) square number (has) to (some) square number.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 86 established.



2.87 Proposition 87

Statement

To find a third apotome.

Proof Actions

1. Let the rational (straight-line) A be laid down.
2. And let the three numbers, E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down.
3. And let CB have to BD the ratio which (some) square number (has) to (some) square number.
4. And let it have been contrived that as E (is) to BC, so the square on A (is) to the square on FG, and as BC (is) to CD, so the square on FG (is) to the (square) on GH [Prop. 10.6 corr.].
5. Therefore, since as E is to BC, so the square on A (is) to the square on FG, the square on A is thus commensurable with the square on FG [Prop. 10.6].
6. And the square on A (is) rational.
7. Thus, the (square) on FG (is) also rational.
8. Thus, FG is a rational (straight-line).
9. And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the square on A thus does not have to the [square] on FG the ratio which (some) square number (has) to (some) square number either.
10. Thus, A is incommensurable in length with FG [Prop. 10.9].
11. Again, since as BC is to CD, so the square on FG is to the (square) on GH, the square on FG is thus commensurable with the (square) on GH [Prop. 10.6].
12. And the (square) on FG (is) rational.

13. Thus, the (square) on GH (is) also rational.

14. Thus, GH is a rational (straightline).

15. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either.

16. Thus, FG is incommensurable in length with GH [Prop. 10.9].

17. And both are rational (straight-lines).

18. FG and GH are thus rational (straight-lines which are) commensurable in square only.

19. Thus, FH is an apotome [Prop. 10.73].

20. So, I say that (it is) also a third (apotome).

21. For since as E is to BC, so the square on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on HG, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on HG [Prop. 5.22].

22. And E does not have to CD the ratio which (some) square number (has) to (some) square number.

23. Thus, the (square) on A does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either.

24. A (is) thus incommensurable in length with GH [Prop. 10.9].

25. Thus, neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A.

26. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.].

27. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as BC is to BD, so the square on FG (is) to the square on K [Prop. 5.19 corr.].

28. And BC has to BD the ratio which (some) square number (has) to (some) square number.

29. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number FG is thus commen- (has) to (some) square number. surable in length with K [Prop. 10.9].

30. And the square on FG is (thus) greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG).

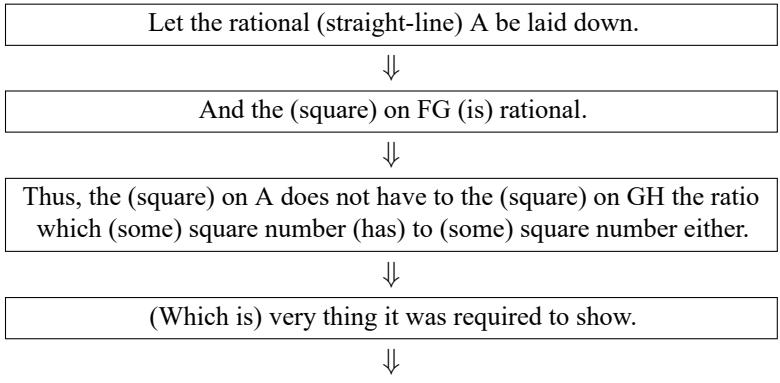
31. And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A.

32. Thus, FH is a third apotome [Def. 10.13].

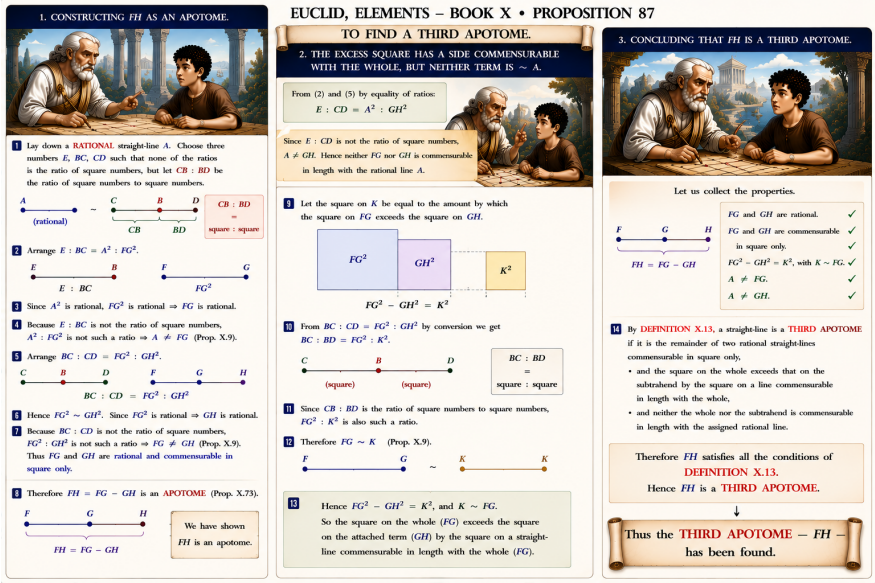
33. Thus, the third apotome FH has been found.

34. (Which is) very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 87 established.



2.88 Proposition 88

Statement

To find a fourth apotome.

Proof Actions

1. Let the rational (straight-line) A, and BG (which is) commensurable in length with A, be laid down.

2. Thus, BG is also a rational (straight-line).

3. And let the two numbers DF and FE be laid down such that the whole, DE, does not have to each of DF and EF the ratio which (some) square number (has) to (some) square number.

4. And let it have been contrived that as DE (is) to EF, so the square on BG (is) to the (square) on GC The (square) on BG is thus com- [Prop. 10.6 corr.]. mensurable with the (square) on GC [Prop. 10.6].

5. And the (square) on BG (is) rational.

6. Thus, the (square) on GC (is) also rational.

7. Thus, GC (is) a rational (straightline).

8. And since DE does not have to EF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either.

9. Thus, BG is incommensurable in length with GC [Prop. 10.9].

10. And they are both rational (straight-lines).

11. Thus, BG and GC are rational (straight-lines which are) commensurable in square only.

12. Thus, BC is an apotome [Prop. 10.73].

13. [So, I say that (it is) also a fourth (apotome).] Now, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC Therefore, since as DE is to EF, [Prop. 10.13 lem.]. so the (square) on BG (is) to the (square) on GC, thus, also, via conversion, as ED is to DF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.].

14. And ED does not have to DF the ratio which (some) square number (has) to (some) square number.

15. Thus, the (square) on GB does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

16. Thus, BG is incommensurable in length with H [Prop. 10.9].

17. And the square on BG is greater than (the square on) GC by the (square) on H.

18. Thus, the square on BG is greater than (the square) on GC by the (square) on (some straight-line) incommensurable (in length) with (BG).

19. And the whole, BG, is commensurable in length with the the (previously) laid down rational (straightline) A.

20. Thus, BC is a fourth apotome [Def. 10.14].

21. Thus, a fourth apotome has been found.

22. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let the rational (straight-line) A, and BG (which is) commensurable in length with A, be laid down.

⇓

And since DE does not have to EF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either.

⇓

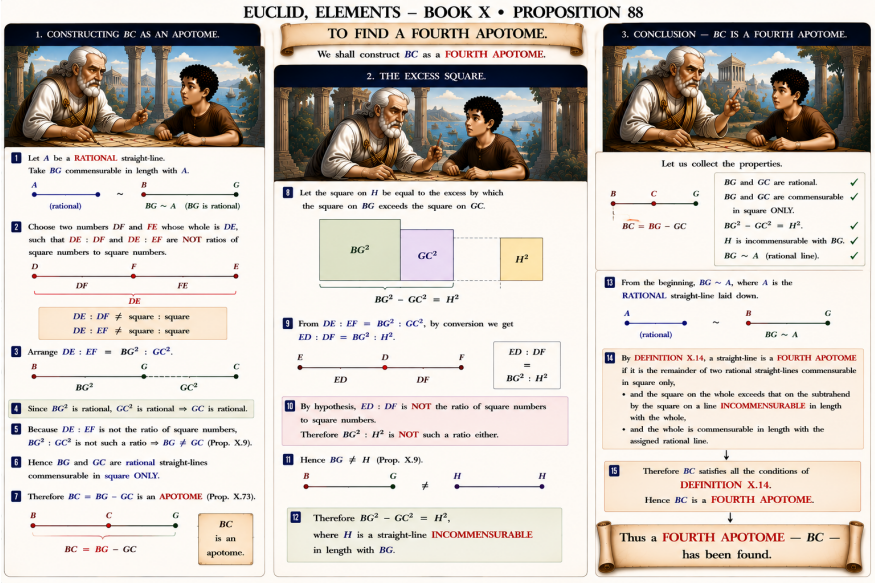
Thus, the (square) on GB does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 88 established.



2.89 Proposition 89

Statement

To find a fifth apotome.

Proof Actions

1. Let the rational (straight-line) A be laid down, and let CG be commensurable in length with A.
2. Thus, CG [is] a rational (straight-line).
3. And let the two numbers DF and FE be laid down such that DE again does not have to each of DF and FE the ratio which (some) square number (has) to (some) square number.
4. And let it have been contrived that as FE (is) to ED, so the (square) on CG (is) to the (square) on GB.
5. Thus, the (square) on GB (is) also rational [Prop. 10.6].
6. Thus, BG is also rational.
7. And since as DE is to EF, so the (square) on BG (is) to the (square) on GC.
8. And DE does not have to EF the ratio which (some) square number (has) to (some) square number.
9. The (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either.
10. Thus, BG is incommensurable in length with GC [Prop. 10.9].
11. And they are both rational (straight-lines).
12. BG and GC are thus rational (straight-lines which are) commensurable in square only.
13. Thus, BC is an apotome [Prop. 10.73].
14. So, I say that (it is) also a fifth (apotome).

15. For, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.].

16. Therefore, since as the (square) on BG (is) to the (square) on GC, so DE (is) to EF, thus, via conversion, as ED is to DF, so the (square) on BG (is) to the (square) on H [Prop. 5.19 corr.].

17. And ED does not have to DF the ratio which (some) square number (has) to (some) square number.

18. Thus, the (square) on BG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either.

19. Thus, BG is incommensurable in length with H [Prop. 10.9].

20. And the square on BG is greater than (the square on) GC by the (square) on H.

21. Thus, the square on GB is greater than (the square on) GC by the (square) on (some straight-line) incommensurable in length with (GB).

22. And the attachment CG is commensurable in length with the (previously) laid down rational (straightline) A.

23. Thus, BC is a fifth apotome [Def. 10.15].

24. Thus, the fifth apotome BC has been found.

25. (Which is) the very thing it was required to show.

Complete Cinematic Logic

Let the rational (straight-line) A be laid down, and let CG be commensurable in length with A.

⇓

The (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either.

⇓

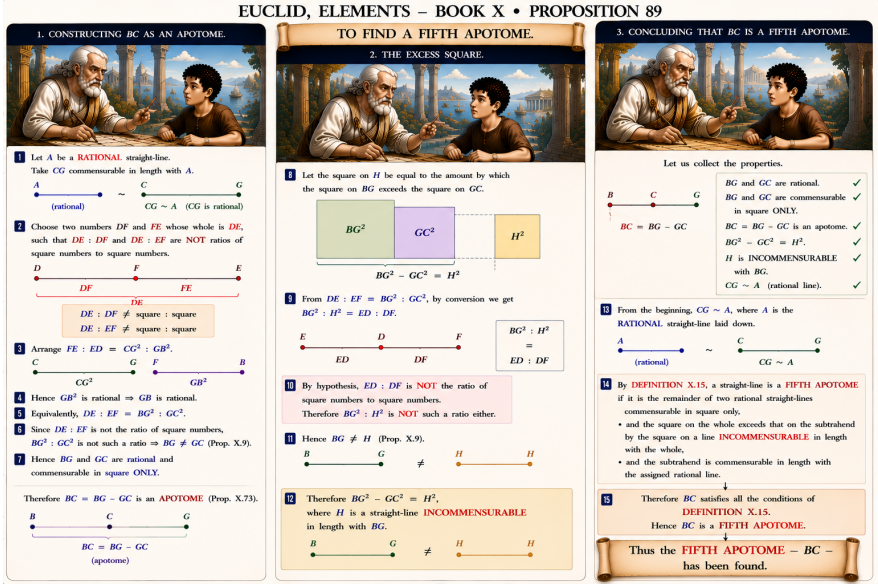
And ED does not have to DF the ratio which (some) square number (has) to (some) square number.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 89 established.



2.90 Proposition 90

Statement

To find a sixth apotome.

Proof Actions

1. Let the rational (straight-line) A, and the three numbers E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down.

2. Furthermore, let CB also not have to BD the ratio which (some) square number (has) to (some) square number.

3. And let it have been contrived that as E (is) to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].

4. Therefore, since as E is to BC, so the (square) on A (is) to the (square) on FG, the (square) on A (is) thus commensurable with the (square) on FG [Prop. 10.6].

5. And the (square) on A (is) rational.

6. Thus, the (square) on FG (is) also rational.

7. Thus, FG is also a rational (straight-line).

8. And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on FG the ratio which (some) square number Thus, A is in- (has) to (some) square number either. commensurable in length with FG [Prop. 10.9].

9. Again, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus commensurable with the (square) on GH [Prop. 10.6].

10. And the (square) on FG (is) rational.

11. Thus, the (square) on GH (is) also rational.

12. Thus, GH (is) also rational.

13. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square (number) has to (some) square (number) either.

14. Thus, FG is incommensurable in length with GH [Prop. 10.9].

15. And both are rational (straightThus, FG and GH are rational (straight-lines lines). which are) commensurable in square only.

16. Thus, FH is an apotome [Prop. 10.73].

17. So, I say that (it is) also a sixth (apotome).

18. For since as E is to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on GH [Prop. 5.22].

19. And E does not have to CD the ratio which (some) square number (has) to (some) square number.

20. Thus, the (square) on A does not have to the (square) GH the ratio which (some) square number (has) to (some) square number either.

21. A is thus incommensurable in length with GH [Prop. 10.9].

22. Thus, neither of FG and GH is commensurable in length with the rational (straight-line) A.

23. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.].

24. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as CB is to BD, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.].

25. And CB does not have to BD the ratio which (some) square number (has) to (some) square number.

26. Thus, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either.

27. FG is thus incommensurable in length with K [Prop. 10.9].

28. And the square on FG is greater than (the square on) GH by the (square) on K.

29. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straightline) incommensurable in length with (FG).

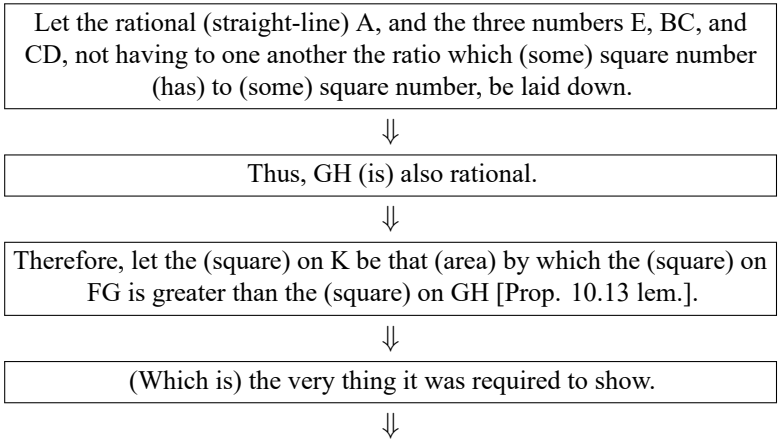
30. And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A.

31. Thus, FH is a sixth apotome [Def. 10.16].

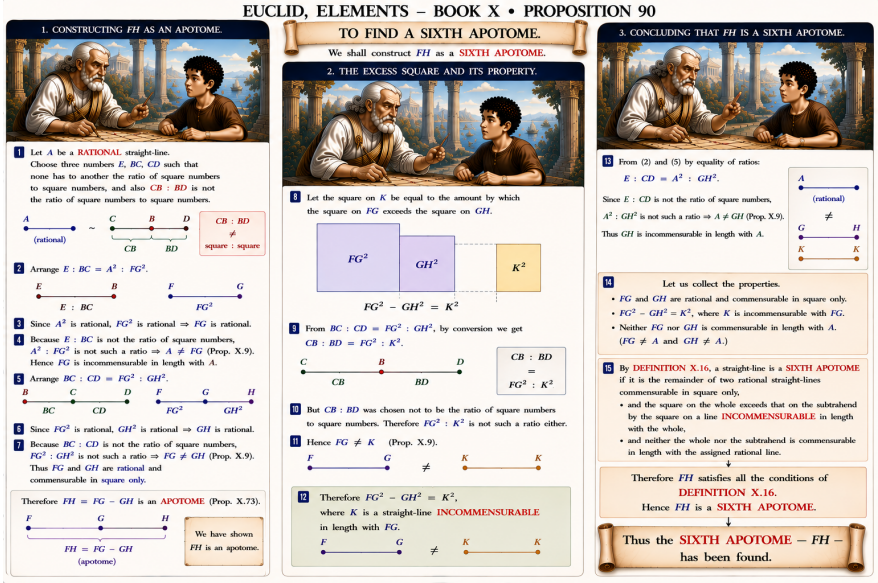
32. Thus, the sixth apotome FH has been found.

33. (Which is) the very thing it was required to show.

Complete Cinematic Logic



Conclusion: Proposition 90 established.



2.91 Proposition 91

Statement

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the first apotome AD.

2. I say that the square-root of area AB is an apotome.

3. For since AD is a first apotome, let DG be its attachment.

4. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73].

5. And the whole, AG, is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on AG is greater

than (the square on) GD by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.11].

6. Thus, if (an area) equal to the fourth part of the (square) on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17].

7. Let DG have been cut in half at E.

8. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure.

9. And let it be the (rectangle contained) by AF and FG.

10. AF is thus commensurable (in length) with FG.

11. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC.

12. And since AF is commensurable in length with FG, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15].

13. But AG is commensurable (in length) with AC.

14. Thus, each of AF and FG is also commensurable in length with AC [Prop. 10.12].

15. And AC is a rational (straight-line).

16. Thus, AF and FG (are) each also rational (straight-lines).

17. Hence, AI and FK are also each rational (areas) [Prop. 10.19].

18. And since DE is commensurable in length with EG, DG is thus also commensurable in length with each of DE and EG And DG (is) rational, and incommen- [Prop. 10.15]. surable in length with AC.

19. DE and EG (are) thus each rational, and incommensurable in length with AC [Prop. 10.13].

20. Thus, DH and EK are each medial (areas) [Prop. 10.21].

21. So let the square LM, equal to AI, be laid down.

22. And let the square NO, equal to FK, have been sub387 tracted (from LM), having with it the common angle LPM.

23. Thus, the squares LM and NO are about the Let PR be their (comsame diagonal [Prop. 6.26]. mon) diagonal, and let the (rest of the) figure have been drawn.

24. Therefore, since the rectangle contained by AF and FG is equal to the square EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17].

25. But, as AF (is) to EG, so AI (is) to EK, and as EG (is) to FG, so EK is to KF [Prop. 6.1].

26. Thus, EK is the mean proportional to AI and KF [Prop. 5.11].

27. And MN is also the mean proportional to LM and NO, as shown before [Prop. 10.53 lem.].

28. And AI is equal to the square LM, and KF to NO.

29. Thus, MN is also equal to EK.

30. But, EK is equal to DH, and MN to LO [Prop. 1.43].

31. Thus, DK is equal to the gnomon UV W and NO.

32. And AK is also equal to (the sum of) the squares LM and NO.

33. Thus, the remainder AB is equal to ST .

34. And ST is the square on LN.

35. Thus, the square on LN is equal to AB.

36. Thus, LN is the square-root of AB.

37. So, I say that LN is an apotome.

38. For since AI and FK are each rational (areas), and are equal to LM and NO (respectively), thus LM and NO—that is to say, the (squares) on each of LP and PN (respectively)—are also each rational (areas).

39. Thus, LP and PN are also each rational (straight-lines).

40. Again, since DH is a medial (area), and is equal to LO, LO Therefore, since LO is is thus also a medial (area). medial, and NO rational, LO is thus incommensurable with NO.

41. And as LO (is) to NO, so LP is to PN [Prop. 6.1].

42. LP is thus incommensurable in length with PN [Prop. 10.11].

43. And they are both rational (straightThus, LP and PN are rational (straight-lines lines). which are) commensurable in square only.

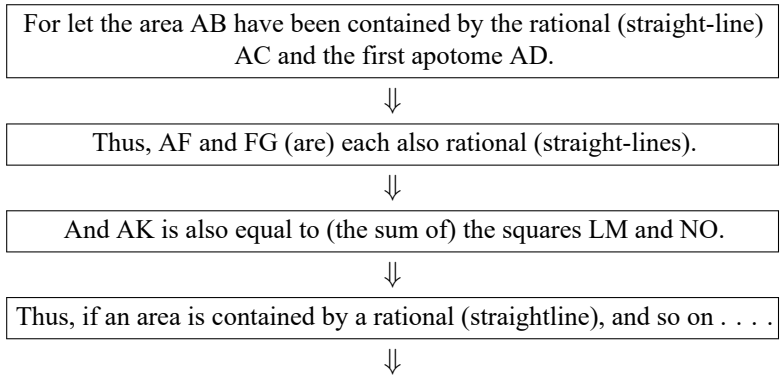
44. Thus, LN is an apotome [Prop. 10.73].

45. And it is the square-root of area AB.

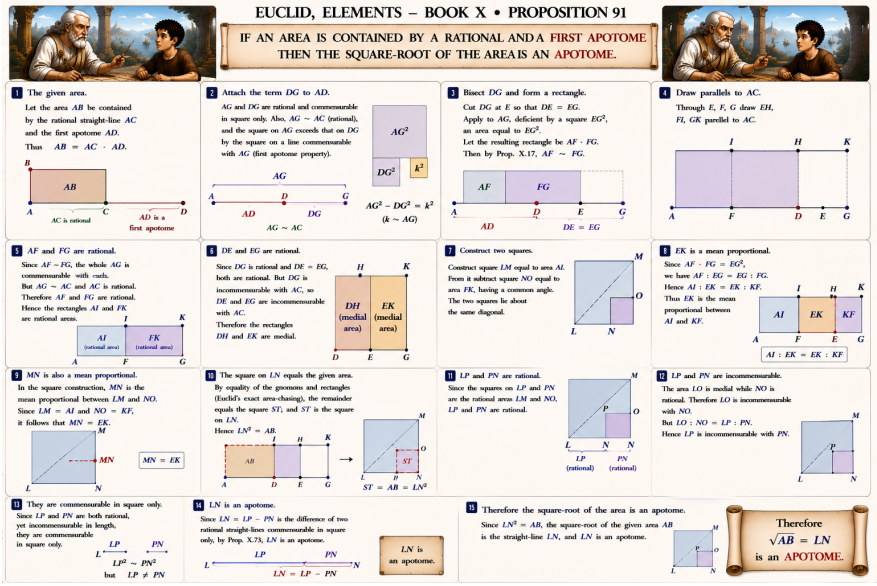
46. Thus, the square-root of area AB is an apotome.

47. Thus, if an area is contained by a rational (straightline), and so on . . . .

Complete Cinematic Logic



Conclusion: Proposition 91 established.



2.92 Proposition 92

Statement

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the second apotome AD.

2. I say that the square-root of area AB is the first apotome of a medial (straight-line).

3. For let DG be an attachment to AD.

4. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment DG is commensurable (in length) with the (previously) laid down rational (straight-line) AC,

and the square on the whole, AG, is greater than (the square on) the attachment, GD, by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.12].

5. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the (square) on GD is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17].

6. Therefore, let DG have been cut in half at E.

7. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure.

8. And let it be the (rectangle contained) by AF and FG.

9. Thus, AF is commensurable in length with FG.

10. AG is thus also commensurable in length with each of AF and FG [Prop. 10.15].

11. And AG (is) a rational (straight-line), and incommensurable in length with AC.

12. AF and FG are thus also each rational (straight-lines), and incommensurable in length with AC [Prop. 10.13].

13. Thus, AI and FK are each medial (areas) [Prop. 10.21].

14. Again, since DE is commensurable (in length) with EG, thus DG is also commensurable (in length) with each of DE and EG [Prop. 10.15].

15. But, DG is commensurable in length with AC [thus, DE and EG are also each rational, and commensurable in length with AC].

16. Thus, DH and EK are each rational (areas) [Prop. 10.19].

17. Therefore, let the square LM, equal to AI, have been constructed.

18. And let NO, equal to FK, which is about the same angle LPM as LM, have been subtracted (from LM).

19. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26].

20. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn.

21. Therefore, since AI and FK are medial (areas), and are equal to the (squares) on LP and PN (respectively), [thus] the (squares) on LP and PN are also medial.

22. Thus, LP and PN are also medial (straight-lines which are) commensurable in square only.

23. And since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 10.17].

24. But, as AF (is) to EG, so AI (is) to EK.

25. And as EG (is) to FG, so EK [is] to FK [Prop. 6.1].

26. Thus, EK is the mean proportional to AI and FK [Prop. 5.11].

27. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.].

28. And AI is equal to LM, and FK to NO.

29. Thus, MN is also equal to EK.

30. But, DH [is] equal to EK, and LO equal to MN [Prop. 1.43].

31. Thus, the whole (of) DK is equal to the gnomon UV W and NO.

32. Therefore, since the whole (of) AK is equal to LM and NO, of which DK is equal to the gnomon UV W and NO, the remainder AB is thus equal to T S.

33. And T S is the (square) on LN.

34. Thus, the (square) on LN is equal to the area AB.

35. LN is thus the square-root of area AB.

36. [So], I say that LN is the first apotome of a medial (straight-line).

37. For since EK is a rational (area), and is equal to LO, LO—that is to say, the (rectangle contained) by LP and PN—is thus a rational (area).

38. And NO was shown (to be) a medial (area).

39. Thus, LO is incommensurable with NO.

40. And as LO (is) to NO, so LP is to PN Thus, LP and PN are incommensurable [Prop. 6.1].

41. LP and PN are thus medial in length [Prop. 10.11].

42. (straight-lines which are) commensurable in square only, and which contain a rational (area).

43. Thus, LN is the first apotome of a medial (straight-line) [Prop. 10.74].

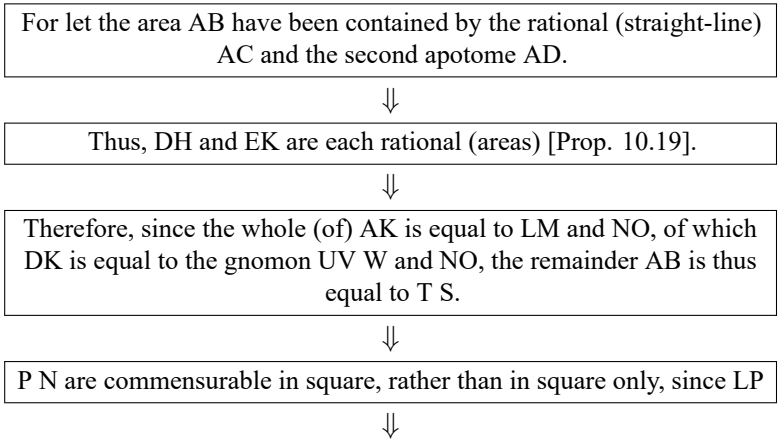
44. And it is the square-root of area AB.

45. Thus, the square root of area AB is the first apotome of a medial (straight-line).

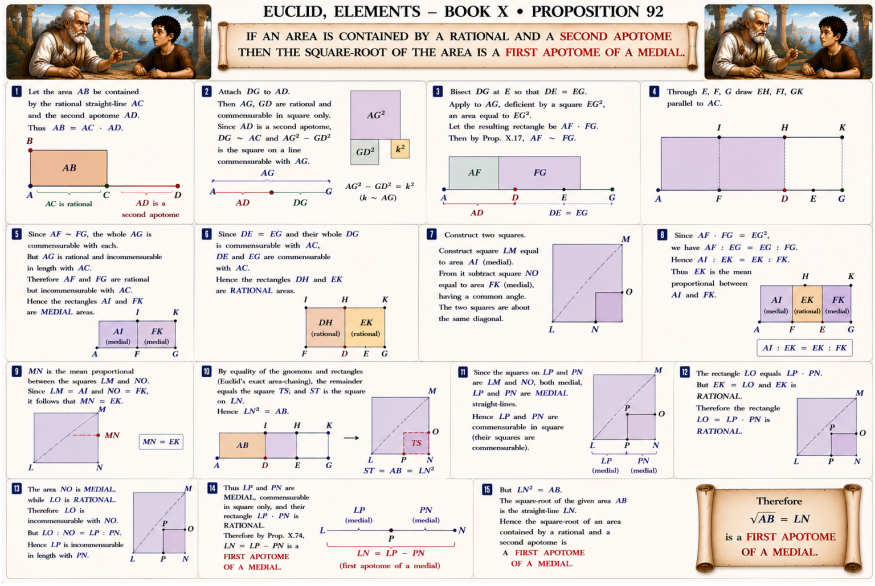
46. (Which is) the very thing it was required to show.

47. P N are commensurable in square, rather than in square only, since LP

Complete Cinematic Logic



Conclusion: Proposition 92 established.



2.93 Proposition 93

Statement

If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the third apotome AD.

2. I say that the square-root of area AB is the second apotome of a medial (straight-line).

3. For let DG be an attachment to AD.

4. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of AG and GD is commensurable in length with the (previ390 ously) laid down rational (straight-line)

AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straightline) commensurable (in length) with (AG) [Def. 10.13].

5. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the square on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17].

6. Therefore, let DG have been cut in half at E.

7. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure.

8. And let it be the (rectangle contained) by AF and FG.

9. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC.

10. Thus, AF and FG are commensurable (in length).

11. AI (is) thus also commensurable with FK [Props. 6.1, 10.11].

12. And since AF and FG are commensurable in length, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15].

13. And AG (is) rational, and incommensurable in length with AC.

14. Hence, AF and FG (are) also (rational, and incommensurable in length with AC) [Prop. 10.13].

15. Thus, AI and FK are each medial (areas) [Prop. 10.21].

16. Again, since DE is commensurable in length with EG, DG is also commensurable in length with each of DE and EG [Prop. 10.15].

17. And GD (is) rational, and incommensurable in length with AC.

18. Thus, DE and EG (are) each also rational, and incommensurable in length with AC [Prop. 10.13].

19. DH and EK are thus each medial (areas) [Prop. 10.21].

20. And since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD.

21. But, AG is commensurable in length with AF, and DG with EG.

22. Thus, AF is incommensurable in length with EG [Prop. 10.13].

23. And as AF (is) to EG, so AI is to EK [Prop. 6.1].

24. Thus, AI is incommensurable with EK [Prop. 10.11].

25. Therefore, let the square LM, equal to AI, have been constructed.

26. And let NO, equal to FK, which is about the same angle as LM, have been subtracted (from LM).

27. Thus, LM and NO are about the same diagonal [Prop. 6.26].

28. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn.

29. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17].

30. But, as AF (is) to EG, so AI is to EK [Prop. 6.1].

31. And as EG (is) to FG, so EK is to FK [Prop. 6.1].

32. And thus as AI (is) to EK, so EK (is) to FK [Prop. 5.11].

33. Thus, EK is the mean proportional to AI and FK.

34. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.].

35. And AI is equal to LM, and FK to NO.

36. Thus, EK is also equal to MN.

37. But, MN is equal to LO, and EK [is] equal to DH [Prop. 1.43].

38. And thus the whole of DK is equal to the gnomon UV W and NO.

39. And AK (is) also equal to LM and NO.

40. Thus, the remainder AB is equal to ST —that is to say, to the square on LN.

41. Thus, LN is the square-root of area AB.

42. I say that LN is the second apotome of a medial (straight-line).

43. For since AI and FK were shown (to be) medial (areas), and are equal to the (squares) on LP and PN (respectively), the (squares) on each of LP and PN (are) thus also medial.

44. Thus, LP and PN (are) each medial And since AI is commensurable with (straight-lines).

45. FK [Props. 6.1, 10.11], the (square) on LP (is) thus also commensurable with the (square) on PN.

46. Again, since AI was shown (to be) incommensurable with EK, LM is thus also incommensurable with MN—that is to say, the (square) on LP with the (rectangle contained) by LP and PN.

47. Hence, LP is also incommensurable in length with PN [Props. 6.1, 10.11].

48. Thus, LP and PN are medial (straight-lines which are) commensurable in square only.

49. So, I say that they also contain a medial (area).

50. For since EK was shown (to be) a medial (area), and is equal to the (rectangle contained) by LP and PN, the (rectangle contained) by LP and PN is thus also medial.

51. Hence, LP and PN are medial (straight-lines which are) commensurable in square only, and which contain a medial (area).

52. Thus, LN is the second apotome of a medial (straight-line) [Prop. 10.75].

53. And it is the square-root of area AB.

54. Thus, the square-root of area AB is the second apotome of a medial (straight-line).

55. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the area AB have been contained by the rational (straight-line) AC and the third apotome AD.

⇓

DH and EK are thus each medial (areas) [Prop. 10.21].

⇓

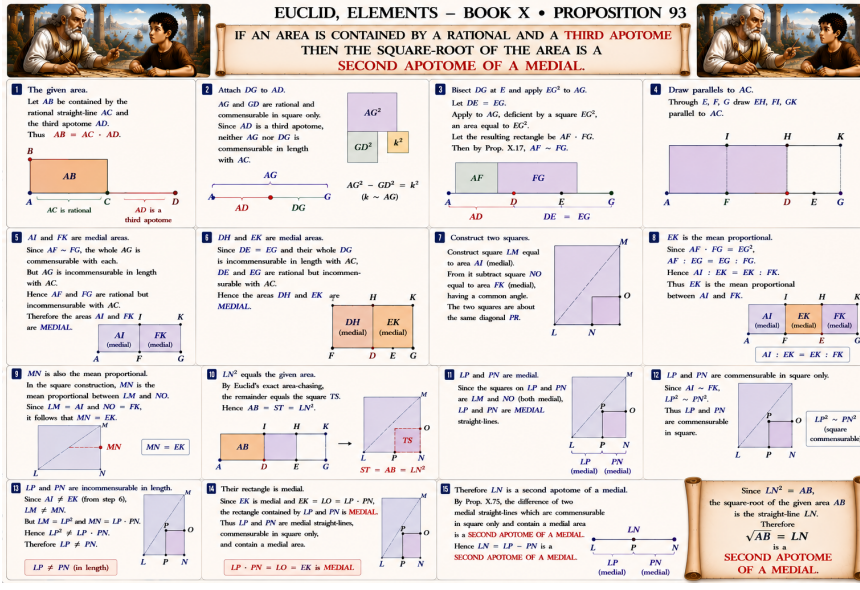
But, MN is equal to LO, and EK [is] equal to DH [Prop. 1.43].

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 93 established.



2.94 Proposition 94

Statement

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the fourth apotome AD.

2. I say that the square-root of area AB is a minor (straight392 line).

3. For let DG be an attachment to AD.

4. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and AG is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the square on (some straight-line) incommensurable in length with (AG) [Def. 10.14].

5. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18].

6. Therefore, let DG have been cut in half at E, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG.

7. Thus, AF is incommensurable in length with FG.

8. Therefore, let EH, FI, and GK have been drawn through E, F, and G (respectively), parallel to AC and BD.

9. Therefore, since AG is rational, and commensurable in length with AC, the whole (area) AK is thus rational [Prop. 10.19].

10. Again, since DG is incommensurable in length with AC, and both are rational (straight-lines), DK is thus a medial (area) [Prop. 10.21].

11. Again, since AF is incommensurable in length with FG, AI (is) thus also incommensurable with FK [Props. 6.1, 10.11].

12. Therefore, let the square LM, equal to AI, have been constructed.

13. And let NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from LM).

14. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26].

15. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn.

16. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus, proportionally, as AF is to EG, so EG (is) to FG [Prop. 6.17].

17. But, as AF (is) to EG, so AI is to EK, and as EG (is) to FG, so EK is to FK [Prop. 6.1].

18. Thus, EK is the mean proportional to AI and FK [Prop. 5.11].

19. And MN is also the mean proportional to the squares LM and NO [Prop. 10.13 lem.], and AI is equal to LM, and FK to NO.

20. EK is thus also equal to MN.

21. But, DH is equal to EK, and LO is equal to MN [Prop. 1.43].

22. Thus, the whole of DK is equal to the gnomon UV W and NO.

23. Therefore, since the whole of AK is equal to the (sum of the) squares LM and NO, of which DK is equal to the gnomon UV W and the square NO, the remainder AB is thus equal to ST —that is to say, to the square on LN.

24. Thus, LN is the square-root of area AB.

25. I say that LN is the irrational (straight-line which is) called minor.

26. For since AK is rational, and is equal to the (sum of the) squares LP and PN, the sum of the (squares) on LP and PN is thus rational.

27. Again, since DK is medial, and DK is equal to twice the (rectangle contained) by LP and PN, thus twice the (rectangle contained) by LP and PN is medial.

28. And since AI was shown (to be) incommensurable with FK, the square on LP (is) thus also incommensurable with the square on PN.

29. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial.

30. LN is thus the irrational (straight-line) called minor [Prop. 10.76].

31. And it is the square-root of area AB.

32. Thus, the square-root of area AB is a minor (straightline).

33. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the area AB have been contained by the rational (straight-line) AC and the fourth apotome AD.

⇓

Therefore, let the square LM, equal to AI, have been constructed.

⇓

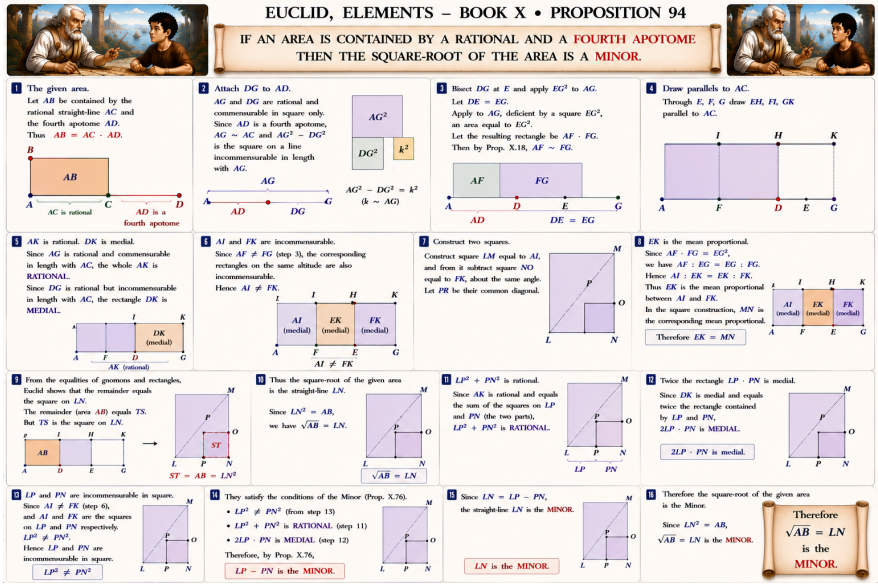
Therefore, since the whole of AK is equal to the (sum of the) squares LM and NO, of which DK is equal to the gnomon UV W and the square NO, the remainder AB is thus equal to ST —that is to say, to the square on LN.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 94 established.



2.95 Proposition 95

Statement

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the fifth apotome AD. say that the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole.

2. For let DG be an attachment to AD.

3. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment GD is commensurable in length the the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment,

DG, by the (square) on (some straight-line) incommensurable (in length) with (AG) [Def. 10.15].

4. Thus, if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18].

5. Therefore, let DG have been divided in half at point E, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG.

6. Thus, AF is incommensurable in length with FG.

7. And since AG is incommensurable in length with CA, and both are rational (straight-lines), AK is thus a medial (area) [Prop. 10.21].

8. Again, since DG is rational, and commensurable in length with AC, DK is a rational (area) [Prop. 10.19].

9. Therefore, let the square LM, equal to AI, have been constructed.

10. And let the square NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from NO).

11. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26].

12. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn.

13. So, similarly (to the previous propositions), we can show that LN is the square-root of area AB.

14. I say that LN is that (straight-line) which with a rational (area) makes a medial whole.

15. For since AK was shown (to be) a medial (area), and is equal to (the sum of) the squares on LP and PN, the sum of the (squares) on LP and PN is thus medial.

16. Again, since DK is rational, and is equal to twice the (rectangle contained) by LP and PN, (the latter) is also rational.

17. And since AI is incommensurable with FK, the (square) on LP is thus also incommensurable with the (square) on PN.

18. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational.

19. Thus, the remainder LN is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77].

20. And it is the square-root of area AB.

21. Thus, the square-root of area AB is that (straightline) which with a rational (area) makes a medial whole.

22. (Which is) the very thing it was required to show.

Complete Cinematic Logic

For let the area AB have been contained by the rational (straight-line) AC and the fifth apotome AD. say that the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole.

⇓

Again, since DG is rational, and commensurable in length with AC, DK is a rational (area) [Prop. 10.19].

⇓

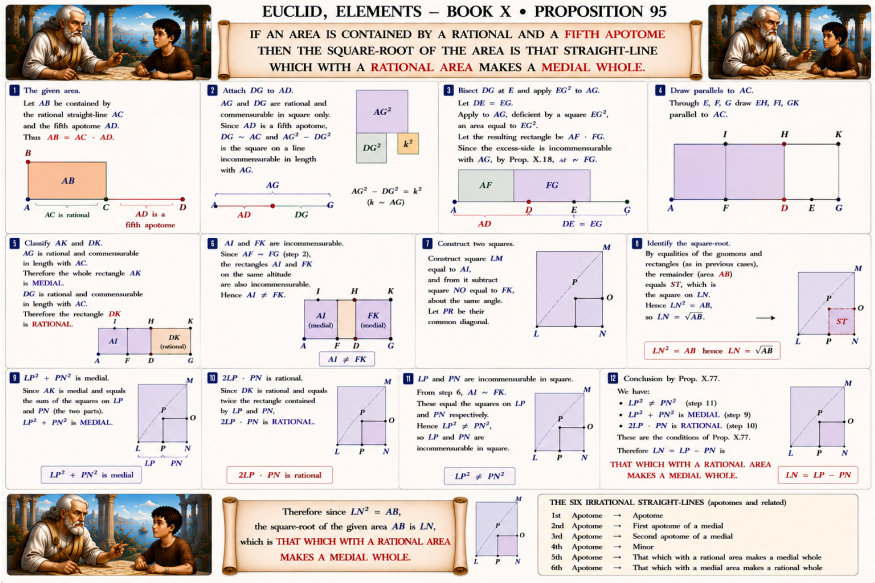
For since AK was shown (to be) a medial (area), and is equal to (the sum of) the squares on LP and PN, the sum of the (squares) on LP and PN is thus medial.

⇓

(Which is) the very thing it was required to show.

⇓

Conclusion: Proposition 95 established.



2.96 Proposition 96

Statement

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.

Proof Actions

1. For let the area AB have been contained by the rational (straight-line) AC and the sixth apotome AD.

2. I say that the square-root of area AB is that (straight-line) which with a medial (area) makes a medial whole.

3. For let DG be an attachment to AD.

4. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) AC, and the

square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) incommensurable in length with (AG) [Def. 10.16].

5. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of square on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18].

6. Therefore, let DG have been cut in half at [point] E.

7. And let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure.

8. And let it be the (rectangle contained) by AF and FG.

9. AF is thus incommensurable in length with FG.

10. And as AF (is) to FG, so AI is to FK [Prop. 6.1].

11. Thus, AI is incommensurable with FK [Prop. 10.11].

12. And since AG and AC are rational (straight-lines which are) commensurable in square only, AK is a medial (area) [Prop. 10.21].

13. Again, since AC and DG are rational (straight-lines which are) incommensurable in length, DK is also a medial (area) [Prop. 10.21].

14. Therefore, since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD.

15. And as AG (is) to GD, so AK is to KD [Prop. 6.1].

16. Thus, AK is incommensurable with KD [Prop. 10.11].

17. Therefore, let the square LM, equal to AI, have been constructed.

18. And let NO, equal to FK, (and) about the same angle, have been subtracted (from LM).

19. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26].

20. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn.

21. So, similarly to the above, we can show that LN is the square-root of area AB.

22. I say that LN is that (straight-line) which with a medial (area) makes a medial whole.

23. For since AK was shown (to be) a medial (area), and is equal to the (sum of the) squares on LP and PN, the sum of the (squares) on LP and PN is medial.

24. Again, since DK was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by LP and PN, twice the (rectangle contained) by LP and PN is also medial.

25. And since AK was shown (to be) incommensurable with DK, [thus] the (sum of the) squares on LP and PN is also incommensurable with twice the (rectangle contained) by LP and PN.

26. And since AI is incommensurable with FK, the (square) on LP (is) thus also incommensurable with the (square) on PN.

27. Thus, LP and PN are (straight-lines which are) incommensu396 rable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by Thus, LN is the irrational (straight-line) called them. that which with a medial (area) makes a medial whole [Prop. 10.78].

28. And it is the square-root of area AB.

29. Thus, the square-root of area (AB) is that (straightline) which with a medial (area) makes a medial whole.

30. (Which is) the very thing it was required to show.

Complete Cinematic Logic

$AB = AC \cdot AD$, with AC rational and AD a sixth apotome.

$\Downarrow$

AG, GD are rational and commensurable in square only; $AG \nsim AC$ and $GD \nsim AC$.

$\Downarrow$

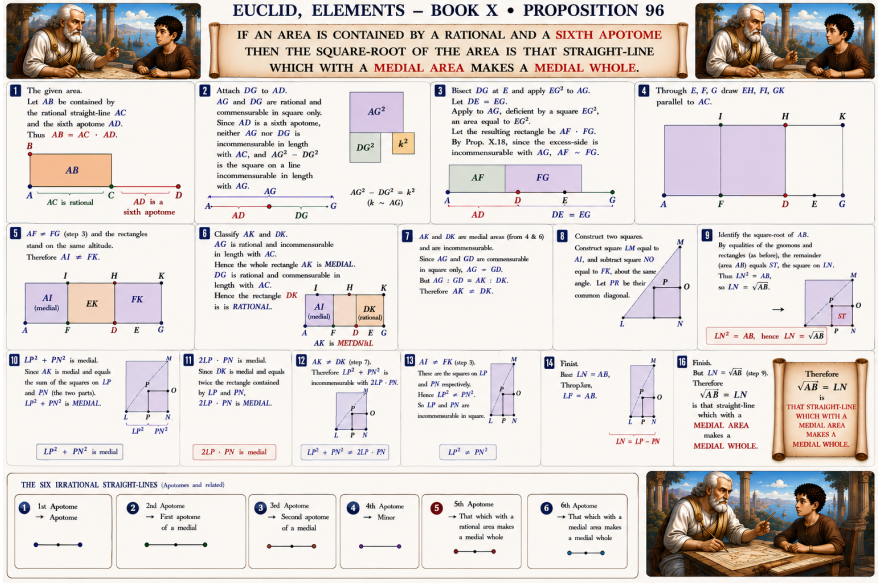
AK medial, DK medial, and $AK \nsim DK$.

$\Downarrow$

$LN^2 = AB$, while $LP^2 + PN^2$ and $2LP \cdot PN$ are medial and incommensurable.

$\Downarrow$

$LN = \sqrt{AB} =$ that which with a medial area makes a medial whole



2.97 Proposition 97

Statement

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.

Proof Actions

1. Let AB be an apotome, and CD a rational (straightline).
2. And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth.
3. I say that CF is a first apotome.
4. For let BG be an attachment to AB.
5. Thus, AG and GB are rational (straight-lines which are) commensurable in square only [Prop. 10.73].
6. And let CH, equal to the (square) on AG, and KL, (equal) to the (square) on BG, have been applied to CD.
7. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB.
8. The remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7].
9. Let FM have been cut in half at point N.
10. And let NO have been drawn through N, parallel to CD.
11. Thus, FO and LN are each equal to the (rectangle contained) by AG and GB.
12. And since the (sum of the squares) on AG and GB is rational, and DM is equal to the (sum of the squares) on AG and GB, DM is thus rational.
13. And it has been applied to the rational (straight-line) CD, producing CM as breadth.

14. Thus, CM is rational, and commensurable in length with CD [Prop. 10.20].

15. Again, since twice the (rectangle contained) by AG and GB is medial, and FL (is) equal to twice the (rectangle contained) by AG and GB, FL (is) thus a medial (area).

16. And it is applied to the rational (straight-line) CD, producing FM as breadth.

17. FM is thus rational, and incommensurable in length with CD [Prop. 10.22].

18. And since the (sum of the squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB.

19. And CL is equal to the (sum of the squares) on AG and GB, and FL to twice the (rectangle contained) by AG and GB.

20. DM is thus incommensurable with FL.

21. And as DM (is) to FL, so CM is to FM [Prop. 6.1].

22. CM is thus incommensurable in length with FM [Prop. 10.11].

23. And both are rational (straightThus, CM and MF are rational (straight-lines lines). which are) commensurable in square only.

24. CF is thus an apotome [Prop. 10.73].

25. So, I say that (it is) also a first (apotome).

26. For since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on BG, and NL to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL.

27. Thus, as CH is to NL, so NL (is) to KL.

28. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1].

29. Thus, the (rectangle contained) by CK and KM is equal to the (square) on NM— that is to say, to the fourth part of the (square) on FM [Prop. 6.17].

30. And since the (square) on AG is commensurable with the (square) on GB, CH [is] also commensurable with KL.

31. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1].

32. CK is thus commensurable (in length) with KM [Prop. 10.11].

33. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and CK is commensurable (in length) with KM, the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17].

34. And CM is commensurable in length with the (previously) laid down rational (straight-line) CD.

35. Thus, CF is a first apotome [Def. 10.15].

36. Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.

37. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB apotome, CD rational; apply AB^2 to CD and obtain breadth CF.

$\Downarrow$

CM and FM are rational, $CM \nsim FM$, hence $CF = CM - FM$ is an apotome.

$\Downarrow$

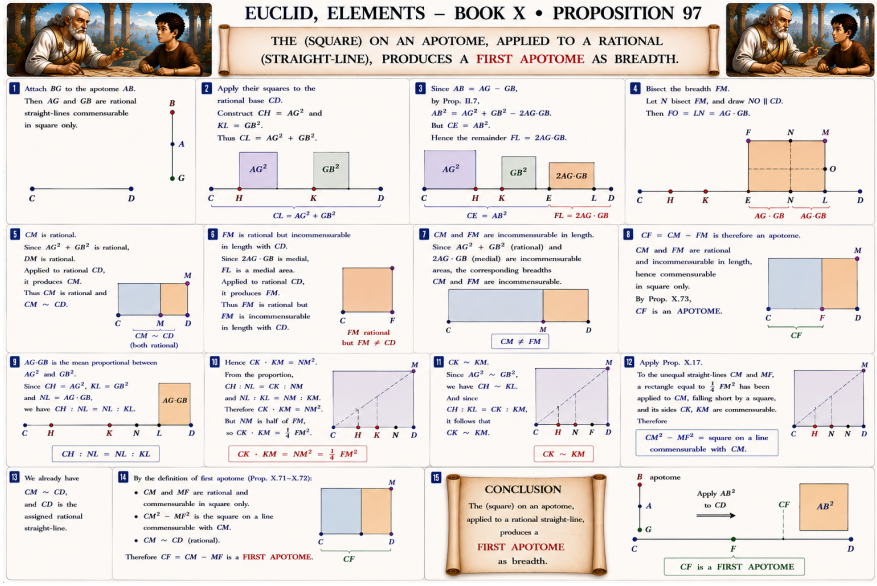
$$CK \cdot KM = NM^2 = \frac{1}{4}FM^2$$

$\Downarrow$

$$CM^2 - MF^2 = \square(\text{line } \sim CM), \qquad CM \sim CD$$

$\Downarrow$

CF is a FIRST APOTOME



2.98 Proposition 98

Statement

The (square) on a first apotome of a medial (straightline), applied to a rational (straight-line), produces a second apotome as breadth.

Proof Actions

1. Let AB be a first apotome of a medial (straight-line), and CD a rational (straight-line).

2. And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth.

3. I say that CF is a second apotome.

4. For let BG be an attachment to AB.

5. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74].

6. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on GB, producing KM as breadth.

7. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB.

8. Thus, CL (is) also a medial (area) [Props. 10.15, 10.23 corr.].

9. And it is applied to the rational (straight-line) CD, producing CM as breadth.

10. CM is thus rational, and incommensurable in length with CD [Prop. 10.22].

11. And since CL is equal to the (sum of the squares) on AG and GB, of which the (square) on AB is equal to CE, the remainder, twice the (rectangle contained) by AG and GB, is thus equal to FL [Prop. 2.7].

12. And twice the (rectangle contained) by AG and GB [is] rational. Thus, FL (is) rational.

13. And it is applied to the rational (straight-line) FE, producing FM as breadth.

14. FM is thus also rational, and commensurable in length with CD [Prop. 10.20].

15. Therefore, since the (sum of the squares) on AG and GB—that is to say, CL—is medial, and twice the (rectangle contained) by AG and GB— that is to say, FL—(is) rational, CL is thus incommensurable with FL.

16. And as CL (is) to FL, so CM is to FM [Prop. 6.1].

17. Thus, CM (is) incommensurable in length with FM [Prop. 10.11].

18. And they are both rational (straight-lines).

19. Thus, CM and MF are rational (straight-lines which are) commensurable in square only.

20. CF is thus an apotome [Prop. 10.73].

21. So, I say that (it is) also a second (apotome).

22. For let FM have been cut in half at N.

23. And let NO have been drawn through (point) N, parallel to CD.

24. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB.

25. And since the (rectangle contained) by AG and GB is the mean proportional to the squares on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (rectangle contained) by AG and GB to NL, and the (square) on BG to KL, NL is thus also the mean proportional to CH and KL.

26. Thus, as CH is to NL, so NL (is) to KL [Prop. 5.11].

27. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to MK [Prop. 6.1].

28. Thus, as CK (is) to NM, so NM is to KM [Prop. 5.11].

29. The (rectangle contained) by CK and KM is thus equal to the (square) on NM [Prop. 6.17]—that is to say, to the fourth part of the (square) on FM [and since the (square) on AG is commensurable with the (square) on BG, CH is also commensurable with KL—that is to say, CK with KM].

30. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to the greater CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17].

31. The attachment FM is also commensurable in length with the (previously) laid down rational (straight-line) CD.

32. CF is thus a second apotome [Def. 10.16].

33. Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth.

34. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB first apotome of a medial, CD rational.

$\Downarrow$

CM is rational with $CM \nsim CD$; FM is rational with $FM \sim CD$; and $CM \nsim FM$.

$\Downarrow$

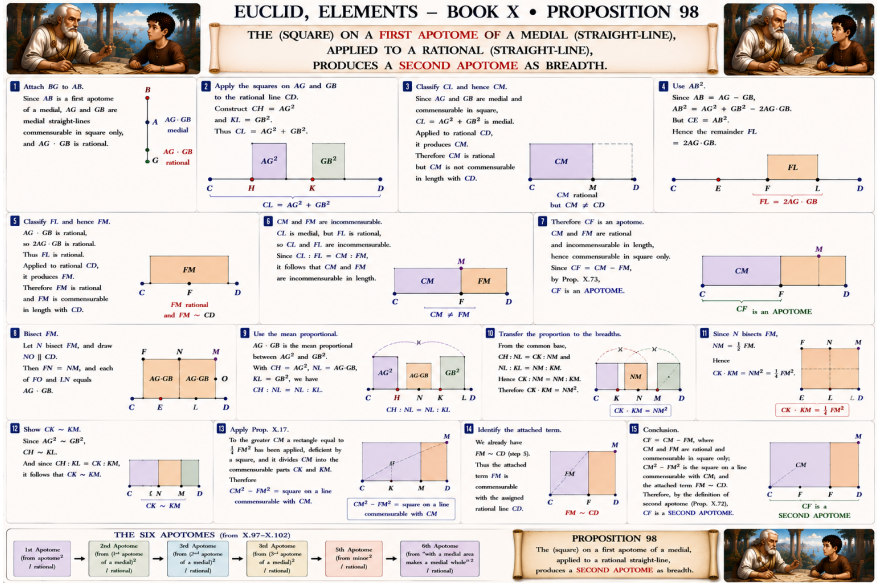
$$CK \cdot KM = NM^2 = \frac{1}{4}FM^2$$

$\Downarrow$

$$CM^2 - MF^2 = \square(\text{line } \sim CM), \qquad FM \sim CD$$

$\Downarrow$

CF is a SECOND APOTOME



2.99 Proposition 99

Statement

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.

Proof Actions

1. Let AB be the second apotome of a medial (straightline), and CD a rational (straight-line).
2. And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth.
3. I say that CF is a third apotome.
4. For let BG be an attachment to AB.
5. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75].
6. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth.
7. And let KL, equal to the (square) on BG, have been applied to KH, producing KM as breadth.
8. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB [and the (sum of the squares) on AG and GB is medial].
9. CL (is) thus also medial [Props. 10.15, 10.23 corr.].
10. And it has been applied to the rational (straight-line) CD, producing CM as breadth.
11. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22].
12. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder LF is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7].

13. Therefore, let FM have been cut in half at point N.

14. And let NO have been drawn parallel to CD.

15. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB.

16. And the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial.

17. And it is applied to the rational (straight-line) EF, producing FM as breadth.

18. FM is thus rational, and incommensurable in length with CD [Prop. 10.22].

19. And since AG and GB are commensurable in square only, AG [is] thus incommensurable in length with GB.

20. Thus, the (square) on AG is also incommensurable with the (rectangle contained) by AG and GB [Props. 6.1, 10.11].

21. But, the (sum of the squares) on AG and GB is commensurable with the (square) on AG, and twice the (rectangle contained) by AG and GB with the (rectangle contained) by AG and GB.

22. The (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.13].

23. But, CL is equal to the (sum of the squares) on AG and GB, and FL is equal to twice the (rectangle contained) by AG and GB.

24. Thus, CL is incommensurable with FL.

25. And as CL (is) to FL, so CM is to FM [Prop. 6.1].

26. CM is thus incommensurable in length with FM [Prop. 10.11].

27. And they are both rational (straight-lines).

28. Thus, CM and MF are rational (straight-lines which are) commensurable in square only.

29. CF is thus an apotome [Prop. 10.73].

30. So, I say that (it is) also a third (apotome).

31. For since the (square) on AG is commensurable with the (square) on GB, CH (is) thus also commensurable with KL.

32. Hence, CK (is) also (commensurable in length) with KM [Props. 6.1, 10.11].

33. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL.

34. Thus, as CH is to NL, so NL (is) to KL.

35. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM (is) to KM [Prop. 6.1].

36. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11].

37. Thus, the (rectangle contained) by CK and KM is equal to the [(square) on MN—that is to say, to the] fourth part of the (square) on FM [Prop. 6.17].

38. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable (in length) with (CM) [Prop. 10.17].

39. And neither of CM and MF is commensurable in length with the (previously) laid down rational (straight-line) CD.

40. CF is thus a third apotome [Def. 10.13].

41. Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.

42. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB second apotome of a medial, CD rational.

$$\Downarrow$$

CM and FM are rational, both incommensurable with CD, and $CM \nsim FM$.

$$\Downarrow$$

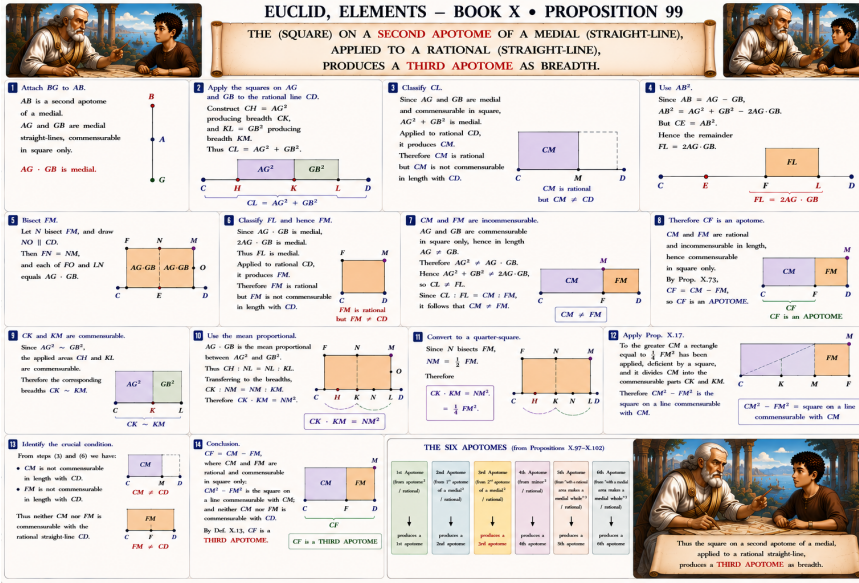
$$CK \cdot KM = NM^2 = \frac{1}{4}FM^2$$

$$\Downarrow$$

$$CM^2 - MF^2 = \square(\text{line } \sim CM), \qquad CM \nsim CD, \quad FM \nsim CD$$

$$\Downarrow$$

CF is a THIRD APOTOME



2.100 Proposition 100

Statement

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.

Proof Actions

1. Let AB be a minor (straight-line), and CD a rational (straight-line).
2. And let CE, equal to the (square) on AB, have been applied to the rational (straight-line) CD, producing CF as breadth.
3. I say that CF is a fourth apotome.
4. For let BG be an attachment to AB.
5. Thus, AG and GB are incommensurable in square, making the sum of the squares on AG and GB rational, and twice the (rectangle contained) by AG and GB medial [Prop. 10.76].
6. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG, producing KM as breadth.
7. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB.
8. And the sum of the (squares) on AG and GB is rational.
9. CL is thus also rational.
10. And it is applied to the rational (straight-line) CD, producing CM as breadth.
11. Thus, CM (is) also rational, and commensurable in length with CD [Prop. 10.20].
12. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7].

13. Therefore, let FM have been cut in half at point N.

14. And let NO have been drawn through N, parallel to either of CD or ML.

15. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB.

16. And since twice the (rectangle contained) by AG and GB is medial, and is equal to FL, FL is thus also medial.

17. And it is applied to the rational (straight-line) FE, producing FM as breadth.

18. Thus, FM is rational, and incommensurable in length with CD [Prop. 10.22].

19. And since the sum of the (squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is [thus] incommensurable with twice the (rectangle contained) by AG and GB.

20. And CL (is) equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB.

21. CL [is] thus incommensurable with FL.

22. And as CL (is) to FL, so CM is to MF [Prop. 6.1].

23. CM is thus incommensurable in length with MF [Prop. 10.11].

24. And both are rational (straight-lines).

25. Thus, CM and MF are rational (straight-lines which are) commensurable in square only.

26. CF is thus an apotome [Prop. 10.73].

27. [So], I say that (it is) also a fourth (apotome).

28. For since AG and GB are incommensurable in square, the (square) on AG (is) thus also incommensurable with the (square) on GB.

29. And CH is equal to the (square) on AG, and KL equal to the (square) on GB.

30. Thus, CH is incommensurable with KL.

31. And as CH (is) to KL, so CK is to KM [Prop. 6.1].

32. CK is thus incommensurable in length with KM [Prop. 10.11].

33. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (square) on GB to KL, and the (rectangle contained) by AG and GB to NL, NL is thus the mean proportional to CH and KL.

34. Thus, as CH is to NL, so NL (is) to KL.

35. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1].

36. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11].

37. The (rectangle contained) by CK and KM is thus equal to the (square) on MN—that is to say, to the fourth part of the (square) on FM [Prop. 6.17].

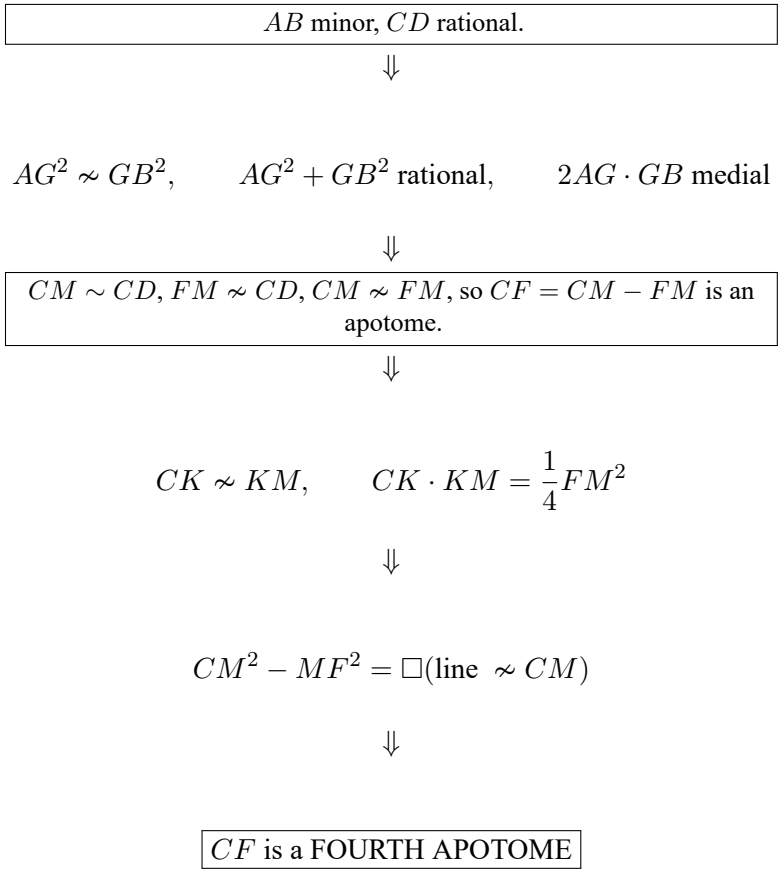
38. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18].

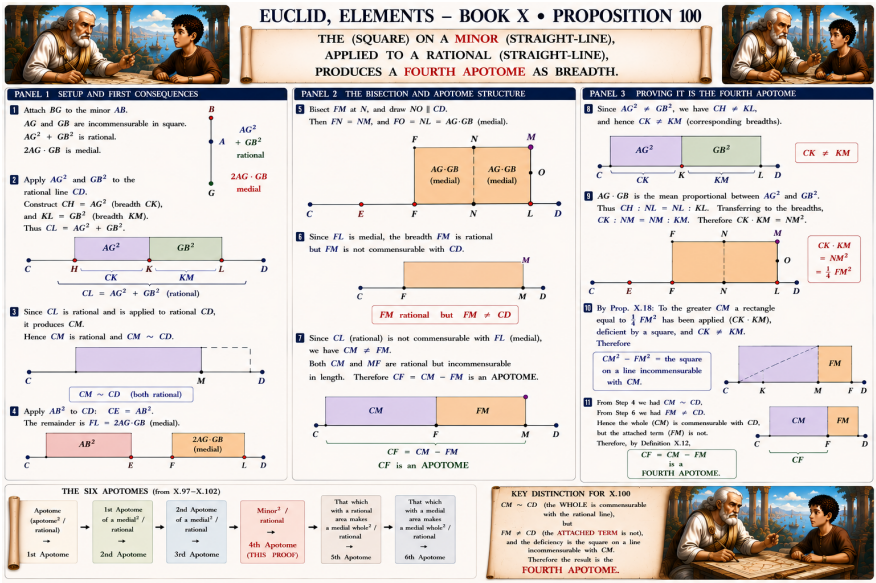
39. And the whole of CM is commensurable in length with the (previously) laid down rational (straight-line) CD.

40. Thus, CF is a fourth apotome [Def. 10.14].

41. Thus, the (square) on a minor, and so on . . .

Complete Cinematic Logic





2.101 Proposition 101

Statement

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.

Proof Actions

1. Let AB be that (straight-line) which with a rational (area) makes a medial whole, and CD a rational (straight-line).

2. And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth.

3. I say that CF is a fifth apotome.

4. Let BG be an attachment to AB.

5. Thus, the straightlines AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle con-

tained) by them rational [Prop. 10.77].

6. And let CH, equal to the (square) on AG, have been applied to CD, and KL, equal to the (square) on GB.

7. The whole of CL is thus equal to the (sum of the squares) on AG and GB.

8. And the sum of the (squares) on AG and GB together is medial. Thus, CL is medial.

9. And it has been applied to the rational (straight-line) CD, producing CM as breadth.

10. CM is thus rational, and incommensurable (in length) with CD [Prop. 10.22].

11. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7].

12. Therefore, let FM have been cut in half at N.

13. And let NO have been drawn through N, parallel to either of CD or ML.

14. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB.

15. And since twice the (rectangle contained) by AG and GB is rational, and [is] equal to FL, FL is thus rational.

16. And it is applied to the rational (straight-line) EF, producing FM as breadth.

17. Thus, FM is rational, and commensurable in length with CD [Prop. 10.20].

18. And since CL is medial, and FL rational, CL is thus incommensurable with FL.

19. And as CL (is) to FL, so CM (is) to MF [Prop. 6.1].

20. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational.

21. Thus, CM and MF are rational (straightlines which are) commensurable in square only.

22. CF is thus an apotome [Prop. 10.73].

23. So, I say that (it is) also a fifth (apotome).

24. For, similarly (to the previous propositions), we can show that the (rectangle contained) by CKM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM.

25. And since the (square) on AG is incommensurable with the (square) on GB, and the (square) on AG (is) equal to CH, and the (square) on GB to KL, CH (is) thus incommensurable with KL.

26. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1].

27. Thus, CK (is) incommensurable in length with KM [Prop. 10.11].

28. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18].

29. And the attachment FM is commensurable with the (previously) laid down rational (straightline) CD.

30. Thus, CF is a fifth apotome [Def. 10.15].

31. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB is that which with a rational area makes a medial whole; CD rational.

$$\Downarrow$$

$$AG^2 \nsim GB^2, \quad AG^2 + GB^2 \text{ medial}, \quad 2AG \cdot GB \text{ rational}$$

$$\Downarrow$$

$CM \nsim CD$, $FM \sim CD$, $CM \nsim FM$, hence CF is an apotome.

$$\Downarrow$$

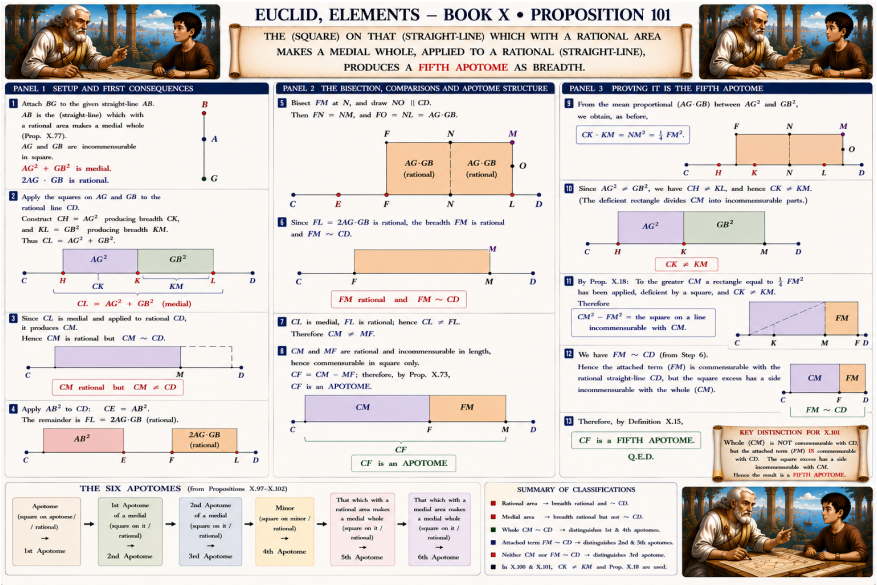
$$CK \nsim KM, \qquad CK \cdot KM = \frac{1}{4} FM^2$$

$$\Downarrow$$

$$CM^2 - MF^2 = \square(\text{line } \nsim CM), \qquad FM \sim CD$$

$$\Downarrow$$

CF is a FIFTH APOTOME



2.102 Proposition 102

Statement

> *The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.*

Proof Actions

1. Let AB be that (straight-line) which with a medial (area) makes a medial whole, and CD a rational (straight-line).

2. And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth.

3. I say that CF is a sixth apotome.

4. For let BG be an attachment to AB.

5. Thus, AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by AG and

GB medial, and the (sum of the squares) on AG and GB incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.78].

6. Therefore, let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG.

7. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB.

8. CL [is] thus also medial.

9. And it is applied to the rational (straight-line) CD, producing CM as breadth.

10. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22].

11. Therefore, since CL is equal to the (sum of the squares) on AG and GB, of which CE (is) equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7].

12. And twice the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial.

13. And it is applied to the rational (straight-line) FE, producing FM as breadth.

14. FM is thus rational, and incommensurable in length with CD [Prop. 10.22].

15. And since the (sum of the squares) on AG and GB is incommensurable with twice the (rectangle contained) by AG and GB, and CL equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB, CL [is] thus incommensurable with FL.

16. And as CL (is) to FL, so CM is to MF [Prop. 6.1].

17. Thus, CM is incommensurable in length with MF [Prop. 10.11].

18. And they are both rational.

19. Thus, CM and MF are rational (straight-lines which are) commensurable in square only.

20. CF is thus an apotome [Prop. 10.73].

21. So, I say that (it is) also a sixth (apotome).

22. For since FL is equal to twice the (rectangle contained) by AG and GB, let FM have been cut in half at N, and let NO have been drawn through N, parallel to CD.

23. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB.

24. And since AG and GB are incommensurable in square, the (square) on AG is thus incommensurable with the (square) on GB.

25. But, CH is equal to the (square) on AG, and KL is equal to the (square) on GB.

26. Thus, CH is incommensurable with KL.

27. And as CH (is) to KL, so CK is to KM [Prop. 6.1].

28. Thus, CK is incommensurable (in length) with KM [Prop. 10.11].

29. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL.

30. Thus, as CH is to NL, so NL (is) to KL.

31. And for the same (reasons as the preceding propositions), the square on CM is greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18].

32. And neither of them is commensurable with the (previously) laid down rational (straight-line) CD.

33. Thus, CF is a sixth apotome [Def. 10.16].

34. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB is that which with a medial area makes a medial whole; CD rational.

$$\Downarrow$$

$$AG^2 \nsim GB^2, \quad AG^2 + GB^2 \text{ medial}, \quad 2AG \cdot GB \text{ medial}$$

$$\Downarrow$$

$CM \nsim CD$, $FM \nsim CD$, and $CM \nsim FM$; therefore CF is an apotome.

$$\Downarrow$$

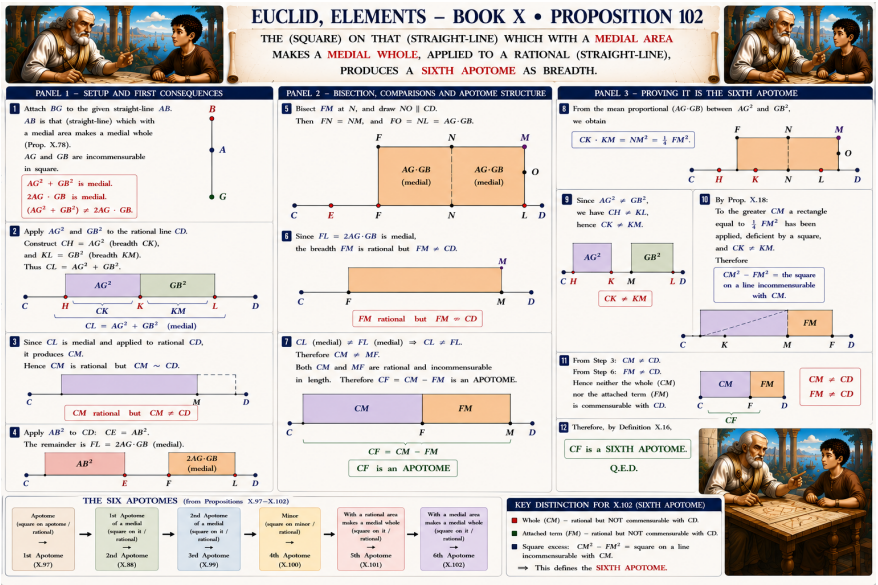
$$CK \nsim KM, \qquad CK \cdot KM = \frac{1}{4} FM^2$$

$$\Downarrow$$

$$CM^2 - MF^2 = \square(\text{line } \nsim CM)$$

$$\Downarrow$$

CF is a SIXTH APOTOME



2.103 Proposition 103

Statement

> *A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.*

Proof Actions

1. Let AB be an apotome, and let CD be commensurable in length with AB.
2. I say that CD is also an apotome, and (is) the same in order as AB.
3. For since AB is an apotome, let BE be an attachment to it.
4. Thus, AE and EB are rational (straight-lines which are) commensurable in square only [Prop. 10.73].
5. And let it have been contrived that the (ratio) of BE to DF is the same as the ratio of AB to CD [Prop. 6.12].
6. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12].

7. And thus as the whole AE is to the whole CF, so AB (is) to CD.

8. And AB (is) commensurable in length with CD.

9. AE (is) thus also commensurable (in length) with CF, and BE with DF [Prop. 10.11].

10. And AE and BE are rational (straight-lines which are) commensurable in square only.

11. Thus, CF and FD are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [CD is thus an apotome.

12. So, I say that (it is) also the same in order as AB.] Therefore, since as AE is to CF, so BE (is) to DF, thus, alternately, as AE is to EB, so CF (is) to FD [Prop. 5.16].

13. So, the square on AE is greater than (the square on) EB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (AE).

14. Therefore, if the (square) on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14].

15. And if AE is commensurable in length with a (previously) laid down rational (straight-line) then so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13].

16. And if the (square) on AE is greater [than (the square on) EB] by the (square) on (some straight-line) incommensurable (in length) with (AE) then the (square) on CF will also be greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14].

17. And if AE is commensurable in length with a (previously) laid down rational (straightline), so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13].

18. Thus, CD is an apotome, and (is) the same in order as AB [Defs. 10.11—10.16].

19. (Which is) the very thing it was required to show.

Complete Cinematic Logic

$$\boxed{AB \text{ apotome and } AB \sim CD.}$$

$$\Downarrow$$

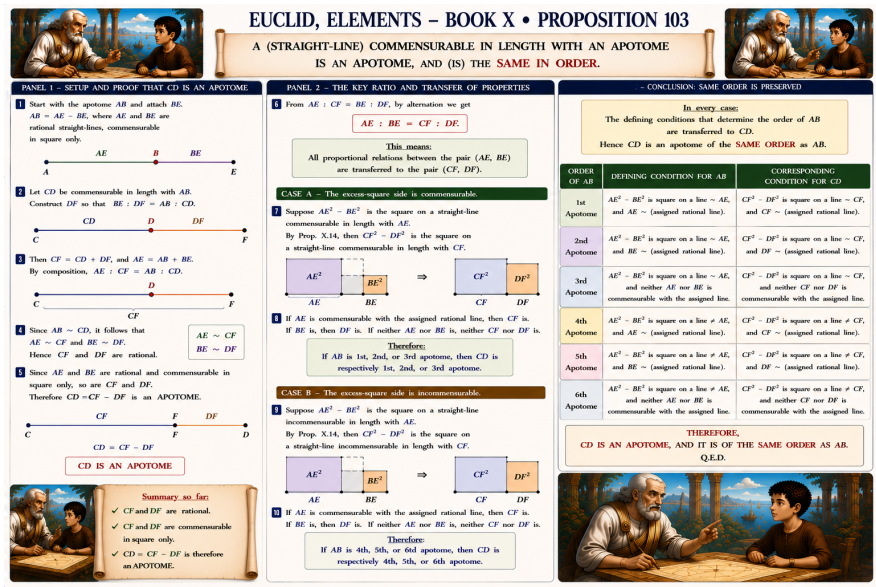
$$AB = AE - BE, \qquad CD = CF - DF, \qquad AE : BE = CF : DF$$

$$\Downarrow$$

$$\boxed{\begin{array}{c}\text{Rationality, square-only commensurability, and the square-excess}\\ \text{condition transfer from } AE, BE \text{ to } CF, DF.\end{array}}$$

$$\Downarrow$$

$$\boxed{\text{order}(CD) = \text{order}(AB)}$$



2.104 Proposition 104

Statement

A (straight-line) commensurable (in length) with an apotome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.

Proof Actions

1. Let AB be an apotome of a medial (straight-line), and let CD be commensurable in length with AB.
2. I say that CD is also an apotome of a medial (straight-line), and (is) the same in order as AB.
3. For since AB is an apotome of a medial (straightline), let EB be an attachment to it.
4. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75].
5. And let it have been contrived that as AB is to CD, so BE (is) to DF Thus, AE [is] also commensurable (in [Prop. 6.12]. length) with CF, and BE with DF [Props. 5.12, 10.11].
6. And AE and EB are medial (straight-lines which are) CF and FD are thus commensurable in square only. also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13].
7. Thus, CD is an apotome of a medial (straight-line) [Props. 10.74, 10.75].
8. So, I say that it is also the same in order as AB.
9. [For] since as AE is to EB, so CF (is) to FD [Props. 5.12, 5.16] [but as AE (is) to EB, so the (square) on AE (is) to the (rectangle contained) by AE and EB, and as CF (is) to FD, so the (square) on CF (is) to the (rectangle contained) by CF and FD], thus as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF also (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.] [and, alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AE and EB (is) to the (rectangle contained) by CF and FD].

10. And the (square) on AE (is) commensurable with the (square) on CF.

11. Thus, the (rectangle contained) by AE and EB is also commensurable with the (rectangle contained) by CF and FD [Props. 5.16, 10.11].

12. Therefore, either the (rectangle contained) by AE and EB is rational, and the (rectangle contained) by CF and FD will also be rational [Def. 10.4], or the (rectangle contained) by AE and EB [is] medial, and the (rectangle contained) by CF and FD [is] also medial [Prop. 10.23 corr.].

13. Therefore, CD is the apotome of a medial (straightline), and is the same in order as AB [Props. 10.74, 10.75].

14. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB is an apotome of a medial and $AB \sim CD$.

$$\Downarrow$$

$$AB = AE - EB, \qquad CD = CF - DF, \qquad AE : EB = CF : FD$$

$$\Downarrow$$

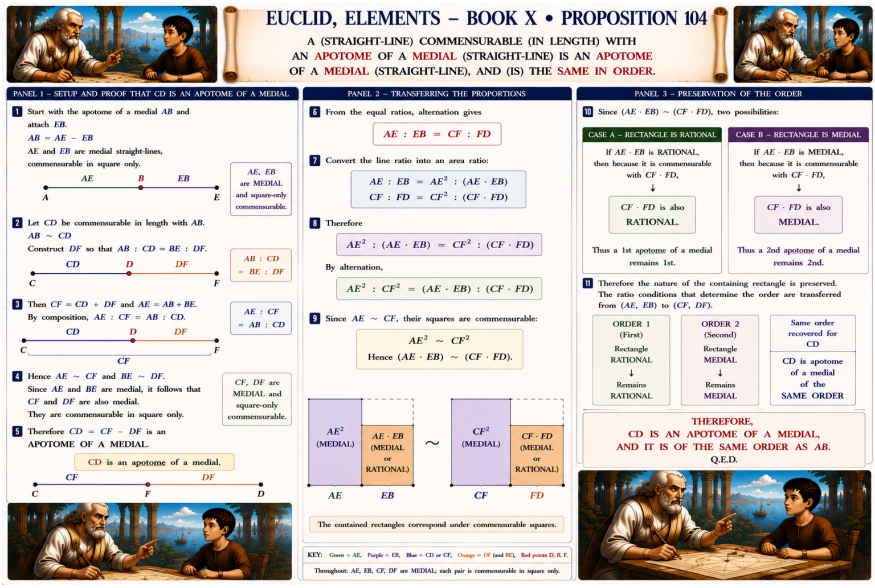
$$AE \cdot EB \sim CF \cdot FD$$

$$\Downarrow$$

If the first rectangle is rational (or medial), so is the second.

$$\Downarrow$$

CD is an apotome of a medial of the SAME ORDER as AB



2.105 Proposition 105

Statement

> *A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).*

Proof Actions

1. For let AB be a minor (straight-line), and (let) CD (be) commensurable (in length) with AB.

2. I say that CD is also a minor (straight-line).

3. For let the same things have been contrived (as in And since AE and EB are the former proposition).

4. (straight-lines which are) incommensurable in square [Prop. 10.76], CF and FD are thus also (straight-lines which are) incommensurable in square [Prop. 10.13].

5. Therefore, since as AE is to EB, so CF (is) to FD [Props. 5.12, 5.16], thus also as the (square) on AE is to the (square) on EB, so the (square) on CF (is) to the (square) on FD [Prop. 6.22].

6. Thus, via composition, as the (sum of the squares) on AE and EB is to the (square) on EB, so the (sum of the squares) on CF and FD (is) to the (square) on FD [Prop. 5.18], [also alternately].

7. And the (square) on BE is commensurable with the (square) on DF [Prop. 10.104].

8. The sum of the squares on AE and EB (is) thus also commensurable with the sum of the squares on CF and FD [Prop. 5.16, 10.11].

9. And the sum of the (squares) on AE and EB is rational [Prop. 10.76].

10. Thus, the sum of the (squares) on CF and FD is also rational [Def. 10.4].

11. Again, since as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.], and the square on AE (is) commensurable with the square on CF, the (rectangle contained) by AE and EB is thus also commensurable with the (rectangle contained) by CF and FD.

12. And the (rectangle contained) by AE and EB (is) medial [Prop. 10.76].

13. Thus, the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.].

14. CF and FD are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

15. Thus, CD is a minor (straight-line) [Prop. 10.76].

16. (Which is) the very thing it was required to show.

Complete Cinematic Logic

$$\boxed{AB \text{ minor and } AB \sim CD.}$$

$$\Downarrow$$

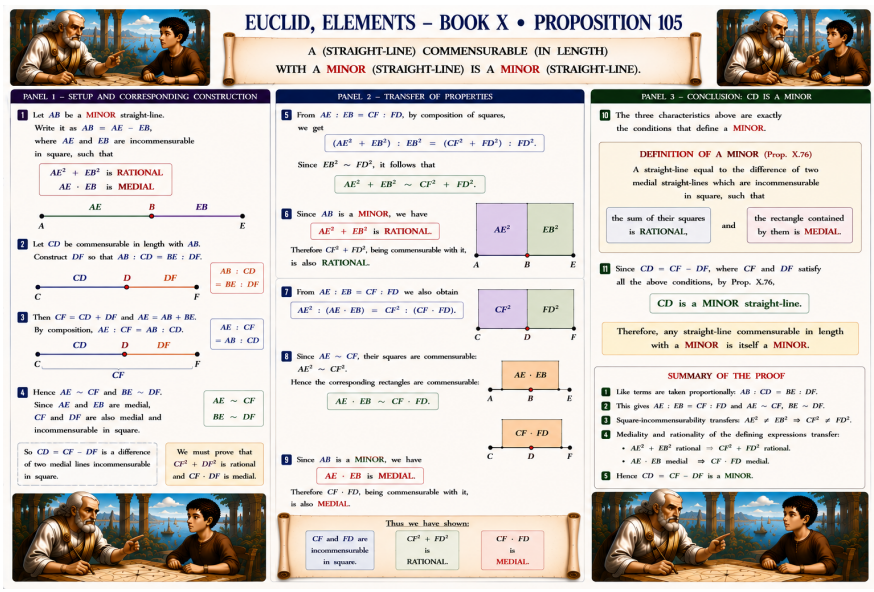
$$AB = AE - EB, \qquad CD = CF - FD, \qquad AE : EB = CF : FD$$

$$\Downarrow$$

$$CF^2 \nsim FD^2, \qquad CF^2 + FD^2 \text{ rational}, \qquad CF \cdot FD \text{ medial}$$

$$\Downarrow$$

$$\boxed{CD \text{ is a MINOR}}$$



2.106 Proposition 106

Statement

A (straight-line) commensurable (in length) with a (straight-line) which with a rational (area) makes a medial whole is a (straight-line) which with a rational (area) makes a medial whole.

Proof Actions

1. Let AB be a (straight-line) which with a rational (area) makes a medial whole, and (let) CD (be) commensurable (in length) with AB.

2. I say that CD is also a (straight-line) which with a rational (area) makes a medial (whole).

3. For let BE be an attachment to AB.

4. Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on AE and EB medial, and the (rectangle contained) by them rational [Prop. 10.77].

5. And let the same construction have been made (as in the previous propositions).

6. So, similarly to the previous (propositions), we can show that CF and FD are in the same ratio as AE and EB, and the sum of the squares on AE and EB is commensurable with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD.

7. Hence, CF and FD are also (straightlines which are) incommensurable in square, making the sum of the squares on CF and FD medial, and the (rectangle contained) by them rational.

8. CD is thus a (straight-line) which with a rational (area) makes a medial whole [Prop. 10.77].

9. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB is "with a rational area makes a medial whole" and $AB \sim CD$.

$\Downarrow$

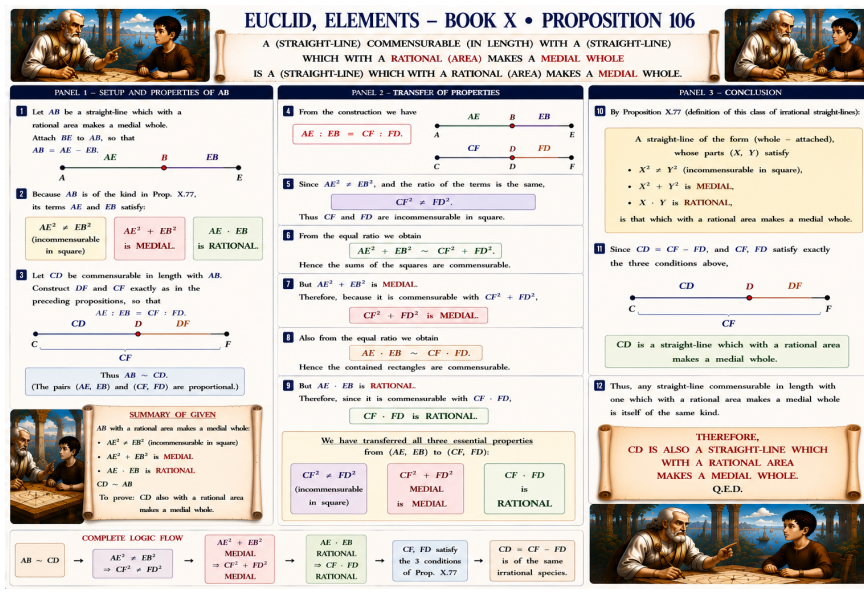
$AE^2 \nsim EB^2$, $AE^2 + EB^2$ medial, $AE \cdot EB$ rational

$\Downarrow$

$CF^2 \nsim FD^2$, $CF^2 + FD^2$ medial, $CF \cdot FD$ rational

$\Downarrow$

CD is of the same irrational species



2.107 Proposition 107

Statement

A (straight-line) commensurable (in length) with a (straight-line) which with a medial (area) makes a medial whole is itself also a (straight-line) which with a medial (area) makes a medial whole.

Proof Actions

1. Let AB be a (straight-line) which with a medial (area) makes a medial whole, and let CD be commensurable (in length) with AB.

2. I say that CD is also a (straight-line) which with a medial (area) makes a medial whole.

3. For let BE be an attachment to AB.

4. And let the same construction have been made (as in the previous propoThus, AE and EB are (straight-lines which sitions). are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78].

5. And, as was shown (previously), AE and EB are commensurable (in length) with CF and FD (respectively), and the sum of the squares on AE and EB with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD.

6. Thus, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

7. Thus, CD is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78].

8. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB is "with a medial area makes a medial whole" and $AB \sim CD$.

$\Downarrow$

$AE^2 \nsim EB^2$, $\quad AE^2 + EB^2$ medial, $\quad AE \cdot EB$ medial

$\Downarrow$

$(AE^2 + EB^2) \nsim AE \cdot EB$

$\Downarrow$

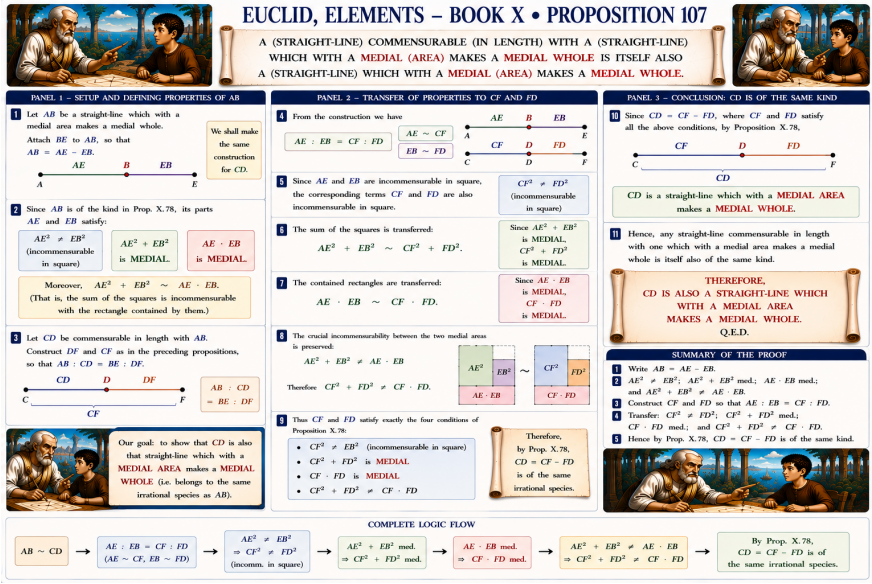
$CF^2 \nsim FD^2$, $\quad CF^2 + FD^2$ medial, $\quad CF \cdot FD$ medial

$\Downarrow$

$(CF^2 + FD^2) \nsim CF \cdot FD$

$\Downarrow$

CD is of the same irrational species



2.108 Proposition 108

Statement

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area—either an apotome, or a minor (straight-line).

Proof Actions

1. For let the medial (area) BD have been subtracted from the rational (area) BC.

2. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either an apotome, or a minor (straight-line).

3. For let the rational (straight-line) FG have been laid out, and let the right-angled parallelogram GH, equal to BC, have been applied to FG, and let GK, equal to DB, have been subtracted (from GH).

4. Thus, the remainder EC is equal to LH.

5. Therefore, since BC is a rational (area), and BD a medial (area), and BC (is) equal to GH, and BD to GK, GH is thus a rational (area), and GK a medial (area).

6. And they are applied to the rational (straight-line) FG.

7. Thus, FH (is) rational, and commensurable in length with FG [Prop. 10.20], and FK (is) also rational, and incommensurable in length with FG [Prop. 10.22].

8. Thus, FH is incommensurable in length with FK [Prop. 10.13].

9. FH and FK are thus rational (straight-lines which are) commensurable in square only.

10. Thus, KH is an apotome [Prop. 10.73], and KF an atSo, the square on HF is greater than tachment to it.

11. (the square on) FK by the (square) on (some straightline which is) either commensurable, or not (commensurable), (in length with HF).

12. First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with HF).

13. And the whole of HF is commensurable in length with the (previously) laid down rational (straight-line) FG.

14. Thus, KH is a first apotome [Def. 10.1].

15. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91].

16. Thus, the square-root of LH—that is to say, (of) EC—is an apotome.

17. And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) incommensurable (in length) with (HF), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fourth apotome [Prop. 10.14].

18. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94].

19. (Which is) the very thing it was required to show.

Complete Cinematic Logic

$$\boxed{BC \text{ rational, } BD \text{ medial, } EC = BC - BD.}$$

$$\Downarrow$$

$$FH \sim FG, \qquad FK \nsim FG, \qquad KH = FH - FK \text{ is an apotome}$$

$$\Downarrow$$

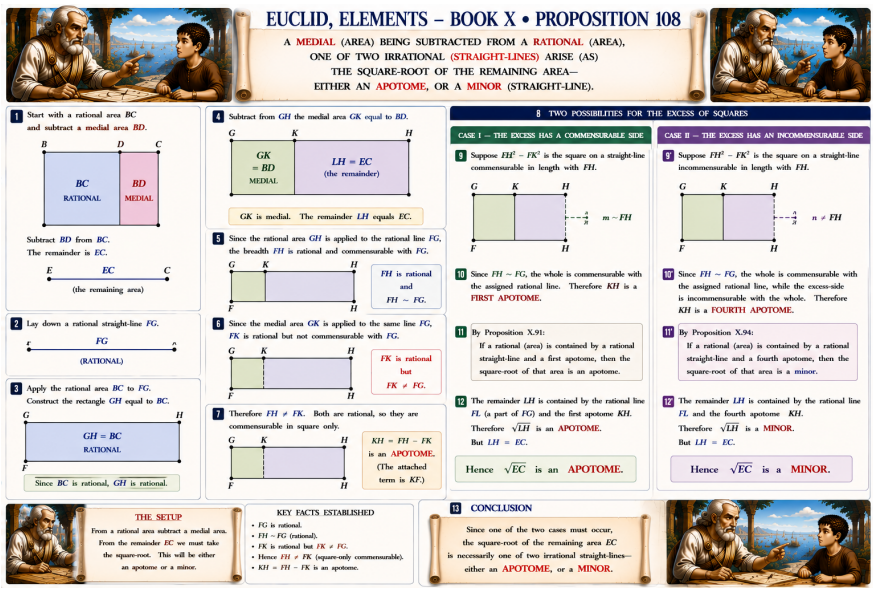
$$FH^2 - FK^2 = \square(\text{line} \sim FH) \qquad FH^2 - FK^2 = \square(\text{line} \nsim FH)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$KH = \text{1st apotome} \qquad KH = \text{4th apotome}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sqrt{EC} = \text{apotome} \qquad \sqrt{EC} = \text{minor}$$



2.109 Proposition 109

Statement

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

Proof Actions

1. For let the rational (area) BD have been subtracted from the medial (area) BC.

2. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

3. For let the rational (straight-line) FG be laid down, and let similar areas (to the preceding proposition) have been applied (to it).

4. So, accordingly, FH is rational, and incommensurable in length with FG, and KF (is) also rational, and commensurable in length with FG.

5. Thus, FH and FK are rational (straight-lines which are) com412 KH is thus mensurable in square only [Prop. 10.13]. an apotome [Prop. 10.73], and FK an attachment to it.

6. So, the square on HF is greater than (the square on) FK either by the (square) on (some straight-line) commensurable (in length) with (HF), or by the (square) on (some straight-line) incommensurable (in length with HF).

7. Therefore, if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a second apotome [Def. 10.12]. And FG (is) rational.

8. Hence, the square-root of LH—that is to say, (of) EC—is a first apotome of a medial (straight-line) [Prop. 10.92].

9. And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) incommensurable (in length with HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fifth apotome [Def. 10.15].

10. Hence, the square-root of EC is that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95].

11. (Which is) the very thing it was required to show.

Complete Cinematic Logic

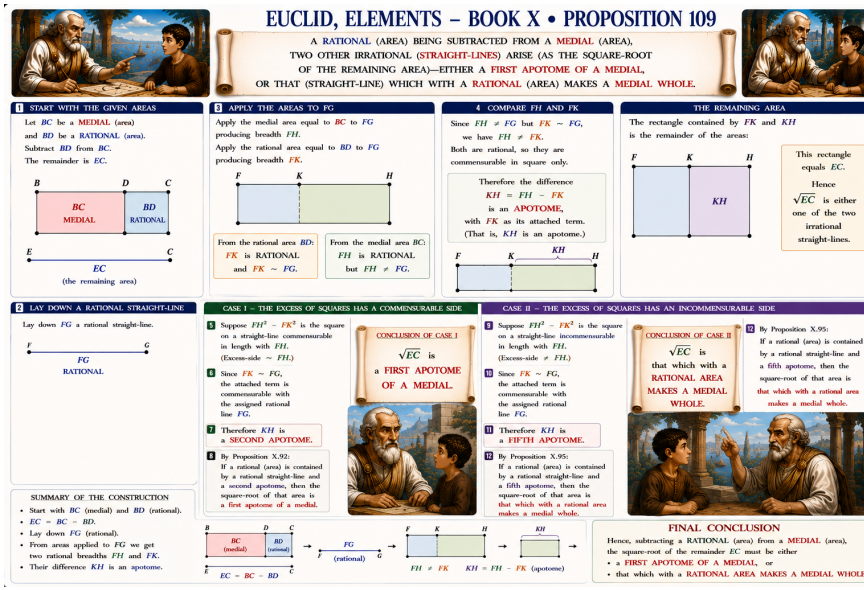
BC medial, BD rational, $EC = BC - BD$.

$\Downarrow$

$FH \nsim FG, \qquad FK \sim FG, \qquad KH = FH - FK$ is an apotome

$\Downarrow$

$FH^2 - FK^2 = \square(\text{line } \sim FH)$	$FH^2 - FK^2 = \square(\text{line } \nsim FH)$
$\Downarrow$	$\Downarrow$
$KH =$ 2nd apotome	$KH =$ 5th apotome
$\Downarrow$	$\Downarrow$
$\sqrt{EC} =$ 1st apotome of a medial	$\sqrt{EC} =$ rational-area medial whole



2.110 Proposition 110

Statement

> *A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area)—either a second apotome of a medial (straightline), or that (straight-line) which with a medial (area) makes a medial whole.*

Proof Actions

1. For, as in the previous figures, let the medial (area) BD, incommensurable with the whole, have been subtracted from the medial (area) BC.

2. I say that the squareroot of EC is one of two irrational (straight-lines)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.

3. For since BC and BD are each medial (areas), and BC (is) incommensurable with BD, accordingly, FH and FK will each be rational (straight-

lines), and incommensurable in length with FG [Prop. 10.22].

4. And since BC is incommensurable with BD—that is to say, GH with GK—HF (is) also incommensurable (in length) with FK [Props. 6.1, 10.11].

5. Thus, FH and FK are rational (straight-lines which are) commensurable in square only.

6. KH is thus as apotome [Prop. 10.73], [and FK an attachment (to it).

7. So, the square on FH is greater than (the square on) FK either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (FH).] So, if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (FH), and (since) neither of FH and FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a third apotome [Def. 10.3]. And KL (is) rational.

8. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93].

9. Hence, the squareroot of LH—that is to say, (of) EC—is a second apotome of a medial (straight-line).

10. And if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) incommensurable [in length] with (FH), and (since) neither of HF and FK is commensurable in length with FG, KH is a sixth apotome [Def. 10.16].

11. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96].

12. Thus, the square-root of LH—that is to say, (of) EC—is that (straight-line) which with a medial (area) makes a medial whole.

13. (Which is) the very thing it was required to show.

Complete Cinematic Logic

$$\boxed{BC \text{ medial, } BD \text{ medial, } BC \nsim BD, \text{ and } EC = BC - BD.}$$

$$\Downarrow$$

$$FH \nsim FG, \qquad FK \nsim FG, \qquad FH \nsim FK$$

$$\Downarrow$$

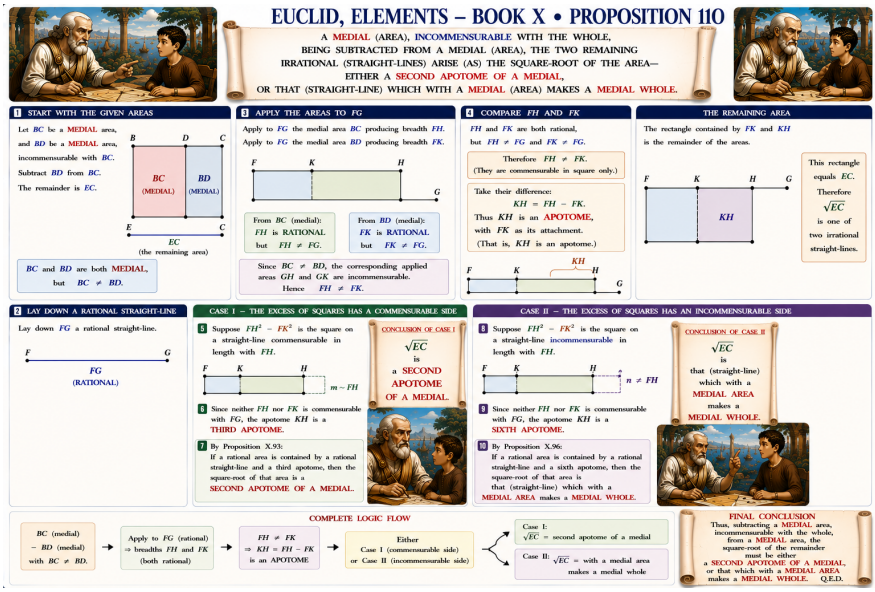
$$FH^2 - FK^2 = \square(\text{line} \sim FH) \qquad FH^2 - FK^2 = \square(\text{line} \nsim FH)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$KH = \text{3rd apotome} \qquad KH = \text{6th apotome}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sqrt{EC} = \text{2nd apotome of a medial} \qquad \sqrt{EC} = \text{medial-area medial whole}$$



2.111 Proposition 111

Statement

An apotome is not the same as a binomial.

Proof Actions

1. Let AB be an apotome.
2. I say that AB is not the same as a binomial.
3. For, if possible, let it be (the same).
4. And let a rational (straight-line) DC be laid down.
5. And let the rectangle CE, equal to the (square) on AB, have been applied to CD, producing DE as breadth.
6. Therefore, since AB is an apotome, DE is a first apotome [Prop. 10.97].
7. Let EF be an attachment to it.
8. Thus, DF and FE are rational (straight-lines which are) commensurable in square only, and the square on DF is greater than (the square on) FE by the (square) on (some straight-line) commensurable (in length) with (DF), and DF is commensurable in length with the (previously) laid down rational (straightline) DC [Def. 10.10].
9. Again, since AB is a binomial, DE is thus a first binomial [Prop. 10.60].
10. Let (DE) have been divided into its (component) terms at G, and let DG be the greater term.
11. Thus, DG and GE are rational (straight-lines which are) commensurable in square only, and the square on DG is greater than (the square on) GE by the (square) on (some straight-line) commensurable (in length) with (DG), and the greater (term) DG is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.5].
12. Thus, DF is also commensurable in length with DG [Prop. 10.12].

13. The remainder GF is thus commensurable in length with DF [Prop. 10.15].

14. [Therefore, since DF is commensurable with GF, and DF is rational, GF is thus also rational.

15. Therefore, since DF is commensurable in length with GF,] DF (is) incommensurable in length with EF.

16. Thus, FG is also incommensurable in length with EF [Prop. 10.13].

17. GF and FE [are] thus rational (straightlines which are) commensurable in square only.

18. Thus, EG is an apotome [Prop. 10.73].

19. But, (it is) also rational.

20. The very thing is impossible.

21. Thus, an apotome is not the same as a binomial.

22. (Which is) the very thing it was required to show.

23. [Corollary] The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

24. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22].

25. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97].

26. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98].

27. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straightline), produces as breadth a third apotome [Prop. 10.99].

28. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100].

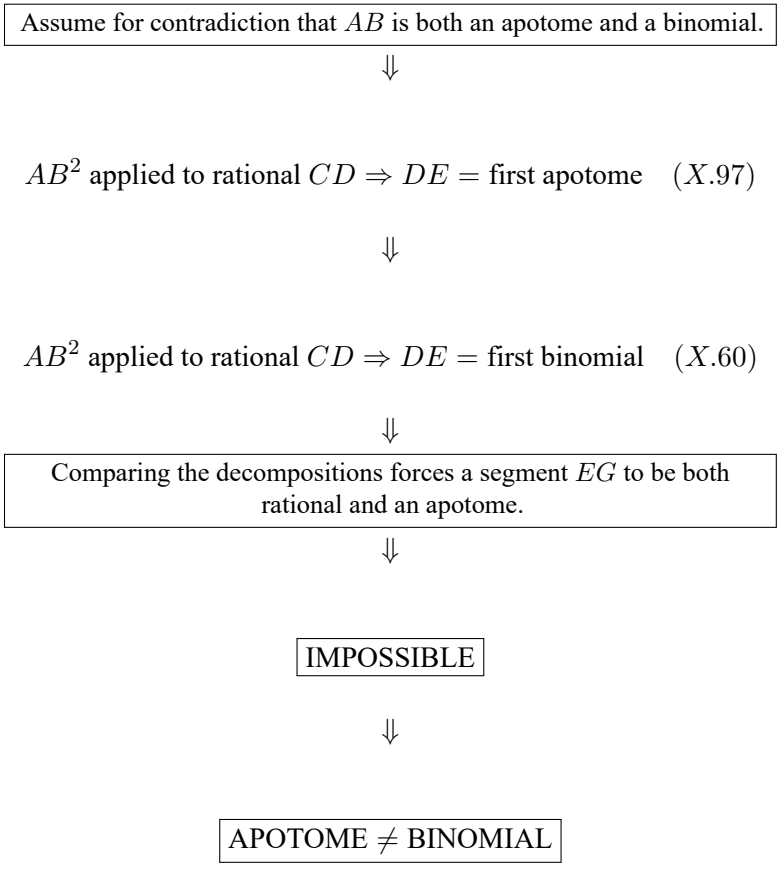
29. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101].

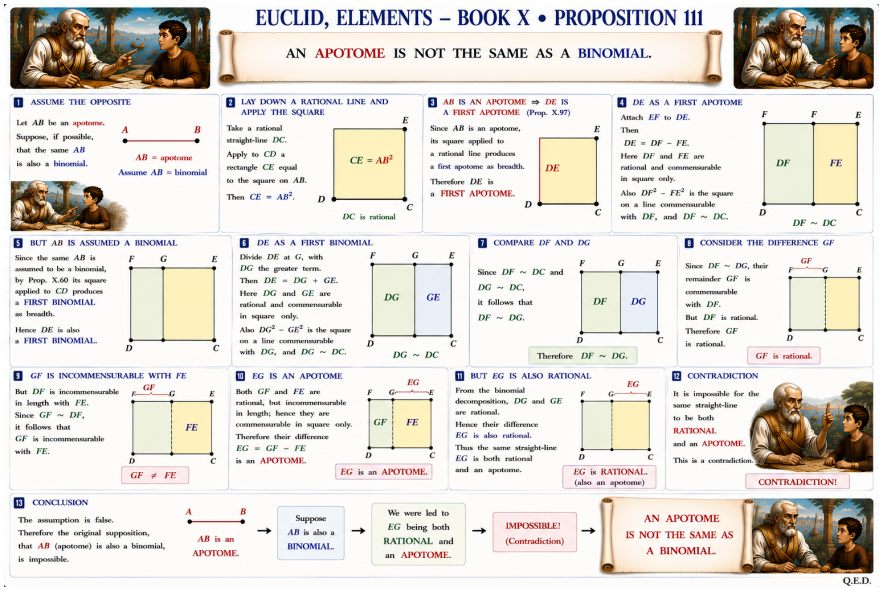
30. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102].

31. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational, and from one another since they are not the same in order—clearly, the irrational (straightlines) themselves also differ from one another.

32. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, in order, 13 irrational (straight-lines) in all: Medial, Binomial, First bimedial, Second bimedial, Major, Square-root of a rational plus a medial (area), Square-root of (the sum of) two medial (areas), Apotome, First apotome of a medial, Second apotome of a medial, Minor, That which with a rational (area) produces a medial whole, That which with a medial (area) produces a medial whole.

Complete Cinematic Logic





2.112 Proposition 112

Statement

The (square) on a rational (straight-line), applied to a binomial (straight-line), produces as breadth an apotome whose terms are commensurable (in length) with the terms of the binomial, and, furthermore, in the same ratio. Moreover, the created apotome will have the same order as the binomial.

Proof Actions

1. Let A be a rational (straight-line), and BC a binomial (straight-line), of which let DC be the greater term.

2. And let the (rectangle contained) by BC and EF be equal to the (square) on A.

3. I say that EF is an apotome whose terms are commensurable (in length) with CD and DB, and in the same ratio, and, moreover, that EF will have the same order as BC.

4. For, again, let the (rectangle contained) by BD and G be equal to the (square) on A.

5. Therefore, since the (rectangle contained) by BC and EF is equal to the (rectangle contained) by BD and G, thus as CB is to BD, so G (is) to EF [Prop. 6.16].

6. And CB (is) greater than BD.

7. Thus, G is also greater than EF [Props. 5.16, 5.14]. Let EH be equal to G.

8. Thus, as CB is to BD, so HE (is) to EF.

9. Thus, via separation, as CD is to BD, so HF (is) to FE [Prop. 5.17].

10. Let it have been contrived that as HF (is) to FE, so FK (is) to KE.

11. And, thus, the whole HK is to the whole KF, as FK (is) to KE.

12. For as one of the leading (proportional magnitudes is) to one of the following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12].

13. And as FK (is) to KE, so CD is to DB [Prop. 5.11].

14. And, thus, as HK (is) to KF, so CD is to DB [Prop. 5.11].

15. And the (square) on CD (is) commensurable with the (square) on DB [Prop. 10.36].

16. The (square) on HK is thus also commensurable with the (square) on KF [Props. 6.22, 10.11].

17. And as the (square) on HK is to the (square) on KF, so HK (is) to KE, since the three (straight-lines) HK, KF, and KE are proportional [Def. 5.9].

18. HK is thus commensurable in length with KE [Prop. 10.11].

19. Hence, HE is also commensurable in length with EK [Prop. 10.15].

20. And since the (square) on A is equal to the (rectangle contained) by EH and BD, and the (square) on A is rational, the (rectangle contained) by EH and BD is thus also rational.

21. And it is applied to the rational (straight-line) BD.

22. Thus, EH is rational, and commensurable in length with BD [Prop. 10.20].

23. And, hence, the (straight-line) commensurable (in length) with it, EK, is also rational [Def. 10.3], and commensurable in length with BD [Prop. 10.12].

24. Therefore, since as CD is to DB, so FK (is) to KE, and CD and DB are (straight-lines which are) commensurable in square only, FK and KE are also commensurable in square only [Prop. 10.11]. And KE is rational.

25. Thus, FK is also rational.

26. FK and KE are thus rational (straight-lines which are) commensurable in square only.

27. Thus, EF is an apotome [Prop. 10.73].

28. And the square on CD is greater than (the square on) DB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (CD).

29. Therefore, if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) commensurable (in length) with [CD] then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) commensurable (in length) with (FK) [Prop. 10.14].

30. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, And if BD (is commensurable), (so) also (is) 10.12]. KE [Prop. 10.12].

31. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE.

32. And if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) incommensurable (in length) with (CD) then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) incommensurable (in length) with (FK) [Prop. 10.14].

33. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12].

34. And if BD (is commensurable), (so) also (is) KE [Prop. 10.12].

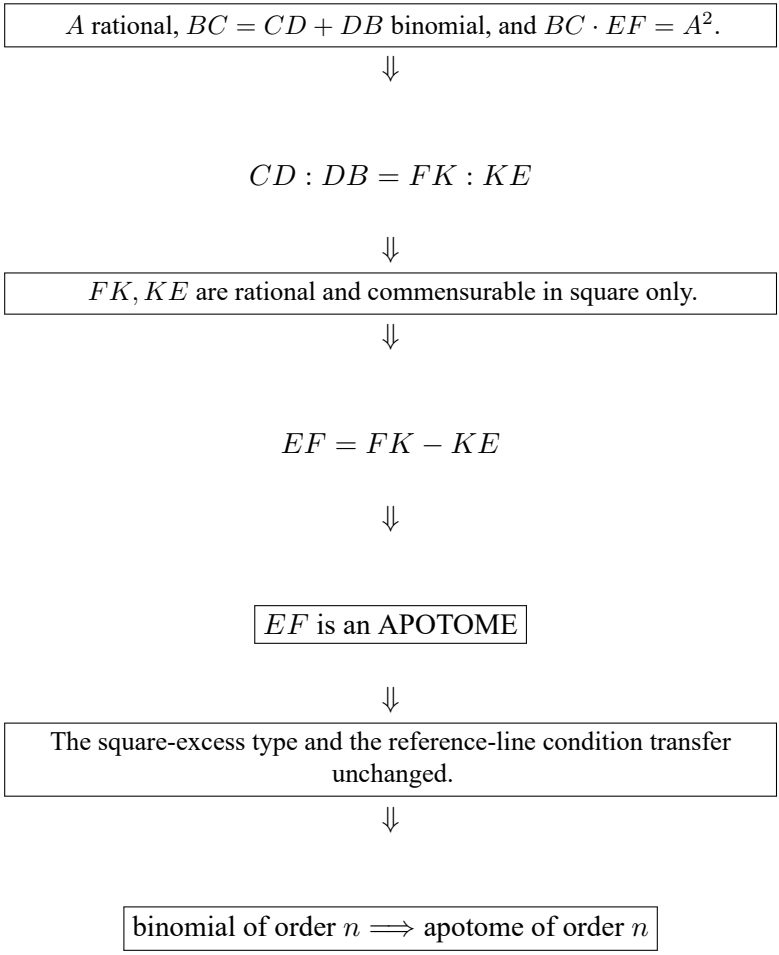
35. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE.

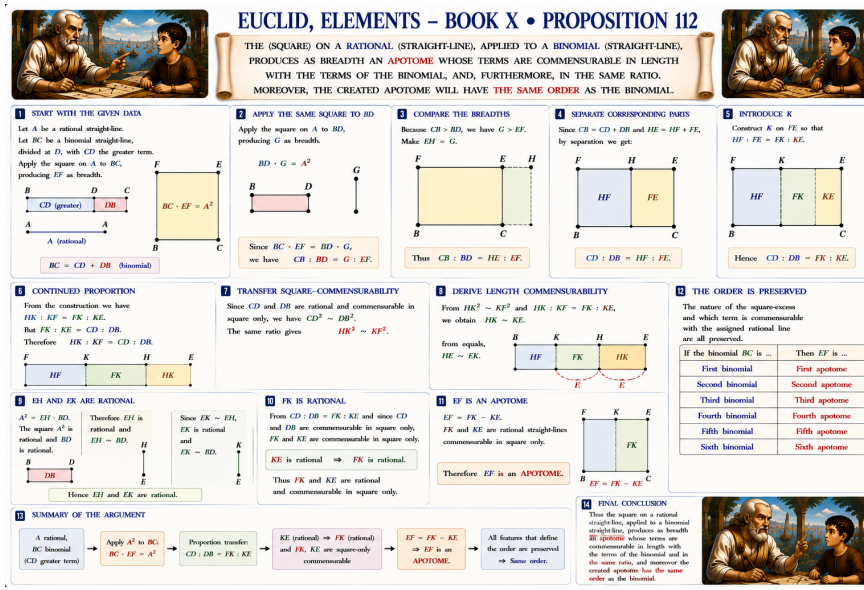
36. Hence, FE is an apotome whose terms, FK and KE, are commensurable (in length) with the terms, CD and DB, of the binomial, and in the same ratio.

37. And (FE) has the same order as BC [Defs. 10.5—10.10].

38. (Which is) the very thing it was required to show. early interpolations into the original text.

Complete Cinematic Logic





2.113 Proposition 113

Statement

> *The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.*

Proof Actions

1. Let A be a rational (straight-line), and BD an apotome.

2. And let the (rectangle contained) by BD and KH be equal to the (square) on A, such that the square on the rational (straight-line) A, applied to the apotome BD, produces KH as breadth.

3. I say that KH is a binomial whose terms are commensurable with the terms of BD, and in the same ratio, and, moreover, that KH has the same order as BD.

4. For let DC be an attachment to BD.

5. Thus, BC and CD are rational (straight-lines which are) commensurable in square only [Prop. 10.73].

6. And let the (rectangle contained) by BC and G also be equal to the (square) on A.

7. And the (square) on A (is) rational.

8. The (rectangle contained) by BC and G (is) thus also rational.

9. And it has been applied to the rational (straight-line) BC.

10. Thus, G is rational, and commensurable in length with BC [Prop. 10.20].

11. Therefore, since the (rectangle contained) by BC and G is equal to the (rectangle contained) by BD and KH, thus, proportionally, as CB is to BD, so KH (is) to G [Prop. 6.16].

12. And BC (is) greater than BD.

13. Thus, KH (is) also greater than G [Prop. 5.16, 5.14].

14. Let KE be made equal to G.

15. KE is thus commensurable in length with BC.

16. And since as CB is to BD, so HK (is) to KE, thus, via conversion, as BC (is) to CD, so KH (is) to HE [Prop. 5.19 corr.].

17. Let it have been contrived that as KH (is) to HE, so HF (is) to FE.

18. And thus the remainder KF is to FH, as KH (is) to HE—that is to say, [as] BC (is) to CD [Prop. 5.19].

19. And BC and CD [are] commensurable in square only.

20. KF and FH are thus also commensurable in square only [Prop. 10.11].

21. And since as KH is to HE, (so) KF (is) to FH, but as KH (is) to HE, (so) HF (is) to FE, thus, also as KF (is) to FH, (so) HF (is) to FE [Prop. 5.11].

22. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9].

23. And thus as KF (is) to FE, so the (square) on KF (is) to the (square) on FH.

24. And the (square) on KF is commensurable with the (square) on FH.

25. For KF and FH are commensurable in square.

26. Thus, KF is also commensurable in length with FE [Prop. 10.11].

27. Hence, KF [is] also commensurable in length with KE [Prop. 10.15].

28. And KE is rational, and commensurable in length with BC.

29. Thus, KF (is) also rational, and commensurable in length with BC [Prop. 10.12].

30. And since as BC is to CD, (so) KF (is) to FH, alternately, as BC (is) to KF, so DC (is) to FH [Prop. 5.16].

31. And BC (is) commensurable (in length) with KF.

32. Thus, FH (is) also commensurable in length with CD [Prop. 10.11].

33. And BC and CD are rational (straight-lines which are) commensurable in square only.

34. KF and FH are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13].

35. Thus, KH is a binomial [Prop. 10.36].

36. Therefore, if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) commensurable (in length) with (BC), then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) commensurable (in length) with (KF) [Prop. 10.14].

37. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12].

38. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FH [Prop. 10.12].

39. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

40. And if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) incommensurable (in length) with (BC) then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) incommensurable (in length) with (KF) [Prop. 10.14].

41. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12].

42. And if CD is commensurable, (so) also (is) FH [Prop. 10.12].

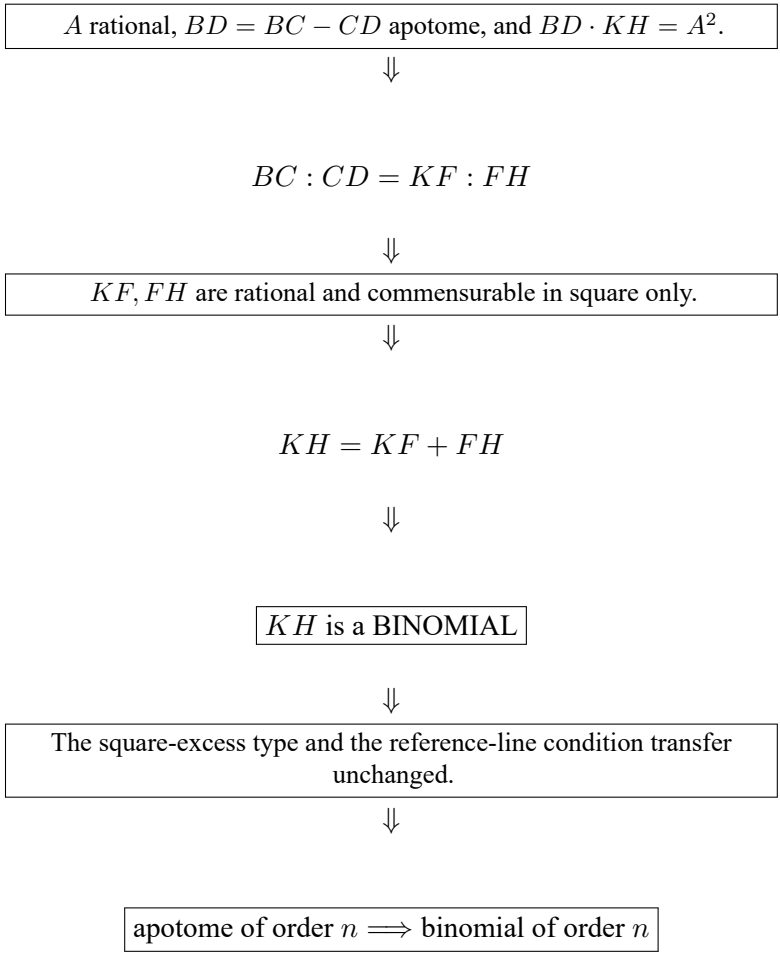
43. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

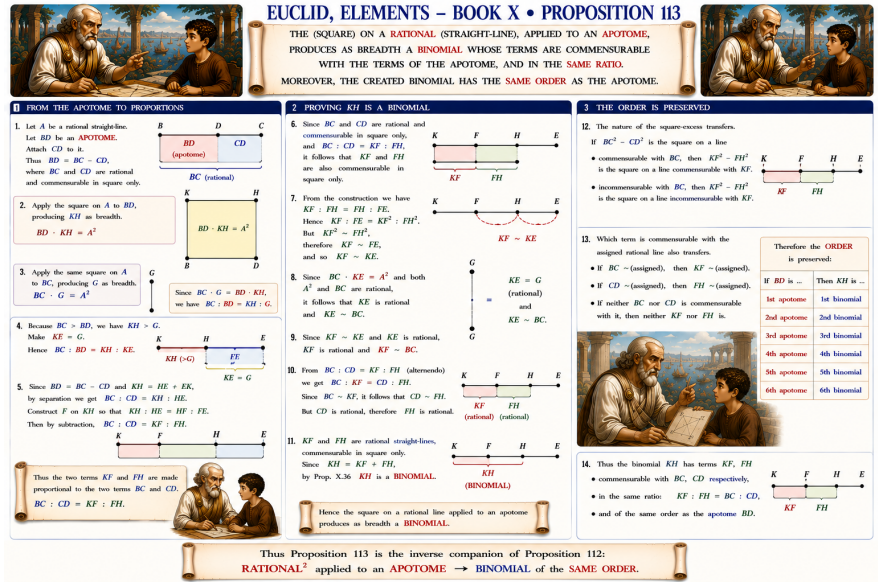
44. KH is thus a binomial whose terms, KF and FH, [are] commensurable (in length) with the terms, BC and CD, of the apotome, and in the same ratio.

45. Moreover, KH will have the same order as BC [Defs. 10.5—10.10].

46. (Which is) the very thing it was required to show.

Complete Cinematic Logic





2.114 Proposition 114

Statement

> *If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).*

Proof Actions

1. For let an area, the (rectangle contained) by AB and CD, have been contained by the apotome AB, and the binomial CD, of which let the greater term be CE.

2. And let the terms of the binomial, CE and ED, be commensurable with the terms of the apotome, AF and FB (respectively), and in the same ratio.

3. And let the square-root of the (rectangle contained) by AB and CD be G.

4. I say that G is a rational (straight-line).

5. For let the rational (straight-line) H be laid down.

6. And let (some rectangle), equal to the (square) on H, have been applied to CD, producing KL as breadth.

7. Thus, KL is an apotome, of which let the terms, KM and ML, be commensurable with the terms of the binomial, CE and ED (respectively), and in the same ratio [Prop. 10.112].

8. But, CE and ED are also commensurable with AF and FB (respectively), and in the same ratio.

9. Thus, as AF is to FB, so KM (is) to ML.

10. Thus, alternately, as AF is to KM, so BF (is) to LM [Prop. 5.16].

11. Thus, the remainder AB is also to the remainder KL as AF (is) to KM [Prop. 5.19].

12. And AF (is) commensurable with KM [Prop. 10.12].

13. AB is thus also commensurable with KL [Prop. 10.11].

14. And as AB is to KL, so the (rectangle contained) by CD and AB (is) to the (rectangle contained) by CD and KL [Prop. 6.1].

15. Thus, the (rectangle contained) by CD and AB is also commensurable with the (rectangle contained) by CD and KL [Prop. 10.11].

16. And the (rectangle contained) by CD and KL (is) equal to the (square) on H.

17. Thus, the (rectangle contained) by CD and AB is commensurable with the (square) on H.

18. And the (square) on G is equal to the (rectangle contained) by CD and AB.

19. The (square) on G is thus commensurable with the (square) on H.

20. And the (square) on H (is) rational.

21. Thus, the (square) on G is also rational. G is thus rational.

22. And it is the square-root of the (rectangle contained) by CD and AB.

23. Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

24. Corollary And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines.

25. (Which is) the very thing it was required to show.

Complete Cinematic Logic

AB apotome, CD binomial, with corresponding terms commensurable and in the same ratio.

$$\Downarrow$$

$$G^2 = AB \cdot CD$$

$$\Downarrow$$

Choose rational H and apply H^2 to CD, producing an apotome KL by X.112.

$$\Downarrow$$

$$AB \sim KL$$

$$\Downarrow$$

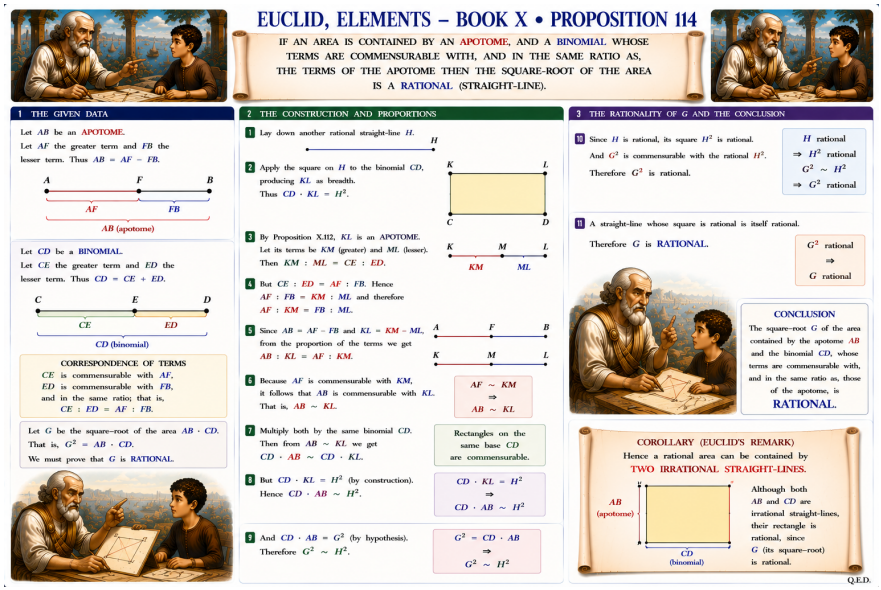
$$CD \cdot AB \sim CD \cdot KL = H^2$$

$$\Downarrow$$

$$G^2 \sim H^2, \qquad H^2 \text{ rational}$$

$$\Downarrow$$

G is RATIONAL



2.115 Proposition 115

Statement

An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).

Proof Actions

1. Let A be a medial (straight-line).
2. I say that an infinite (series) of irrational (straight-lines) can be created from A, and that none of them is the same as any of the preceding (straight-lines).
3. Let the rational (straight-line) B be laid down.
4. And let the (square) on C be equal to the (rectangle contained) by B and A.
5. Thus, C is irrational [Def. 10.4].

6. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20].

7. And (C is) not the same as any of the preceding (straight-lines).

8. For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth.

9. So, again, let the (square) on D be equal to the (rectangle contained) by B and C.

10. Thus, the (square) on D is irrational [Prop. 10.20].

11. D is thus irrational [Def. 10.4].

12. And (D is) not the same as any of the preceding (straight-lines).

13. For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces C as breadth.

14. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines).

15. (Which is) the very thing it was required to show.

Complete Cinematic Logic

A medial. Fix a rational straight-line B.

$$\Downarrow$$

$$C^2 = B \cdot A$$

$$\Downarrow$$

$$D^2 = B \cdot C$$

$$\Downarrow$$

$$E^2 = B \cdot D$$

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$$x_{n+1}^2 = B\,x_n$$

$$\Downarrow$$

Every new term is irrational and differs from all preceding terms.

$$\Downarrow$$

INFINITELY MANY DISTINCT IRRATIONAL STRAIGHT-LINES

EUCLID, ELEMENTS – BOOK X • PROPOSITION 115

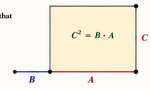
AN INFINITE (SERIES) OF IRRATIONAL (STRAIGHT-LINES) CAN BE CREATED FROM A MEDIAL (STRAIGHT-LINE), AND NONE OF THEM IS THE SAME AS ANY OF THE PRECEDING (STRAIGHT-LINES).

1 START WITH A MEDIAL STRAIGHT-LINE

1 Let A be a **MEDIAL** straight-line.

2 Lay down a **RATIONAL** straight-line B.

3 Construct C such that $C^2 = B \cdot A$.



4 Since A is irrational and B is rational, the rectangle $B \cdot A$ is irrational. Hence C^2 is irrational, therefore C is **IRRATIONAL**.

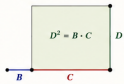
5 C cannot be the same as any of the previously known irrational straight-lines, because the square on none of them, when applied to a rational straight-line, produces a medial straight-line as breadth.

But here $\frac{C^2}{B} = A$ (medial).

Therefore C is a new irrational straight-line.

2 REPEAT THE CONSTRUCTION

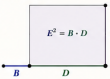
6 Construct D such that $D^2 = B \cdot C$.



7 Since C is irrational and B is rational, $B \cdot C$ is irrational. Hence D^2 is irrational, therefore D is **IRRATIONAL**.

8 If D were the same as A or C, then the square on that preceding straight-line, applied to the rational B, would produce the other preceding straight-line as breadth. This is impossible. Therefore D is different from A and C.

9 Construct E such that $E^2 = B \cdot D$.



10 E is irrational and different from A, C and D.

THE RECURSIVE RULE

From any irrational x_n in the sequence, construct x_{n+1} by $x_{n+1}^2 = B \cdot x_n$,

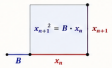
where B remains the fixed rational straight-line.

3 INFINITE SERIES OF DISTINCT IRRATIONAL STRAIGHT-LINES

11 At every stage:

If x_n is irrational and B is rational, then $B \cdot x_n$ is irrational.

Hence $x_{n+1}^2 = B \cdot x_n$ is irrational, so x_{n+1} is irrational.

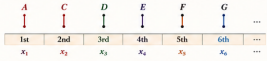


12 Also, $\frac{x_{n+1}^2}{B} = x_n$.

Therefore if x_{n+1} were the same as any preceding straight-line x_k ($k \le n$), then the square on x_k applied to B would produce x_n as breadth.

But this never happens in the construction. Thus x_{n+1} is different from all preceding straight-lines.

13 Continue this process without end.



GENERAL FORM

$x_1 = A$ (medial), $x_{n+1}^2 = B \cdot x_n$ for $n = 1, 2, 3, \ldots$

All x_n are irrational and mutually distinct.

CONCLUSION

From the medial straight-line A and the fixed rational straight-line B we can create an **INFINITE SERIES of IRRATIONAL STRAIGHT-LINES**, and none of them is the same as any of the preceding.

THE INFINITE GENERATION

A (medial) → $C^2 = B \cdot A$ → $D^2 = B \cdot C$ → $E^2 = B \cdot D$ → $F^2 = B \cdot E$ → $G^2 = B \cdot F$ → No repetition. Infinitely many irrational straight-lines.

End of Book X